Recent Advances in Banach lattices

Gerard Buskes (University of Mississippi)
Ben de Pagter (Delft University of Technology)
Vladimir Troitsky (University of Alberta)
Anthony Wickstead (Queen's University Belfast)

April 29 – May 4, 2018

1 Overview

This workshop focused on recent developments in the area of Banach lattices. The goal of the workshop was to bring together leading experts and active young researchers to discuss the current and future directions of these developments and to identify potential applications and the main open problems. We planned to understand the "big picture" of connections between these developments and other areas of Functional Analysis.

The workshop focused on the following topics:

- Tensor products and polynomials in Banach lattices
- Free Banach lattices
- Multi-normed spaces and their connections to Banach lattices
- Stochastic analysis in vector and Banach lattices
- Unbounded convergences
- Applications to Mathematical Economics/Finance

Each day (or half-day) of the workshop was devoted to one subject. Experts in these subjects presented mini-course style presentations.

2 Overview and history of the field

A vector lattice (Riesz spaces) is a vector space which is also a lattice, such that the two structures are compatible in a certain natural way. If, in addition, the space is a Banach space (and, again, a certain natural compatibility axiom is satisfied), one gets the concept of a Banach lattice. Most classical spaces that appear in modern Functional Analysis are vector and Banach lattices.

We start with the "old" history of the subject. The unofficial beginning of this area was the address of F. Riesz at the ICM in 1928, followed by works of H. Freudenthal and L. Kantorovich on vector lattices. In its early days (around the 1950s), most activities were devoted to vector lattices and were concentrated in several major schools: the Russian school (L. Kantorovich and the Krein brothers), the Japanese school (H. Nakano and T. Ando), the German school (H.H. Schaefer), and the Dutch school (A.C. Zaanen and W.Luxemburg). Since then, the area has expanded and has developed connections with many other areas of mathematics, including Banach spaces, which lead to the rise of Banach lattices. Several major advances in Banach lattices were made around the 1970s by H.H. Schaefer, P. Meyer-Nieberg, J. Lindenstrauss, L. Tsafriri, W. Johnson, and N. Ghoussoub. To mention a few other important connections (which were not addressed at the workshop): Function spaces, Spectral Theory of positive operators, Perron-Frobenius Theory of non-negative matrices, Ergodic Theory, Operator Semigroups, Convex Analysis, Inequalities, Banach lattice algebras, etc. There have been numerous applications of vector and Banach lattice to Math Economics and Math Finance. These applications go back to works of L. Kantorovich and, more recently, of C. Aliprantis and I. Polyrakis. In these applications, vector and Banach lattices are used to model markets and the theory of positivity is used in search of market equilibrium.

The subject area of vector and Banach lattices (which is nowadays often referred to as "*Positivity*") is old and well established, and many parts of it have become classical. Members of the Positivity community have been distinguished in various ways. L. Kantorovich received the Nobel Prize in Economics for applications of Positivity in Economics. H. Schaefer was a member of Mathematics and Natural Sciences Class of the Heidelberg Academy of Sciences and of the Academy of Sciences in Zaragoza. C. Aliprantis was a distinguished professor of Economics and Mathematics at Purdue University; he was also the founding editor of the journals Economic Theory and Annals of Finance. W. Luxemburg has been a Fellow of the American Mathematical Society; he, A. Zaanen, and H. Freudenthal have been members of the Royal Netherlands Academy of Arts and Sciences. In 2000, International Commission on Mathematical Instruction at ICU instituted a Hans Freudenthal Medal. B. Johnson was awarded a Stefan Banach Medal in 2007. G. Curbera was a Curator at IMU over the period of 2011–2014.

There have been many regular and irregular conferences and workshops. In the 70s–80s, there was a series of workshops on Riesz spaces in Oberwolfach: June 24–30 1973, June 22–28 1975, June 19–25 1977, June 24–30 1979, June 27 – July 3 1982, July 1–5 1985, and April 23–29 Apr 1989. Currently, *Positivity* conferences are normally held every second year; they are the main events for the Positivity community. Below we list the locations of past Positivity conferences.

- Edmonton, Canada, 2017
- Chengdu, China, 2015
- Leiden, Netherlands, 2013
- Madrid, Spain, 2009
- Belfast, UK, 2007
- Dresden, Germany, 2005
- Rhodos, Greece, 2003
- Nijmegen, Netherlands, 2001
- Ankara, Turkey, 1999

Here we list a few recent "irregular" conferences and workshops:

- Positivity in Functional Analysis and Applications session at the 2006 Summer Meeting of the Canadian Mathematical Society, Calgary, Alberta.
- Symposium on Positivity and Its Applications in Science and Economics, Bolu, Turkey, September 2008.
- Workshop on Nonnegative Matrix Theory, American Institute of Mathematics, Palo Alto, California, December 2008.
- Ordered Spaces and Applications Conference, National Technical University of Athens, Greece, 2011.

- Ordered Banach Algebras Workshop, Lorentz Centre, Leiden, Netherlands, July 2014.
- Workshop on Operators and Banach lattices I and II, Universidad Complutense de Madrid, October 2012 and November 2016.

For further information about the field, we refer the reader to the following classical monographs: [1, 2, 36, 37, 39, 43].

Several new areas of vector and Banach lattice theory and applications have emerged lately and have been very active during the last decade. The workshop was focused on some of these areas.

3 Tensor products of Banach lattices and polynomials on Banach lattices

In [5], Y. Benyamini, S. Lassalle, and J.G. Llavona revived the topic of homogeneous orthogonally additive polynomials on Banach lattices, which had originally been started by K. Sundaresan in [42]. They connected the study of such polynomials with concavifications and this has subsequently led to a very rapid development in [7, 8, 9]. Newer developments have used the theory of Fremlin tensor products [14, 15], which provides "natural" tensor products for the categories of vector lattices and Banach lattices. This created the link with an already well-established theory of homogeneous polynomials and multilinear maps on Banach spaces.

At the workshop, G. Buskes presented a mini-course (two lectures) on this subject. In his mini-course, new directions and further explorations were proposed:

- For every reasonable cross norm p on on the algebraic tensor product of two Banach lattices E and F, there exists a reasonable cross norm $\hat{p}$ on that tensor product for which the completion is a Banach lattice with the $\hat{p}$-closure of the Fremlin cone as its positive cone. Though any Banach lattice tensor product can be obtained this way (a result due to C. Labuschagne), the map that sends p to $\hat{p}$ has otherwise not been studied. For instance, what can be said about the image of the set of the natural Grothendieck norms?
- Banach lattice tensor products are appearing in the study of holomorphic functions in the works of R. Ryan et al. For instance, Boyd, Ryan, and Snigireva showed in [41] that the radius of analyticity for a power series of positive polynomials on Banach lattices equals its radius of uniform convergence. The connection between analytic functions and tensor products is part of a larger program that studies polynomials on Banach lattices.
- Preservation of Banach space properties under formation of Banach lattice tensor products and a study of geometric and other properties of Banach lattices of spaces of polynomials on Banach lattices via tensor products have been started in e.g. [7, 8, 9, 30].

4 Free Banach lattices

Free vector lattices, — the free objects in the category of vector lattices, — have been known for a long time, see, e.g., [4]. A vector lattice X is a free vector lattice over a subset A if every map $f\colon A \to Y$, where Y is another vector lattice, extends uniquely to a lattice homomorphism $T\colon X \to Y$; we write $X = FVL(A)$. In a major recent paper [40], B. de Pagter and A.Wickstead proved the existence of free Banach lattices and characterized some of their properties. A Banach lattice X is a free Banach lattice over a subset A if every bounded map $f\colon A \to Y$, where Y is another Banach lattice, extends uniquely to a lattice homomorphism $T\colon X \to Y$ with $\|T\| = \sup_{a\in A}\|f(a)\|$; we then write $X = FBL(A)$. It was observed in [40] that $FBL(A)$ has a very interesting and rich structure. In [3], A.Avilés, J.Rodríguez, and P.Tradacete came up with an alternative approach to the subject. Namely, instead of $FBL(A)$, they consider $FBL[E]$, the free Banach lattice generated by a given Banach space E, defined as follows: a Banach lattice X is a free Banach lattice over a closed subspace E if every bounded linear operator $S\colon E \to Y$, where Y is another Banach lattice, extends uniquely to a lattice homomorphism $T\colon X \to Y$ with $\|T\| = \|S\|$. This approach allowed them to solve an open problem of J.Diestel.

P.Tradacete presented a mini-course on this subject at the workshop, consisting of three lectures. The first lecture was primarily focused on free vector lattices, the second lecture was focused on free Banach lattices generated by sets and by Banach spaces; the third lecture was focused on applications of this technique.

5 Multinormed spaces

The idea of a multinorm is very natural: while a norm measures the "size" of a single vector in a normed space, a multinorm measures the size of a "multivector", i.e., a finite sequence of vectors. Multinorms have been implicitly used in Functional Analysis for a long time. During the last few years, the concept of a multinorm has been formalized and a systematic study of multinorms and, more generally, of p-multinorms, has been developed in [10, 11, 12, 13, 6]. Multinormes are naturally related to tensor products of Banach spaces. Many well known concepts in Functional Analysis may be expressed naturally in the language of multinorms (e.g., absolutely summing operators). In the last few years, several representation theorems for multinormed spaces have been developed, some of them going back to unpublished work of G. Pisier, see [38, 13]. Namely, multinormed spaces may be represented as closed subspaces of Banach lattices. Recently, T. Oikhberg extended this result by proving that certain p-multinormed spaces may be represented as subquotients of Banach lattices.

At the workshop, G. Dales presented an overview of the theory of multinorms (including its history and numerous examples), N.Laustsen introduced and discussed p-multinorms and discussed representation theorems for multinorms. A. Helemski presented a continuous version of p-multinorms, essentially replacing multivectors with multifunctions. Finally, T.Oikhberg discussed injective and projective objects in the category of p-multinormed spaces, as well as a representation theorem for p-multinorms.

6 Stochastic processes

Historically, Stochastic Processes have been modelled using tools of Probability Theory. Around 2005, a new approach was started in a series of papers by J.Grobler, C.Labuschagne, V. Troitsky, B. Watson, and others, in which stochastic processes are modelled using tools of vector or Banach lattices. In this approach, a filtration is a sequence of positive projection on the lattice, while a stochastic process is a sequence of vectors in the lattice, which is adapted to the filtration. This "measure-free" approach generalizes the classical approach. Many facts of the classical theory have been "migrated" into the "measure-free" setting. In particular, Doob's Martingale Convergence theorem, Brownian Motion, and the theory of stochastic integration has been extended to the measure-free setting; see [22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 16].

There is a considerable interplay going on between the "measure-free" approach and the classical theory of stochastic processes. It combines various techniques and facts of Probability Theory with power of Functional Analysis.

At the workshop, W. Kuo presented an overview of the theory of stochastic processes on vector lattices, including conditional expectations, martingales, and filtrations. J. Grobler presented several vector lattice techniques that have been developed in the last few years, which are critical for this theory and its applications. In particular, he discussed universal and $\sup$-completions, Brownian Motion, and Dobrakov and Itô integration.

7 Unbounded convergences

Uo-convergence (where "uo" stands for "unbounded order") in a vector lattice is defined as follows: $x_\alpha \xrightarrow{uo} x$ if $|x_\alpha - x| \wedge u$ converges in order to zero for every positive vector u. Un-convergence (unbounded norm convergence) in a Banach lattice is defined similarly by replacing the order convergence in the definition with norm convergence. These two convergences have been known for a long time, but have became very popular in the last few years due to some recent discoveries that led to numerous applications. In particular, that uo-convergence is stable under passing to and from regular sublattices; see [19]. It follows that in Banach Function Spaces, uo-convergence agrees with almost everywhere convergence, while un-convergence agrees with convergence in measure. This allowed extending many results from Measure Theory and Banach Function Spaces to general vector Banach lattices; see [16, 17, 19, 20].

At the workshop, V. Troitsky presented an overview of uo-convergence, M. Kandić presented an overview of un-convergence and the corresponding un-topology. J. van der Walt presented a characterization of uo-convergence in spaces of continuous functions. He essentially proved that in these spaces, uo-convergence may still be interpreted as convergence "almost everywhere", where "almost everywhere" is understood in a certain topological sense.

M. Taylor presented generalizations of unbounded convergences to locally solid vector lattices. Unfortunately, due to health reasons that surfaced shortly before the workshop, M. Taylor could not attend in person, so his presentation was pre-recorded and then played out at the workshop.

E.Emelyanov presented a generalization of unbounded convergences to lattice-normed spaces.

8 Applications to Math Finance

Some of the techniques that have been recently developed in vector and Banach lattices have been actively applied to Math Finance. In particular, measure-free stochastic theory replaces the classical stochastic theory in option pricing models, with uo-convergence replacing almost everywhere convergence.

There were two expository talks on this subject at the workshop. The first one was about using measure-free stochastic analysis for option pricing. This project represents a major program of replacing (or, rather, generalizing) the huge machinery of classical Stochastic Analysis with measure-free analogues in the setting of vector lattices. This program is now close to completion. Initially, this talk was supposed to be presented by C. Labuschagne. However, unfortunately, C. Labuschagne had to cancel his particilation on a short notice due to a sudden serious medical condition; his talk was presented by J. Grobler instead. It was based on [34] and on some recent unpublished work of C. Labuschagne and J. Grobler.

The second talk was an overview applications of uo-convergence to Math Finance; in particular, to risk measures on Orlicz spaces; it was presented by F. Xanthos; the talk was based on [18, 20, 21].

9 Universal Banach Lattice

In the morning session of the last day of the workshop, T. Oikhberg presented an overview of the recent paper [35]. In that paper, the authors construct a separable Banach lattice U which is universal in the class of separable Banach lattices in the following sense: U contains every separable Banach lattice as a sublattice. This is analogous to the classical fact that the space $C[0, 1]$ contains every separable Banach space as a closed subspace. In the same paper, the author also constructed a universal projective Banach lattice, such that every separable Banach lattice is its quotient (over a closed ideal).

10 Conclusion

The meeting went in a cosy and friendly environment, with many informal discussions during talks and after talks. During the workshop, groups of people were seen everywhere in the centre working on joint projects.

Several open questions were mentioned and discussed during the workshop. In particular, E.Emelyanov initiated a discussion whether the Brezis-Lieb Lemma extends to uo-convergence in vector lattices, while F.Dashiell and G.Dales proposed characterizing second preduals of $C(K)$ spaces in Banach lattice terms.

The workshop served as a good mixer: it brought together mathematicians working in areas which are different but sufficiently close, so that people could follow and understand each other's results. In addition, there were many young mathematicians, including graduate students and postdoctoral researchers. Most of the talks were on introductory level; this allowed to introduce the students into the subjects. This workshop helped more experienced mathematicians in exchanging of ideas and learning new subjects; it provided young people with an overview of what is currently active in the area.

References

[1] Y. Abramovich and C.D. Aliprantis, *An invitation to Operator theory*, Vol. 50. Providence, RI: American Mathematical Society, 2002.

[2] C.D. Aliprantis and O. Burkinshaw, *Positive Operators*, Springer 2006.

[3] A. Avilés, J. Rodríguez, P. Tradacete, The free Banach lattice generated by a Banach space. *J. Funct. Anal.*, **274** (2018), no. 10, 2955–2977.

[4] R.D. Bleier, Free vector lattices, *Trans. Amer. Math. Soc.*, **176** (1973), pp. 73–87.

[5] Y. Benyamini, S. Lassalle, and J. G. Llavona, Homogeneous orthogonally additive polynomials on Banach lattices, *Bull. London Math. Soc.*, **38** (2006), 459–469.

[6] O. Blasco, Power-normed spaces, *Positivity*, **21** (2017), 593–632.

[7] Q. Bu, G. Buskes, Polynomials on Banach lattices and positive tensor products, *J. Math. Anal. Appl.*, **388** (2012), 845–862.

[8] Q. Bu, G. Buskes, A. Popov, A. Tcaciuc, V. Troitsky, The 2-concavification of a Banach lattice equals the diagonal of the Fremlin tensor square, *Positivity*, **17** (2013), no. 2, 283–298.

[9] Q. Bu, G. Buskes, Diagonals of projective tensor products and orthogonally additive polynomials. *Studia Math.* **221** (2014), no. 2, 101–115.

[10] H.G. Dales and M.E. Polyakov, Multi-normed spaces, *Dissertationes Math.*, **488** (2012), 165 pp.

[11] H.G. Dales, M. Daws, H.L. Pham, P. Ramsden, Multi-norms and the injectivity of $L_p(G)$, *J. London Math. Soc.*(2), **86** (2012), 779–809.

[12] H.G. Dales, M. Daws, H.L. Pham, P. Ramsden, Equivalence of multi-norms, *Dissertationes Math.*, **498** (2014), 53 pp.

[13] H.G. Dales, N. Laustsen, T. Oikhberg, V.G. Troitsky, Multi-norms and Banach lattices. *Dissertationes Math.*, **524** (2017), 115 pp.

[14] D.H. Fremlin, Tensor products of Archimedean vector lattices, *Amer. J. Math.*, **94** (1972), 777–798.

[15] D.H. Fremlin, Tensor products of Banach lattices, *Math. Ann.*, **211** (1974), 87–106.

[16] N. Gao and F. Xanthos, Unbounded order convergence and application to martingales without probability, *J. Math. Anal. Appl.*, **415** (2014), 931–947.

[17] N. Gao, Unbounded order convergence in dual spaces, *J. Math. Anal. Appl.*, **419**, 2014, 347–354.

[18] N. Gao, F. Xanthos, Option spanning beyond L_p-models, *Math. Financ. Econ.*, **11** (2017), no. 3, 383–391.

[19] N. Gao, V.G. Troitsky, and F. Xanthos, Uo-convergence and its applications to Cesàro means in Banach lattices, *Israel J. Math.*, **220**, (2017), 649–689.

[20] N. Gao, D. Leung, C. Munari, F. Xanthos, Fatou property, representations, and extensions of law-invariant risk measures on general Orlicz spaces, *Finance and Stochastics*, **22**(2), (2018) 395–415.

[21] N. Gao, F. Xanthos, On the C-property and w*-representations of risk measures, *Mathematical Finance*, **28**(2), (2018) 748–754.

[22] J. Grobler, C.C.A. Labuschagne, Itô's rule and Lévy's theorem in vector lattices. *J. Math. Anal. Appl.* **455** (2017), no. 2, 979–1004.

[23] J. Grobler, C.C.A. Labuschagne, The Itô integral for martingales in vector lattices. *J. Math. Anal. Appl.*, **450** (2017), no. 2, 1245–1274.

[24] J. Grobler, C.C.A. Labuschagne, The quadratic variation of continuous time stochastic processes in vector lattices. *J. Math. Anal. Appl.*, **450** (2017), no. 1, 314–329.

[25] J. Grobler, C.C.A. Labuschagne, The Itô integral for Brownian motion in vector lattices: Part 2. *J. Math. Anal. Appl.*, **423** (2015), no. 1, 820–833.

[26] J. Grobler, C.C.A. Labuschagne, The Itô integral for Brownian motion in vector lattices: Part 1. *J. Math. Anal. Appl.*, **423** (2015), no. 1, 797–819.

[27] J. Grobler, C.C.A. Labuschagne, Jensen's and martingale inequalities in Riesz spaces. *Indag. Math. (N.S.)* **25** (2014), no. 2, 275–295.

[28] J.J. Grobler, The Kolmogorov-Čentsov theorem and Brownian motion in vector lattices. *J. Math. Anal. Appl.* **410** (2014), no. 2, 891–901.

[29] J.J. Grobler, Continuous stochastic processes in Riesz spaces: the Doob-Meyer decomposition. *Positivity*, **14** (2010), no. 4, 731–751.

[30] D. Ji, B. Lee, Q. Bu, Diagonals of injective tensor products of Banach lattices with bases. *Positivity*, **21** (2017), no. 3, 975–988.

[31] W. Kuo, M.J.J. Rogans, B.A. Watson, Mixing inequalities in Riesz spaces. *J. Math. Anal. Appl.*, **456** (2017), no. 2, 992–1004.

[32] W. Kuo, J. Vardy, B.A. Watson, Mixingales on Riesz spaces. *J. Math. Anal. Appl.*, **402** (2013), no. 2, 731–738.

[33] C.A.A. Labuschagne, B.A. Watson, Discrete stochastic integration in Riesz spaces. *Positivity*, **14** (2010), no. 4, 859–875.

[34] C.A.A. Labuschagne, H.S.I. Ouerdiane, Risk measures on Orlicz heart spaces. *Commun. Stoch. Anal.*, **9** (2015), no. 2, 169–180.

[35] D.H. Leung, L. Li, T. Oikhberg, M.A. Tursi, Separable universal Banach lattices, *Israel J. of Math.*, to appear.

[36] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II*, Springer-Verlag, Berlin, 1979.

[37] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Publishing Co., Amsterdam-London, 1971, North-Holland Mathematical Library.

[38] J.L. Marcolino Nhani, La structure des sous-espaces de treillis, *Dissertationes Math.*, **397** (2001), 50 pp.

[39] P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991.

[40] B. de Pagter, A.W. Wickstead, Free and projective Banach lattices, *Proc. Roy. Soc. Edinburgh Sect. A*, **145**(1) (2015), pp. 105–143.

[41] C. Boyd, R. Ryan, N. Snigireva, Radius of analyticity of analytic functions on Banach spaces, *J. Math. Anal. Appl.*, **463** (2018), no. 1, 40–49.

[42] K. Sundaresan, Geometry of spaces of homogeneous polynomials on Banach lattices, in: *Applied Geometry and Discrete Mathematics*, DIMACS Ser. Discrete Math. Theoret. Cmput. Sci,.4, Amer. Math. Soc., Providence, RI, 1991.

[43] A. C. Zaanen, *Riesz spaces. II*, North-Holland Publishing Co., Amsterdam-New York, 1983.