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BIRS

Contents

Five-day Workshop Reports

Chapter 1

Mathematical Programming in Data Mining and Machine Learning (07w5078)

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Overview of the Field

The field of Machine Learning (ML) and Data Mining (DM) is focused around the following problem: Given a data domain D we want to approximate an unknown function $y(x)$ on the given data set $X \subset D$ (for which the values of $y(x)$ may or may not be known) by a function f from a given class $\mathcal{F}$ so that the approximation generalizes in the best possible way on all of the (unseen) data $x \in D$. The approximating function f might take real values, as in the case of regression; binary values, as in the case of classification; or integer values, as in some cases of ranking; or this function might be a mapping between ordered subsets of data points and ordered subsets of real, integer or binary values, as in the case of structured object prediction. The quality of approximation by f can be measured by various objective functions. For instance in the case of support vector machine (SVM)[4] classification the quality of the approximating function is estimated by a weighted sum of a regularization term $h(f)$ and the hinge loss term $\sum_{x \in X} \max\{1 - y(x)f(x), 0\}$. Hence, many of the machine learning problems can be posed as an optimization problem where optimization is performed over a given class $\mathcal{F}$ for a chosen objective.

The connection between optimization and machine learning (although always present) became especially evident with the popularity of the SVMs [4], [24], and the kernel methods in general [75]. SVM classification problem is formulated as a convex quadratic program.

$$
(P) \qquad \begin{array}{ll}
\max & -\frac{1}{2}\alpha^T Q\alpha - c\sum_{i=1}^{n} \xi_i \\
\text{s.t.} & -Q\alpha + by + s - \xi = -e, \\
 & 0 \le \alpha \le c, \ s \ge 0, \ \xi \ge 0,
\end{array}
$$

where $\alpha \in \mathbf{R}^n$ is the vector of dual variables, b is the bias (scalar) and s and ξ are the n-dimensional vectors of slack and surplus variables, respectively. y is a vector of labels, e is the vector of all 1's of length n and

c is the penalty parameter of the loss function in the objective. Q is the label encoded kernel matrix, i.e. $Q_{ij} = y_i y_j K(x_i, x_j)$, where $K(\cdot, \cdot)$ the the kernel matrix (or function) which implicitely defines the class $\mathcal{F}$.

The problem is always feasible and, in theory, finding the global solution for this problem is easy, that is it can be done in polynomial time. However, many large-scale practical cases proved to be difficult to handle by standard optimization software. This led to a number of special purpose implementations. First implementations were developed by researchers from the ML community [15], [10] and some of the later implementations were proposed by the mathematical programming (MP) community [7], [6], [5], [17].

Success of the convex optimization for SVMs led to the extensive use of convex optimization models and methods for other machine learning problems in the past 6-7 years, such as learning the kernel [12], [3], where optimization is done over the matrices $K(x_i, x_j)$; and computation of the entire regularization path [8], [16], where soltion is found for all possible values of the penalty parameter c. Beyond classification, optimization model have been used in dimensionality reduction [9], [26]; low rank matrix factorization [20], [11], metric learning [29], [25], structured objects prediction [23], [22] and many others. It became apparent that the connection between the two fields can benefit greatly from the collaboration of the researches from both sides. However, the MP community and the ML/DM community are quite disjoint. They do not typically share conferences or publication venues. To remedy this situation in the past several years there have been several occasions when researches from both fields were brought together in a workshop or a conference. There were two workshop organized specifically on optimization in machine learning and data mining. Both were held in 2005, one held in Trunau,Germany and one at McMaster University in Canada. A special topic on machine learning and large scale optimization was published in the Journal of Machine Learning Research in 2006. The potential for collaboration between the fields have been steadily increasing in the last few years.

The purpose of the Banff workshop was to continue and improve upon the effort of bringing together outstanding researches from the fields of mathematical programming, data mining and statistical machine learning to ignite new collaborations and expose each side to the possibilities available in each field. The purpose is to identify the new problems and to match them with potential solution approaches.

Recent Developments and Open Problems

The amount of literature in the Machine Learning and Data Mining communities that involves optimization models is very extensive. We do not attempt to present a comprehensive survey of existing and possible topics here. We focus, instead, on the work immediately relevant to the results presented at workshop. This work, in fact, is a representative selection of recent developments and open problems in the field, but it is by no means axhaustive.

Nonlinear Classification

Nonlinear classification via kernels is the essence of support vector machines. There are still many unresolved questions, perhaps main of them being, how to find a good kernel. This question has been addressed in recent years with limited success by means of convex optimization [12], [3].

The two issues discussed at the workshop were of a different nature, however. It is well known that the use of kernels in SVM is made possible by the use of ℓ_2 regularization term on the model parameters. It is unclear how to extend kernalization for the ℓ_1 regularization, which otherwise may produce better (more sparse) models. Saharon Rosset in his talk discussed the extension of the ℓ_1 regularization case to the infinite dimensional case which allows the use of nonlinear ℓ_1 classification. Ted Wild addressed topic of exploiting prior knowledge when using a kernel.

Structured objects prediction

Structured objects prediction is a generalization of the standard classification or regression problems. Instead of predicting a label of an object the structured object prediction aim to predict a set of labels of a collection of objects. Structured output classification can be posed similarly to a classification problem but with exponential number of constraints. Due to the very large number of constraints and a somewhat different motivation for structured output prediction the extension of classification approaches in not straightforward. A few talk of the workshop addressed this issue. We list ranking in the category of structured output prediction problems because these problems are related. In both cases the number of constraints is too large to search over exhaustively and in both cases the optimization function (the loss function and the regularization) is more complex to derive than for the classification and regression problems. Of the talks on the topic presented at the conference one talk (by Yasemin Altun) addressed the question of introducing regularization into the structured output prediction problems. Another talk (by Chris Burges) discussed the modeling approaches specifically to the ranking problem and the third talk by Thorsten Joachims addressed an SVM formulation and a fast optimization method for predicting structured objects.

Dimensionality reduction and metric learning

Semidefinite programming (SDP)[28] was by far the most popular optimization topic of the workshop. There is a large variety of machine learning problems that are being posed as semidefinite programming problems. The main application of semidefinite programming arises in dimensionality reduction. If the data is given to us as a set of points in a space of a large dimension it is important to recognize if in actuality it lies on (or near to) a low dimensional surface. Alternatively, the data can be embedded in a low dimensional space by using a proper distance metric. If the data is represented by a matrix, it may be desirable to extract a low rank matrix factorization, or a low rank matrix completion. All of these setting can be addressed via an SDP formulation. In some case the SDP is a relaxation of the original nonconvex problem. Several talks at the workshop presented SDP models for ML problems.

A fundamental difficulty with the SDP approach is that the resulting SDPs are expensive to solve in practice with conventional methods. Recently new methods for solving large scale SDPs were proposed in the optimization community. These method have inferior performance guarantees, compared to the interior point methods, but they can be much faster in practice for certain classes of SDPs, expecially if an accurate solution is not needed [29],[14]. Fast approximate methods which exploit the structure of the specific formulations were discussed in the talk of d'Aspermont and Weinberger. A special case of solving SDPs - the problem of finding minimum volume enclosing ellipsoid was presented by Todd [2].

Clustering

Robust clustering that deals with the uncertainty, noise in the data set, which a major concern in cluster analysis. Dr. Ghosh's talk consider a special case of robust clustering where the target is to find interesting patterns in the data set by ignoring a certain number of outliers. New optimization-based approaches have been proposed. This is different from the traditional approach where the outliers were removed first based on statistical measurements.

In many scenarios such as in biological discovery and diagnosis, we need to find not only the patterns in the data set, but also the features that characterize these patterns. This leads to the so-called bi-clustering problem, which has recently become a very hot research topic in the clustering community. Pardalos' talk proposed an fractional program model to attack this problem. He also showed that the problem is NP-hard and suggested some heuristics to solve the new optimization model applied to biological applications.

Meila's talk considered the robustness of the data set, i.e., under what distribution, a 'good' clustering can converge to the best clustering.

Ben-David's talk addressed the complexity issues of clustering.

Complexity

The issues of empirical and theoretical complexity of many of the machine learning problems are still unresolved. The theoretical complexity bounds for the underlying optimization models are usually known, but they are typically based on the worst case analysis and assume that the goal is to find a fairly accurate solution. It is not always relevant for the ultimate goal of a machine learning problem - good generalization error bounds. Some of these issued were addressed at the workshop. A talk by John Langford (Yahoo! Inc.) focused on reductions of ML problems. The goal is to find relationship between machine learning problems, and construct algorithms which work via reduction. Another talk, by Nathan Srebro addressed the issue of how complexity of a ML problem should depend on the size of the available data set.

Presentation Highlights

We will now list some of the presentations arranged by the topics discussed above.

Nonlinear Classification

Speaker: **Saharon Rosset** (IBM Research)
Title: ℓ_1 *Regularization in Infinite Dimensional Feature Spaces*
Abstract:

In this talk I discuss the problem of fitting ℓ_1 regularized prediction models in infinite (possibly non-countable) dimensional feature spaces. Our main contributions are: a. Deriving a generalization of ℓ_1 regularization based on measures which can be applied in non-countable feature spaces; b. Proving that the sparsity property of ℓ_1 regularization is maintained in infinite dimensions; c. Devising a path-following algorithm that can generate the set of regularized solutions in ”nice” feature spaces; and d. Presenting an example of penalized spline models where this path following algorithm is computationally feasible, and gives encouraging empirical results.

Speaker: **Ted Wild** (U. of Wisconsin)
Title: *Nonlinear Knowledge in Kernel Machines*
Abstract:

We give a unified presentation of recent work in applying prior knowledge to nonlinear kernel approximation and nonlinear kernel classification. In both approaches, prior knowledge over general nonlinear sets is incorporated into nonlinear kernel approximation or classification problems as linear constraints in a linear program. The key tool in this incorporation is a theorem of the alternative for convex functions that converts nonlinear prior knowledge implications into linear inequalities without the need to kernelize these implications. Effectiveness of the proposed approximation formulation is demonstrated on two synthetic examples as well as an important lymph node metastasis prediction problem arising in breast cancer prognosis. Effectiveness of the proposed classification formulation is demonstrated on three publicly available datasets, including a breast cancer prognosis dataset. All these problems exhibit marked improvements upon the introduction of prior knowledge of nonlinear kernel approximation and classification approaches that do not utilize such knowledge.

Structured Objects Prediction

Speaker: **Yasemin Altun** (TTI, Chigaco)
Title: *Regularization in Learning to Predict Structured Objects*
Abstract:

Predicting objects with complex structure is ubiquitous in many application areas. Recent work on machine learning focused on devising different loss functions and algorithms for structured output prediction. Another important component of learning is regularization and it has not been explored in structured output prediction problems till now. However, the complex structure of the outputs results in learning with features with dramatically different properties, which in turn can require different regularizations. Convex analysis tools provide the connection between regularization and approximate moment matching constraints. Motivated with these theoretical results, we explore various regularization schemes in learning to predict structured outputs, in particular hierarchical classification and label sequence learning.

Speaker: **Chris Burges** (Microsoft Research)
Title: *Learning to Rank*
Abstract:

The problem of ranking occurs in many guises. The field of Information Retrieval is largely dependent on ranking: there the problem is, given a query, to sort a (sometimes huge) database of documents in order of relevance. Recommender systems a also often need to rank: given a set of movies or songs that some collaborative filtering algorithm has decided you would probably enjoy, which ones should be at the top of the list? Ranking has been less studied in the machine learning community than classification, but the two are also closely related: for a binary classifier, the area under the ROC curve (the curve of true positives versus false positives) is equal to a simple ranking statistic. In this talk I will give an overview of the problem from the point of view of the needs of a large, commercial search engine. I will describe some recent approaches to solving the ranking problem. Considering this problem highlights a serious problem in machine learning that is rarely addressed: the mismatch between the cost functions we optimize, and the ones we actually care about. I will also describe recent work that is aimed at addressing this ”optimization / target cost mismatch” problem.

Speaker: **Thorsten Joachims** (Cornell University)
Title: *Large-Margin Training for Predicting Structured Outputs*
Abstract:

Over the last decade, much of the research on discriminative learning has focused on problems like classification and regression, where the prediction is a single univariate variable. But what if we need to predict complex objects like trees, orderings, or alignments? Such problems arise, for example, when a natural language parser needs to predict the correct parse tree for a given sentence, when one needs to optimize a multivariate performance measure like the F1-score, or when predicting the alignment between two proteins.

This talk discusses how these complex and structured prediction problems can be formulated as convex programs. In particular, it presents a support vector approach that generalizes conventional classification SVMs to a large range of structured outputs and multivariate loss functions. The resulting optimization problems are convex quadratic, but have an exponential (or infinite) number of constraints. To solves the training problems efficiently, the talk explores a cutting-plane algorithm. The algorithm is implemented in the SVM-Struct software and empirical results will be given for several examples.

Demensionality Reduction and Semidefinite Programming

Speaker: **Alexandre d’Aspermont** (Princeton)
Title: *Semidefinite Optimization with Applications in Sparse Multivariate Statistics*
Abstract:

We use recently developed first order methods for semidefinite programming to solve convex relaxations of combinatorial problems arising in sparse multivariate statistics. We discuss in detail applications to sparse principal component analysis, sparse covariance selection and sparse nonnegative matrix factorization.

Speaker: **Francis Bach** (Ecole des Mines de Paris)
Title: *Low-rank matrix factorization with attributes*
Abstract:

We develop a new collaborative filtering (CF) method that combines both previously known users’ preferences, i.e. standard CF, as well as product/user attributes, i.e. classical function approximation, to predict a given user’s interest in a particular product. Our method is a generalized low rank matrix completion problem, where we learn a function whose inputs are pairs of vectors – the standard low rank matrix completion problem being a special case where the inputs to the function are the row and column indices of the matrix. We solve this generalized matrix completion problem using tensor product kernels for which we also formally generalize standard kernel properties. Benchmark experiments on movie ratings show the advantages of our generalized matrix completion method over the standard matrix completion one with no information about movies or people, as well as over standard multi-task or single task learning methods.

Speaker: **Tony Jebara** (Columbia University)
Title: *Semidefinite Programming for Classification and Dimensionality Reduction*
Abstract:

We propose semidefinite programming (SDP) to improve the support vector machine (SVM) linear classifier by exploiting tighter Vapnik-Chervonenkis (VC) bounds based on an ellipsoidal gap-tolerant classification model. SDPs are used to modify the regularization criterion for a linear classifier which improves its accuracy dramatically without making any additional assumptions on the binary classification problem. A bounding minimum volume ellipsoid is estimated via SDP on the data and used to redefine the margin in an SVM. The technique is fully kernelizable and therefore accommodates nonlinear classification as well. Tighter VC generalization bounds can also be estimated numerically using an iterated variant of SDP.

In addition, a similar iterated variant of SDP is used to improve dimensionality reduction by directly optimizing the eigen-gap. This method is reminiscent of semidefinite embedding which reduces dimensionality of the data by maximizing the trace of a matrix (the sum of the eigenvalues). Our novel method gives rise to a more general linear function of the eigenvalues in the SDP which is handled iteratively by interleaving the SDP with eigen-decomposition. In some cases, only global minima exist for these general linear functions of eigenvalues. Experiments reveal that this is a competitive method for visualizing high dimensional data.

Speaker: **Sam Roweis** (U.Toronto)
Title: *Visualizing Pairwise Similarity via Semidefinite Programming*
Abstract:

Binary pairwise similarity data is available in many domains where quantifying the similarity/difference between objects is extremely difficult or impossible. Nonetheless, it is often desirable to obtain insight into such data by associating each object (record) with a point in some abstract feature space – for visualization purposes this space is often two or three dimensional. We present an algorithm for visualizing such similarity data, which delivers an embedding of each object such that similar objects are always closer in the embedding space than dissimilar ones. Many such mappings may exist, and our method selects amongst them the one in which the mean distance between embedded points is as large as possible. This has the effect of stretching the mapping and, interestingly, favoring embeddings with low effective dimensionality.

We study both the parametric and non-parametric variants of the problem, showing that they both result in convex Semidefinite Programs (SDP). In the non-parametric version, input points may be mapped to any point in space, whereas the parametric version assumes that the mapping is given by some function (e.g. a linear or kernel mapping) of the input. This allows us to generalize the embedding to points not used in the training procedure.

Speaker: **Michael J. Todd** (Cornell University)
Title: *On minimum-volume ellipsoids: From John and Kiefer-Wolfowitz to Khachiyan and Nesterov-Nemirovski*
Abstract:

The problem of finding the minimum-volume ellipsoid containing a set in R^n has arisen in contexts from optimization to statistics and data analysis over the last sixty years. We describe some of these settings and algorithms old and new for solving the problem.

Speaker: **Kilian Weinberger** (University of Pennsylvania)
Title: *Distance Metric Learning via Semidefinite Programming*
Abstract: Many problems in computer science can be simplified by clever representations of sensory or symbolic input. How to discover such representations automatically, from large amounts of data, remains a fundamental challenge. The goal of metric learning is to derive Euclidean representations of labeled or unlabeled inputs from observed statistical regularities. In this talk I will review two recently proposed algorithms for metric learning. Both algorithms rely on modern tools in convex optimization that are proving increasingly useful in many areas of machine learning. In addition to the two metric learning algorithms, I will propose a novel method [27] to approximate large scale SDPs with Laplacian graph regularization.

Clustering

Speaker: **Joydeep Ghosh** (UT Austin)
Title: *Locating a Few Good Clusters: A Tale of Two Viewpoints*
Abstract:

Many applications involve discovering a small number of dense or cohesive clusters in the data while ignoring the bulk of the data. We will discuss two broad approaches to this problem: (a) a generative approach where one determines and fits a suitable probabilistic model to the data, and (b) a non-parametric approach inspired by Wishart's remarkable but obscure mode analysis work from 1968. The pros and cons of the two approaches will be illustrated using results from both artificial and gene expression data analysis.

Speaker: **Marina Meila** (University of Washington)
Title: *The stability of a good clustering*
Abstract: If we have found a ”good” clustering C of data set X, can we prove that C is not far from the (unknown) best clustering C* of this data set? Perhaps surprisingly, the answer to this question is sometimes yes. We can show bounds on the distance(C, C*) for two clustering criteria: the Normalized Cut and the squared distance cost of K-means clustering. These bounds exist in the case when the data X admits a ”good” clustering for the given cost.

Speaker: **Panos Pardalos** (University of Florida)
Title: *Biclustering in Data Mining*
Abstract:

Biclustering consists of simultaneous partitioning of the set of samples and the set of their attributes (features) into subsets (classes). Samples and features classified together are supposed to have a high relevance to each other. We review the most widely used and successful biclustering techniques and their related applications from a theoretical viewpoint emphasizing mathematical concepts that can be met in existing biclustering techniques. Then we define the notion of consistency for biclustering using interrelation between centroids of sample and feature classes. We have shown that consistent biclustering implies separability of the classes by convex cones. While earlier works on biclustering concentrated on unsupervised learning and did not consider employing a training set, whose classification is given, our model represents supervised biclustering, whose consistency is achieved by feature selection. It involves the solution of a fractional 0-1 programming problem. Encouraging computational results on microarray data mining problems are reported.

Complexity

Speaker: **Nathan Srebro** (IBM Research & TTI-Chicago)
Title: *Computational Complexity and Data Set Size*
Abstract:

In devising methods for optimization problems associated with learning tasks, and in studying the runtime of these methods, we usually think of the runtime as increasing with the data set size. However, from a learning performance perspective, having more data available should not mean we need to spend more time optimizing. At the extreme, we can always ignore some of the data if it makes optimization difficult. But perhaps having more data available can actually allow us to spend less time optimizing?

It appears that such behavior exists in several combinatorial problems such as learning the dependency structure of Markov networks, sparse sensing, and Gaussian-mixture clustering. In these problems there appears to be a phase transition, where learning beneath some data threshold is computationally intractable (although it is statistically possible), but learning with more data becomes computationally easy. This threshold was empirically studied and characterized for the problem of Gaussian-mixture clustering [21].

Can perhaps a more continuous, but similar, effect exist in convex optimization problems such as learning Support Vector Machines (SVMs)? A new stochastic gradient decent approach for SVM learning [19] does in-fact display such behavior: the computation time required to obtain a predictor with some target accuracy decreases, rather than increases, with the amount of available data.

Scientific Progress Made

Here we list the impact which the workshop already have had on the work of some of the participants. We are sure that there are other participants whose work also have been or is likely to be affected by the workshop, but we do not have the complete information.

Don Goldfarb's report

Here is some feedback on the conference. Specifically, here are some of the ways that the conference has impacted my own research.

Jong-Shi Pang's talk on bi-level optimization and machine learning introduced me to new applications of bi-level programming. As these problems can be formulated as math programs with equilibrium (or complementarity) constraints, this is important to me as I have been working on developing algorithms for these problems. In fact I just submitted a paper on this subject to SIOPT.

I found the talk by Inderjit Dhillon on ”Machine Learning with Bregman Distances” very useful as I have also used Bregman distances in recent work on image denoising. I expect that I will be able to apply some of my ideas to the topics discussed by Dhillon.

I also found the talks by Kilian Weinberger, Gert Lanckriet, Tony Jabara, Alexandre D'Aspremont and Sam Roweis covering various uses of semidefinite programming in machine learning to be of great interest as this is an area in which I am also working.

Ted Wild's report

The workshop provided me with an excellent opportunity to interact with other researchers. I particularly enjoyed the talk on the representer theorem for 1-norm support vector machines and the talk on parameter selection, both of which gave me ideas I hope to implement in future research.

One topic that seemed to arise frequently at the workshop was dealing with (dis-)similarity data. Applications and methods for processing such data were discussed. Often, the solution involved learning a low-rank kernel matrix or low-dimensional surface via convex programming.

Alexandre D'Aspermont's report

Francis Bach and I wrote a paper while we were in Banff, it's under review at ICML. A title and abstract follow:

”Full Regularization Path for Sparse Principal Component Analysis”

Given a sample covariance matrix, we examine the problem of maximizing the variance explained by a particular linear combination of the input variables while constraining the number of nonzero coefficients in this combination. This is known as sparse principal component analysis and has a wide array of applications in machine learning and engineering. We formulate a new semidefinite relaxation to this problem and derive a greedy algorithm that computes a *full set* of good solutions for all numbers of non zero coefficients, with complexity $O(n^3)$, where n is the number of variables. We then use the same relaxation to derive sufficient conditions for global optimality of a solution, which can be tested in $O(n^3)$. We show on toy examples and biological data that our algorithm does provide globally optimal solutions in many cases.

Nathan Srebro

I can point out several direct benefits of the workshop on my work in the short time since the workshop:

Discussions with Bach and d'Aspremont, including hearing of current work by Bach, that were directly helpful in current work on trace-norm regularization submitted to ICML. This directly lead to the optimization method we are now using, and clarified relationship between different formulations.

Comments and pointers from Zhang on stochastic gradient methods that had large impact on directions of current work on fast SVM optimization also submitted to ICML.

Hearing about recent progress in the optimization literature, most notably about the LR method for SDPs, which directly relates to my work.

Other relevant pointers to specific papers, some of which I have already used directly.

Working with another workshop participant (Rosset) on a paper already accepted for publication.

I except interactions and discussions at the workshop would lead to even more collaborations. This was by far the most productive workshop I've participate in.

Chris Burges's report

Probably the biggest impact for me was generated by John Langford's talk, which I found very intriguing: I had not heard of these results before, and will now follow their development. I also liked Michael Todd's talk, and the fact that the mathematical programming crowd took as a take-home message that we need faster ways to solve SDPs. It was also very valuable simply to chat with various people, some of whom I knew beforehand, some not. Re. your question re. main methods, etc., for ranking, there is really increasing interest in this, from rather standard methods like neural nets, to treating ranking as a structured learning problem a la Joachims etc., to brand new methods (that I touched upon) to solve problems with non-differentiable costs. I think this is a key open problem in machine learning, that the ML community is only now gradually catching on to: the standard method, of coming up with a smooth cost function that attempts to encapsulate the problem, but whose form is chosen as much for computational tractability as for fidelity to the problem being solved, is very likely leading us astray, and is demonstrably doing so, in the case of information retrieval. I don't think the Mathematical Programming community has this as a hot topic, at the moment, though.

Tony Jebara's report

Overall, the workshop was excellent, it really helped to talk to a small yet high-expertise crowd who are actively using and developing the tools we are spending time with. Here are some more specific benefits:

We integrated some elements of my presentation as well as concepts from Gert Lanckriet's talk into 2 lectures in an advanced machine learning course (COMS 6998-4 Learning and Empirical Inference) being offered at Columbia by Vladimir Vapnik, Gerald Tesauro, Irina Rish and myself.

We explored some of the matrix factorization ideas in my lab after seeing some related concepts at BIRS.

After a brief conversation with folks at BIRS, we realized that a more general set of spectral functions can be optimized with our method and have an abstract on this at the Learning Workshop in 2007 in Puerto Rico.

Don Goldfarb and Watao Yin briefly discussed possibilities for collaboration, possibly when the term ends and schedules are less hectic.

Katya Scheinberg's report

The workshop broadened and substatially extended my understanding of the field of machine learning and the current use of optimization.

Two of the participants Wotao Yin and Alexandre D'Aspermont will visit IBM in spring to give presentations and discuss possible research ideas for collaboration with another participant Saharon Rosset and myself.

I was invited to visit Joydeep Ghosh and Indrajit Dhillon at the University of Texas at Austin, where I presented my work on active set methods for support vector machines and where we discussed other optimization approaches for convex QPs arising in nonnegative matrix factorization, which Indrajit is interested in.

The talk by Thorsten Joachims was very interesting for me and I have since read related papers and had a few ideas that I am planning to use in my work.

The talk by Kilian Weinberger also was related to some ideas on metric learning which we were exploring with my colleague at IBM.

Jiming Peng's report

There were quite a few interesting talks. Steve Wright and Inderjit Dhillon have been in contact and collaboration. Following Dr. Ghosh's talk on robust clustering, where the purpose is to find several well-structured clusters that cover a certain portion of the data set, had a 2-hour meeting with Joydeep, and we discussed how his problem can be modeled as 0-1 conic optimization, a fresh optimization model proposed by myself. We are working on a joint project along this line.

The talks by two students are also very interesting. One is by Kilian Weinberger on dimension reduction where the purpose is to represent high-dimensional data in a lower-dimensional space while reserving certain relationships in the original space, another one is by Ted Wild on incorporating prior knowledge into the SVM approaches for better separation.

Jong Shi Pang's talk starts to set up a bridge between machine learning and bi-level optimization, and it seems to have a lot of potential in the future.

Outcome of the Meeting

The meeting was viewed by all of the participants as very successful. Quoting Nathan Srebro " This was by far the most productive workshop I've participate in." The overall quality of the talks was outstanding. The workshop attracted researchers with very closely related interests, yet whose work covers a fairly broad set of machine learning problems. As a result each talk covered a new topic and caused intense and lively discussions. Most of the talk were given one hour time slots, which allowed for many questions during the talks. Some of the talks were thus transformed into informative group discussion and it greatly benefited the understanding by the audience. Such impromptu discussion are only possible in the informal setting of a small workshop such as this.

The schedule contained two long free periods (of five to six hours) which we used for recreational activities such as hiking and skiing and also for uninterrupted collaboration work. There were also 2 presentations/discussion scheduled for after dinner time, which continued on informal basis in the lounge for the remainder of the evenings.

There were 34 participants, including 5 students and 5 women. Along with participants from academia there were participants from IBM, Microsoft, Yahoo! Inc. and TTI Research. Below is the complete list of participant.

List of Participants

Altun, Yasemin (Toyota Technological Institute at Chicago)
Aybat, Serhat (Columbia University)
Bach, Francis (Center of Mathematical Morphology)
Ben-David, Shai (University of Waterloo)
Burges, Chris (Microsoft Research)
d'Aspremont, Alexandre (Princeton University)
Dhillon, Inderjit (University of Texas, Austin)
Ghosh, Joydeep (University of Texas, Austin)
Goldfarb, Donald (Columbia University)
Jebara, Tony (Columbia University)
Joachims, Thorsten (Cornell University)
Lanckriet, Gert (University of California, San Diego)
Langford, John (Yahoo Inc.)
Lee, Sang (University of Wisconsin-Madison)
Meila, Marina (University of Washington)
Mittelmann, Hans (Arizona State University)
Pang, Jong-Shi (Rensselaer Polytechnic Institute)
Pardalos, Panos (University of Florida)
Peng, Jiming (University of Illinois at Urbana-Champaign)
Rosset, Saharon (IBM Research)
Roweis, Sam (University of Toronto)
Scheinberg, Katya (IBM TJ Watson Research Center)
Schuurmans, Dale (University of Alberta)
Srebro, Nathan (University of Toronto)
Todd, Michael (Cornell University)
Tsuchiya, Takashi (Institute of Statistical Mathematics)
Wahba, Grace (Wisconsin-Madison)
Weinberger, Kilian (University of Pennsylvania)

Wild, Ted (University of Wisconsin-Madison)
Yin, Wotao (Rice University)
Zhang, Tong (Yahoo! Inc.)
Zhu, Xiaojin (Jerry) (University of Wisconsin-Madison)
Zhu, Jiaping (McMaster University)

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Chapter 2

Innovations in Mathematics Education via the Arts (07w5062)

Jan 21 - Jan 26, 2007

Organizer(s): Gerda de Vries (University of Alberta), George Hart (Stony Brook University), Reza Sarhangi (Towson University)

Summary Introduction

Our primary objective was to bring together a diverse body of mathematically trained professionals who individually incorporate the arts in their educational activities. As a group, we brainstormed to identify promising areas and techniques for a wider movement of math education via the arts.

The following paragraphs are from participants reports of the experience:

This was a very productive week. I liked the flow of the workshop, that we worked together as a large group to decide on our goals then broke into groups to work on developing the goals, which we then reported back to the group. Then, we discussed other goals for new groups, but we were allowed to also participate in the first groups or the second groups. That we had the freedom to move within groups or stay in groups made it easy to focus on activities targeted towards my interests and talents.

It was intellectually energizing to be a part of a diverse group, comprising people in specialized areas of mathematics and the arts within higher education, teacher education and K-12 school contexts. The challenge of bridging from the specialized areas to making a measurable difference in learning in the K-12 classroom is significant. It involves the ongoing cultivation of multiple perspectives through continuous dialog between all parties.

This workshop was for me a unique experience that provided me with connections to elementary and high school teachers of mathematics that would have been difficult to realize otherwise.

I have formed a new collaboration and started a new project. From presentations by others, I have learned new methods for enhancing mathematical education and new ways to incorporate art into mathematics.

This five day intense workshop within these excellent facilities in BIRS has been a very positive and unique experience for me and, no doubt, for the entire group of participants. It has served not only for my personal professional development but also for reinvigorating the teaching of mathematics through the arts. I believe that this can be a very valuable pedagogical tool. I foresee that in the next few years this workshop will be a reference point in the sense that many of the seminal ideas and personal connections of future projects that involve teaching with some kind of artistic activity started here during these five days.

Workshops

We alternated our discussions with hands-on activities that we felt were models for classroom use.

- CD truncated icosahedron: George Hart
- Birch bark ornament, traditional science: Barb Frazer
- Islamic cutouts: Carol Bier
- L-Systems: Glyn Rimmington and Mara Alagic
- Twiddler, Etch-a-Sketch, and Long-sword: Susan Gerofsky
- Math and Rhythm Godfried Touissant, Paco Gomez, David Rappaport, Susan Gerofsky
- Oulipo: Susan Gerofsky

Outcomes

After brainstorming about many possible outcomes, the group converged on the goal of developing pedagogical materials at various levels. There are various groups of participants who have committed to target their energy towards future projects that were incubated here:

Bob Bosch, Pau Atela, Doug Burkholder, and David Richter will be editing a book of long-term, out-of-class projects that can be incorporated into existing sophomore-junior-senior-level courses. Each project will be a module that builds upon material found in one or more courses in the standard curriculum. Each project will be assigned to a group of students. The final piece of each project will be the creation of a piece of art (a piece of sculpture, for example). In each case, mathematics will be an integral part of the creation process. Carlo Sequin promised to contribute at least two project idea that he will write up in the next few months.

Nat Friedman, Mara Alagic, Glyn Rimmington, Stewart Craven, and Phil Wagner formed a group focused on K 12 education. The group is concerned about activities where art is in some meaningful way ought to be connected to mathematics. Whether the inspiration for the mathematics comes from the art or the mathematics in and of itself leads to artistic representations, there is a need for suggestions/activities for elementary and secondary teachers to use in their classrooms. To this end the group will create a framework that provides the critical information required by teachers to embed these lessons in their programs. They will start by writing 4 6 lessons, field test them, and refine them to be published in an appropriate form. Stewart will initially write a lesson based on the construction of a giant stellated octahedron followed by a series of lessons about students who use their own photographs imported into Geometers Sketchpad to explore transformational geometry. He will additionally submit my workshop plans for two Mathematics and Art sessions that he will be doing over the course of the next four months.

Blake Mellor, Gwen Fisher, Kevin Hartshorn, Doris Schattschneider and Carolyn Yackel formed a collaboration to edit a collection of activities/projects for Mathematics for Liberal Arts. They plan to create some sample projects by July 2007, along with detailed guidelines for the projects, and send out a call for proposals by the end of the summer. They hope to collect a range of projects, on different topics and of different lengths, to be a resource for teachers of college Math for Liberal Arts courses, and possibly also for high school teachers.

Godfried Toussaint, David Rappaport, Paco Gomez, Susan Gerofsky, and Reza Sarhangi started a collaboration to explore the potential of teaching a variety of mathematical concepts through music and rhythm. They are working on an initial article for a mathematics education academic journal (For the Learning of Mathematics or Educational Studies in Mathematics). They will be outlining a program to use Toussaint's innovative circular representations of rhythmic patterns in music to teach concepts in a wide range of mathematical areas, ranging from number theory to geometry, abstract algebra, and combinatorics. They hope to use the analysis from this collaborative article as the basis for the development of a book of lesson ideas and materials for mathematics instructors at a variety of different levels, to encourage thoughtful implementation of mathematics teaching via music. A proposed book may also include a call for articles from other mathematics educators who use music as a means to teach math concepts.

Gwen Fisher wrote a proposal for a mathematical art exhibit Mathematical Expressions: Bead Weaving with Gwen Fisher at the San Jose Museum of Quilts and Textiles in California that Carol Bier will help her submit.

Gene Klotz is writing a proposal to form a wiki for the math/arts community. Workshop participants helped to develop a taxonomy of the types of content to include.

Pau Atela and Philip Wagner have joined forces to disseminate to the larger public an exhibition about the biological phenomenon of phyllotaxis and current mathematical models for the phenomenon. This exhibit was prepared a few years back with biologists, mathematicians, and artists as participants. It has been very popular in a few botanical gardens in Europe but has never been exhibited (outside Smith College) in North America.

Carolyn Yackel, Mara Alagic, and Gwen Fisher plan to develop and conduct an assessment study on the effects of introducing mathematical art in the classroom on spatial reasoning skills.

Also, Pau Atela and Bob Bosch plan to work on a portrait of Fibonacci constructed out of images from Paus phylotaxis research.

In addition, several topics were discussed which we agree the participants should explore further. One is the idea of a joint interenational congress which combines the art and math communities from many countries into one conference. Participants will explore this idea with the organizers of Bridges, ISAMA, ISIS, NEXUS, Katachi, the Math and Design Conference. Another topic discussed was for participants to follow up on the funding opportunities offered by the NFS for the National Science Digital Library.

Detailed Individual Reports

We asked the participants each to write a page on the following topics. Their responses follow.

- Name, affiliation
- Paragraph about experience here
- Description of math education needs you feel are important and whether they were addressed
- L-Systems: Glyn Rimmington and Mara Alagic
- What you see as the long range impact of this week's workshop
- Anything else you think should be mentioned in our final report to BIRS

Mara Alagic & Glyn Rimmington

College of Education, Wichita State University, Kansas, USA

BIRS Experience and Mathematics Education Needs

It was intellectually energizing to be a part of a diverse group, comprising people in specialized areas of mathematics and the arts within higher education, teacher education and K-12 school contexts. The challenge of bridging from the specialized areas to making a measurable difference in learning in the K-12 classroom is significant. It involves the ongoing cultivation of multiple perspectives through continuous dialog between all parties.

Partnerships

We joined the K-12 collaborative group along with Nat, Stewart and Phil to develop a framework for using mathematics and arts in the classroom. It will take into account such issues as prior learning and life experiences of students and teachers. An important part of the framework is the cross-indexing of arts with mathematics resources and vice versa. Such a framework must include information for K-12 teachers on how to integrate the resources into their classes. Two NSF RFPs (NSDL and CLII) were identified and investigated to support the provision of more resources for mathematics and arts teachers. This is consistent with the framework proposal. The group investigating the grant proposal comprises Gary, Gene, Dirk, David, Glyn and Mara.

Accomplishments

We learned more about L-systems in terms of how they may be integrated into classrooms to improve student learning of a range of concepts, such as 3D and 2D geometry, recursion, iteration, branching and evolving structures. The music/rhythm activity will be introduced into elementary mathematics education classes and to instructional leaders. There are a couple of other interesting ideas to take to our classrooms.
Long Range Impact
The vision of improving learning outcomes in the K-12 mathematics classroom can only be accomplished through an ongoing dialog between those with new ideas in mathematics and the arts and the classroom teachers and instructional leaders. The proposed framework for integration of resources will help with this process.
Research Questions
We believe two important research questions that relate to the observations above are: - How is teaching a mathematics concept via art changing/influencing understanding of that concept? - Are these (if yes, how) representations different from traditional/non-art-based representations?

Pau Atela, Smith College

This five day intense workshop within these excellent facilities in BIRS has been a very positive and unique experience for me and, no doubt, for the entire group of participants. It has served not only for my personal professional development but also for reinvigorating the teaching of mathematics through the arts. I believe that this can be a very valuable pedagogical tool. I foresee that in the next few years this workshop will be a reference point in the sense that many of the seminal ideas and personal connections of future projects that involve teaching with some kind of artistic activity started here during these five days.

I am involved in two main partnerships. One, with Philip Wagner, entails the dissemination to the larger public of an exhibition about the biological phenomenon of phyllotaxis and current mathematical models for the phenomenon. This exhibit was prepared a few years back with biologists, mathematicians, and artists as participants. It has been very popular in a few botanical gardens in Europe but has never been exhibited (outside Smith College) in this side of the Atlantic.

The second main partnership involves writing a book that will be a resource for College level faculty. It will contain art-math projects aimed at upper level students that have passed at least a calculus course. The idea is that these projects would be flexible enough so that the teacher will be able to implement them either within a course for the whole class, or as supplementary activities for a subgroup of students. Some will also be suitable for semester-long courses or for independent studies. I have volunteered to be one of the editors, together with Robert Bosch (Oberlin College), David Richter (Western Michigan University) and Doug Burkholder (Lenoir-Rhyne College).

A smaller scale project that could take place in the near future is a collaboration with Robert Bosch involving carefully chosen images of mathematical models of plant spiral patterns involving Fibonacci numbers and a computer generated portrait of Fibonacci using those images as shades of grey with algorithms created by Robert Bosch.

Carol Bier

Mills College, Oakland CA Research Associate, The Textile Museum, Washington DC

The BIRS Workshop, Innovations in Mathematics Education via the Arts, provided an outstanding opportunity for key players in the fledgling field of intersections among mathematics and art to address educational needs and to develop plans for our future development. We arrived at some very basic and profound understandings of shared goals and diverse perspectives. Banff offered a unique environment in which to brainstorm, focus, group, and regroup, allowing inspiration, creative leaps, and cross-fertilization of ideas. The Max Bell Building and Corbett Hall were ideally suited to our needs, and the BIRS staff provided a very supportive and nurturing environment for our group. The catering in Donald Cameron Hall is outstanding, and the facilities of the Professional Development Centre also contributed to the inspiring ambience of our intense intellectual engagement.

Intellectually, I feel that my experience here was encountered by others as well, that our work and professional activities in art-math intersections were affirmed, and that I am inspired to continue to pursue them, in spite of all too frequent resistance from within the establishment. We have an important agenda, and it is

worth pursuing.
My Own Accomplishments at BIRS Workshop
Offered workshop on folding and cutting, playing with the geometry of the circle, to form patterns that are used in Islamic art and architecture Learned proof from Doris Schattschneider why the folding results in a 30 degree angle Invited Gwen F, Daina T, Carolyn Y, Susan G, Barbara F, and others to submit proposals for their work on mathematics related to fiber media, textile technologies, and interlacing (bead-weaving, crocheting, knitting, longsword dancing, birch-bark chomping) for Textile Society of America 2008 Symposium, to be held in Honolulu, Hawaii Deadline for proposals October 1, 2007 Renewed contact with Gary Greenfield for Journal of Mathematics and the Arts, to consider papers derived from our plenary session, "Textiles - Math = 0/Textiles + Math = ∞" organized w/ Dave Masunaga at Textile Society of America 2006 Symposium in Toronto Promoted Textile Society of America interests in textiles and math to art-math community Edited revised draft by Gerda de Vries for publication in TSA 2006 Symposium Proceedings Met Barbara Frazer; acknowledged shared interests in traditional science, and parallels between traditional science and values in classical Islamic world and First Nations in Canada Sent her the syllabus for my course, Sufism, Spirituality, and Science (Fall 2007, SFSU) Introduced Dirk Huylebrouck to work of Eric Broug, www.broug.com Explored Banff Centre as possible symposium venue for TSA 2012; met with Nancy Sande of Conference Sales; sought (and received) proposal

Immediate Plans
Used proof Doris Schattschneider provided in my class on Thursday, 1/25 Edited paper by Reza Sarhangi on Geometric Constructions and their Arts for Bridges 2007 (Friday 1/26) Met with curator of textiles at the De Young Museum to propose art-math workshops for upcoming exhibition of Turkmen rugs (Fall 2007) (Friday 1/26) Take Banff Centre proposal for TSA 2012 to TSA Board for consideration at February meeting Use proof DS provided in future classes and workshops on geometry and Islamic art Submit proposal to San Jose Quilt and Textile Museum for a series of exhibitions called Mathematical Expressions, of which Gwen Fischers Beaded Weavings will be the first Pursue plans to make contacts with venues in the San Francisco Bay Area for art-math activities (De Young Museum; Oakland Museum; Asian Art Museum; SF Craft and Folk Art Museum; Lawrence Hall of Science; Exploratorium; Mathematical Sciences Research Institute; UC-Berkeleys Center for Middle Eastern Studies; San Jose Quilt and Textile Museum; Santa Rosa Gallery) Will welcome support from Bridges Organization. Prepare a workshop proposal for Bridges-to-Teachers/Teachers-for-Bridges 2007 Hoping to introduce Doris Schattschneider to a friend who is a digital artist in NJ in February Seek to apply principles of interlacing used in longsword-dancing for programs with children Engage children in soap film activities using Nat Friedmans knots to form minimal surfaces Establish link on TSA website to Carolyn Ys Knitting Network Encourage Bridges website to establish links to Ethnomathematics Digital Library, and to my students work at the Math Forum Pursuing several references I learned about from colleagues at BIRS: Mathematics and Aesthetics, Mathematics and Beauty, books on Ornament, Experiencing Geometry, Homoestheticus Suggest to colleagues at University of Hawaii, Hawaii Pacific University, Honolulu Academy of Art, Iolani School, Ethnomathematics Digital Library that they sponsor individual speakers on art/math subjects, so they could then participate in the TSA 2008 symposium in Honolulu
Longer-term ideas
Want to consider more contributions I can offer to The Math Forum at Drexel University Want to consider ways I can contribute to Gene Klotz wiki initiative, introduced at BIRS Hope to plan TSA 2012 at Banff Centre; and consider keynote address on Traditional Fiber Technologies among First Nations (bark-chewing; pattern-making; snowshoe construction, skin clothing, etc.); encourage participation of U. Alberta, U. Calgary, Nickle Museum Seek grants/develop proposals to support proposed activities from the BIRS Workshop Put together list of art-math museum exhibits for Bridges website

Robert Bosch, Oberlin College

The BIRS Workshop Innovations in Mathematics Education via the Arts was an amazing, enlightening, and invigorating gathering. I came to it with great excitement, and it did not disappoint; in fact, it exceeded my very high expectations.

I have seen in my own work as a mathematician/artist/educator how beneficial it is to combine math and art. (Ive taken great pleasure in using mathematics to help me create works of visual art, Ive found that combining math and art provides many opportunities for incorporating my students in publishable research projects, and Ive seen that it can inspire students who—beforehand—felt that math was boring and useless.)

I strongly believe that many mathematics educators—at all levels—would jump at the chance to bring some art into their classrooms. All that they need are some resources. And this BIRS workshop has been the genesis of an entire collection of high quality materials.

For me, perhaps the most significant outcome of this workshop is that I will be editing (with Pau Atela, Doug Burkholder, and David Richter) a book of long-term, out-of-class projects that can be incorporated into existing sophomore-/junior-/senior-level courses. Each project will be a module that builds upon material found in one or more courses in the standard curriculum. Each project will be assigned to a group of students. The final piece of each project will be the creation of a piece of art (a piece of sculpture, for example). In each case, mathematics will be an integral part of the creation process.

I also made many contacts. I think that theres a very good chance that some of these will lead to additional collaborations in the near future. (One example: Pau Atela and I have talked about working on a portrait of Fibonacci constructed out of images from his phylotaxis research.)

Doug Burkholder

Experience:
The time spent here in Banff this past week has been a productive week. In addition to building friendships with faculty with common interest, I have formed a new collaboration and started a new project. From presentations by others, I have learned new methods for enhancing mathematical education and new ways to incorporate art into mathematics.
Prior Needs:
As I came to this conference, I was hoping to gain new insights into projects that I could take into my courses. While I am always looking for ideas and projects for all levels of courses, I am specifically interested in projects for the upper-level mathematics courses. Like many faculty at small private liberal arts colleges, I teach a wide range of courses and I find that I do not have the time or expertise to develop projects in all of my courses. Specifically, I want to incorporate more opportunities for visualization of the mathematical material.
Partnerships Formed:
Through brainstorming sessions and breakout group discussion David Richter, Pau Atela, Bob Bosch, and I decided to form a partnership leading to the publication of resource material for upper-level mathematics course. We also identified several others, such as Carlo Sequin, Doris Schattscheider, and Gary Greenfield who are willing to assist us in various aspects of our project such as creation of material and expertise in the publication process.
Project Planned:
Our project is to compile as set of 15 to 20 projects which use art to enhance mathematical instruction in upper-level mathematics courses. These projects will be self contained and ready to distribute to students in traditional upper-level courses. They should enhance the mathematical experience both through the visual arts and through alternate applications and extensions of the mathematical material being taught within the course. Each project should culminate in artwork produced individually or through group effort. Generally, these projects will be designed to take 2-4 weeks, but they should be flexible and open ended to allow situations where they may extend over the semester or where they could be used for undergraduate research projects. There is also the possibility of using these projects, perhaps in a condensed version, in math clubs. There is also the opportunity for these projects to be used as the core of an upper-level seminar course devoted to mathematics and art.
Long Range Impact:
In addition to assisting faculty currently excited about the ability for art to enhance mathematical instruction, the publication of our resource material should attach other faculty to math and art. This should also assist young faculty as they begin their teaching career. Likewise, I anticipate taking advantage of projects begun by other partnerships here are BIRS, such as the liberal arts math and art project.

Stewart Craven, Toronto District School Board

I came into this week wondering if I could make a contribution to the proceedings and wondering about what new things I might learn. I was concerned that my mathematics knowledge would not be sufficient. My concerns were allayed as the week played out. Shortly after I arrived in Banff I encountered Reza and George on the main street here in Banff. We proceeded to a coffee shop and the rich discussions began. The week was orchestrated in such away that the participants grouped and regrouped in a various combinations that of course led to discussing a vast array of topics from numerous perspectives. I discovered that my knowledge of elementary and secondary school teaching and learning was a critical piece of the mosaic particularly given the kinds of products that have been proposed. Nevertheless, I continue to be in awe of those participants whose mathematics understanding and creative abilities in art are seemingly beyond my grasp. It would be remiss of me not to comment on our surroundings. The mountains, the deer, the birds, the Banff Centre, and the village all contribute to an environment where learning and creativity will inevitably flourish. Also, activities such as, the Circus, the Banff Centre tour, the native story telling, the walk up Tunnel Mountain, and the excursion to the hot springs all serve to activate the senses. One last note is to commend the staff at the Banff Centre who have been so friendly and helpful throughout the week.
Accomplishments

First and foremost, I learned about ideas in topology and knots. I learned how to construct a truncated icosahedrons and six-point or eight point stars through paper folding and I learned about how represent rhythm in music as polygons.

Projects

The K 12 group is concerned about activities where art is in some meaningful way ought to be connected to mathematics. Whether the inspiration for the mathematics comes from the art or the mathematics in and of itself leads to artistic representations, there is a need for suggestions/activities for elementary and secondary teachers to use in their classrooms. To this end our group will create a framework that provides the critical information required by teachers to embed these lessons in their programs. We will start by writing 4 - 6 lessons, field test them, and refine them to be published in an appropriate form. I will initially write a lesson based on the construction of a giant stellated octahedron followed by a series of lessons about students who use their own photographs imported into Geometers Sketchpad to explore transformational geometry. I will additionally submit my workshop plans for two Mathematics and Art sessions that I will be doing over the course of the next four months. I will work closely with my group (Nat, Mara, Glyn, and Phil) to achieve our goals.

Gerda de Vries

Department of Mathematical and Statistical Sciences University of Alberta

I learned a lot about the math&art community - what types of activities people are involved in, what their educational interests are, etc.

I am in awe of the leeway that educators have at liberal arts colleges, and have come to the realization that most innovations in education will come from colleagues at such institutions.

The members of the group primarily were academics, and so were most comfortable identifying areas at the undergraduate level where we can have impact.

I think that members of our group have and can develop engaging activities that can have impact on education at the K-12 level when combined with (tested) pedagogical support materials. Unfortunately, due to a lack of critical mass of K-12 education specialists in our group, we were not able to 1) identify areas of the K-12 curriculum that need to be addressed in the first place, nor 2) address the development of pedagogical support materials.

Through discussions with colleagues, I now have ideas about how to improve my outreach activities.

I am inspired to contact my local science museum and find out whether there is interest in math&art there.

I am inspired to contact colleagues in education to develop pedagogical support materials for at least one of my outreach activities (support materials that can be used by classroom teachers to follow up on concepts explored during my classroom visits, for example).

I look forward to receiving updates on the book projects initiated at this workshop - at the moment, I have very vague ideas about how I might contribute, but these ideas may become more concrete after some incubation time.

Connection to a new group of people, with possibilities to collaborate on projects in the future.

This is the most productive workshop I have attended at BIRS. It truly was a workshop, with participants working together to articulate goals and develop plans towards achieving those goals.

Gwen Fisher

Affiliation: Mathematics Department, California Polytechnic State University, San Luis Obispo

Paragraph about experience:
This was a very productive week. I liked the flow of the workshop, that we worked together as a large group to decide on our goals then broke into groups to work on developing the goals, which we then reported back to the group. Then, we discussed other goals for new groups, but we were allowed to also participate in the first groups or the second groups. That we had the freedom to move within groups or stay in groups made it easy to focus on activities targeted towards my interests and talents.

Description of math education needs you feel are important and whether they were addressed: I believe that assessment of math/art programs is an imperative educational need that was not addressed sufficiently here. I proposed a study to assess the effects of our project, and while I received moral support for such work, and suggestions that others might be interested in working on this in the future, nobody had sufficient interest to work with me to develop this project during this week. Most of the final products appear to be the creation and collection of mathematical art resource materials. While I believe that this work is also very important, at some point, the math/art/education group of scholars will likely need to justify that our work is actually teaching mathematical skills and concepts to get more widespread support of our work from our education colleagues and government agencies.

Partnerships formed: Carolyn Yackel, Mercer University (yackel_ca@mercer.edu)
Kevin Hartshorn, Moravian College (hartshorn@moravian.edu)
Doris Schattschneider, Moravian College (schattdo@moravian.edu) Blake Mellor, Loyola Marymount University (bmellor@lmu.edu) Carol Bier, Mills College / The Textile Museum (carol.bier@gmail.com) Mara Alagic, Wichita State University (mara.alagic@wichita.edu)

Accomplishments: We wrote a proposal for a book of activities joining mathematics and art in a liberal arts environment.

I wrote a proposal for a mathematical art exhibit Mathematical Expressions: Bead Weaving with Gwen Fisher at the San Jose Museum of Quilts and Textiles in California that Carol will help me submit, and hopefully get accepted.

Projects planned: Carolyn, Kevin, Blake, Doris and I plan to complete the book of activities described above: I will be providing at least one of the activities, and I will be the illustrator for the book. The other four will be the co-editors.

Carolyn, Mara, and I plan to develop and conduct a study on the effects of studying mathematical art on spatial reasoning skills, at least I hope so.

What you see as the long range impact of this week's workshop: I believe that the impact will be the creation and dissemination of more mathematical art projects/lessons/information to the general public, especially students.

Anything else you think should be mentioned in our final report to BIRS: The facilities are wonderful, and the staff has been very kind and helpful.

Nathaniel Friedman

Dept. of Mathematics, Univ. at Albany-SUNY, Albany, NY 12222

(1) I had a very important experience at the BIRS workshop on Innovations in Mathematics Education via the Arts. The workshop sessions went very well and all facilities were first-rate.

(2) There were many innovative projects introduced that reflected my basic concern with mathematical understanding through visualization having an arts component. The level of discussion was very high and a variety of significant ideas were introduced. These ideas were definitely developed in a practical manner during the workshop.

(3) I formed a partnership on the development of projects for K-12 education with Mara Alagic and Glyn Rimmington of Wichita State University, Kansas, Stewart Craven of the Toronto School Board, Canada, and Philip Wagner of the Fusion Project, San Francisco. We have very complementary skills and I look forward to our collaboration.

I also plan to collaborate with Doris Schattschneider of Moravian College, Pennsylvania, on her project for developing a collection of iconic images relating mathematics and art for a CD.

Thirdly, I plan to collaborate with Gene Klotz of Swarthmore College, Pennsylvania, on his Web-Based Math Forum project.

(4) I organized the first Arts/ Mathematics Conference at the University at Albany in June, 1992. There have been annual conferences organized by myself as well as Reza Sarhangi and others every year since 1992. I consider this first BIRS Workshop on Mathematics Education via the Arts of historical significance. I envision this Workshop as strongly accelerating the movement in education relating mathematics and art at all levels from K-college. I am totally grateful to BIRS and the Banff Center for making this possible.

Susan Gerofsky, Curriculum Studies

Faculty of Education, University of British Columbia Vancouver, BC, Canada.

I am very pleased to be a participant in the BIRS workshop, Innovations in Mathematics Education Via the Arts. It has been a wonderful opportunity to connect with an exciting and creative group of like-minded colleagues in a place where we could concentrate on our collaborative work without distractions. To quote Banff Centre Service Director Jim Olver, ”there are no excuses” at BIRS to delay the work that you must do. BIRS provides all the necessities and trimmings for a highly successful academic workshop: comfortable accommodations, excellent meeting facilities, delicious meals, computers, internet access, scanning and copying facilities, lounges, a reading room, and the most helpful staff imaginable, all in this spectacularly beautiful setting. With the support of this infrastructure in place, we were able to work very productively and accomplish a great deal during our five-day residency. Working to establish connections between mathematics and the arts in a faculty of education, I often feel isolated from both mathematicians and artists at my own university. I think that many of us do occupy the position of the lone ”math/arts” advocate in our own institutions. It is both a necessity and a delight to gather together for an intensive working session like this one. Our workshop addressed the need for connections between mathematics and the arts at all levels of education: K-12, college liberal arts courses, university undergraduate mathematics courses, and in terms of lifelong learning through museums, television programs, books, CDs, websites, traveling math/ art shows and other media. My own professional interest centers on secondary school mathematics education, and this was certainly addressed in all our sessions and in the outcomes of the workshop. I was also happy to expand my own view of mathematics education to take in ages ”zero to infinity”. One of the most exciting outcomes of this BIRS workshop was a partnership several of us formed around the potential of teaching a very wide variety of mathematical concepts through music and rhythm. I am now collaborating with colleagues Godfried Toussaint (McGill), David Rappaport (Queens), Paco Gomez (Univ. Politecnico de Madrid, visiting at McGill), and Reza Sarhangi (Towson University, Baltimore) on an initial article for a mathematics education academic journal (For the Learning of Mathematics or Educational Studies in Mathematics). We will be outlining a program to use Toussaint’s innovative circular representations of rhythmic patterns in music to teach concepts in a wide range of mathematical areas, ranging from number theory to geometry, abstract algebra, and combinatorics. We hope to use the analysis from this collaborative article as the basis for the development of a book of lesson ideas and materials for mathematics instructors at a variety of different levels, to encourage thoughtful implementation of mathematics teaching via music. A proposed book may also include a call for articles from other mathematics educators who use music as a means to teach math concepts. Workshop leaders and participants are planning to pitch the idea of a book series offering math and art lesson plans and connections aimed at different levels of schooling (say, high school, elementary school, or undergraduate math courses) and different artistic media (math and music, or math and sculpture for example). This is a very exciting prospect, and may result in a coordinated series of resource materials which will promote a practical implementation of an enriched mathematics teaching via the arts, with the potential to reach many more students through an embodied, humanistic grasp of abstract concepts. Thanks to BIRS for providing us the opportunity to come together to accomplish this important work, in this beautiful place.

Paco Gómez

Department of Applied Mathematics School of Computer Science, Polytechnique University of Madrid, Madrid, Spain.

Participating in the BIRS workshop, Innovations in Mathematics Education via the Arts has been a great experience for many reasons. For one, it has been a unique opportunity to meet very creative people with my own interests. Sometimes, it is hard for those who like mathematics and art. Also, the settings and facilities were perfect, with no obstacle to attention, allowing all us to concentrate on our work. I really identify myself with what Susan Gerofsky says in her report. I also feel very isolated from both mathematicians and artists. They both seem to have prejudices on each other. In workshops like this, those prejudices are broken down. I have been in touch with mathematicians who love art and try to apply to their work. They do not do that for the fun of it, but because there are many connections. A good idea to extend this workshop would be to bring over artists so that it becomes a truly interdisciplinary workshop. In particular, at this workshop I worked on connections between mathematics and the arts at all levels of mathematical education, ranging from K-12 to college liberal arts courses and university undergraduate mathematics courses. Since I have mostly worked with university students (and a little bit with babies), one of the outcomes I was expecting from this workshop was to learn about other levels of education. Excitingly enough, I am now collaborating with Susan Gerofsky, Godfried Toussaint, David Rappaport, and Reza Sarhangi. We are writing a paper on teaching mathematics via musical rhythm. Our main idea is using geometry, in particular circular representations, in order to teach mathematics concepts. We will concentrate our efforts on both the content and the methodology.

Gary Greenfield

University of Richmond, Richmond, Virginia, USA

This is my third trip to Banff. The previous two visits were for conferences, but this one was for a workshop. Banff is arguably one of the finest venues in the world, and being involved with a small focused group is infinitely more rewarding than being associated with the ”sprawl” of a conference. My only regret is that our schedule precluded taking better advantage of the excellent facilities and recreational diversions.

It is seminal that a group of educators was organized to consider the broader picture of the role, impact, integration, and future of mathematics and art in education. The fact that so many different levels (K through ∞!) were considered attests to the effort to address the perceived need and its value at all levels.

Group Projects Planned: - Contributions to ”activities projects” materials for both K-12 and post calculus math classes;

Individual Projects Planned: - Further investigation of NSF funding opportunities - Math and art research into L-systems art making and optimization-based art making

Partnerships Formed: In my role as editor of the JMA, solicitations for - individual contributions - report to be published about the activities of the workshop.

A concerted effort to promote mathematics and art education initiatives.

The unique inspirational, informational, hands-on activities to energize and engage participants led by George Hart (CD polytope sculpture), Barb Fraser (birch bark ornament), Carol Bier (Islamic art constructions), Glyn Rimmington and Mara Alagic (L-parser), Susan Gerofsky (long sword patterns), and Godfried Toussaint (mathematics of rhythm).

George W. Hart, Stony Brook University

This was an outstanding week for us. A wonderful mix of creative people participated, with varying backgrounds, diverse areas of expertise, and experience working with different levels of students. But we all feel a passion for mathematics education via art. Many of us have had great teaching experiences with art-based math education activities that we have developed. After much sharing and brainstorming, we decided that the greatest need that we as a group could fill is to provide resources for educators at all levels. Other teachers will use these ideas in their classes if we write them up in a useful manner, which conveys some of our excitement and contains clear procedures to follow.

There are three main projects formed, plus smaller groups of participants formed a number of additional partnerships. The main projects are edited collections of activities, one aimed at the K-12 level, one aimed at college-level liberal arts students, and one aimed at post-calculus students. We expect that each collection will be published in the form of a book, probably with an electronic supplement on a CD or a web site. We discussed representative activities for each collection and developed a format that contributors can follow when proposing activities to the editors. The groups of editors for the three collections will format some initial activities and with them as models, announce a call for additional solicitations. Many of the workshop participants will be contributors.

I am certain that everyone left feeling creatively invigorated, with a sense that our work will have a very positive long-term impact on mathematics education.

Kevin Hartshorn (Moravian College)

This was a tremendously rewarding experience. I've made contacts with many people connected to math/art activities, rejuvinated my enthusiasm for teaching my "Math for Design" course (a quantitative literacy course at Moravian), gained many excellent ideas for using art to bring mathematics to a broader community, and established a new and promising book project.

Coming to the conference, my main focus was developing projects and ideas for my Math for Design course. I wanted projects to energize the students (mostly art majors) while teaching them some genuine mathematical concepts. In a broader scope, I wanted projects with a public face - projects that would bring mathematical ideas to the broader campus community.

We (Blake Mellor, Carolyn Yackel, Gwen Fisher, Doris Schattschneider and myself) formed a very enthusiastic coalition. We have all been hungry for activities for a liberal arts mathematics course. Together, we have made some solid plans to develop a book of activities and projects. Blake, Carolyn, and I will offer our energy to the project, Doris has been a font of knowledge and insight from her previous experience, and Gwen has had excellent ideas to create a very strong product.

That book is certainly going to be the long-range output for this project. We hope to have some significant progress, including a proposal for publishers, available by the summer. At that point, we will be soliciting for contributions from the math/art community.

This is a perfect facility for collaboration, creativity and really solid mathematical work. We are removed from the distractions of the day-to-day work. There is a truly inspiring environment in the Banff community. And the facilities are top notch - amazing meeting rooms, internet and computing capabilities, staff, etc. I really look forward to a chance to attend another workshop here.

William Higginson

Queen's University at Kingston Banff, 2007 01 26

I have been interested in the academic area defined by the overlap of mathematics and art for more than a decade. During that time I have attended several national and international conferences and have written on both theoretical and practical aspects of certain issues in the field. From that perspective my sense is that this workshop may well come to be seen in the future as a distinct developmental landmark in the evolution of an interesting subfield of mathematics with important implications for education. The breadth and intensity of the interests and experience of workshop participants was positively exploited by the ongoing presentation of a rich set of examples to the group. These same characteristics presented some challenges as the workshop progressed in the construction a common vision/sense of purpose. The workshop was well planned and effectively led. The BIRS facilities were outstanding.

I am concerned about the public image of mathematics and the effect it has in limiting possibilities for learners. I think that the interests of members of this group have considerable potential as one direction from which to address this problem. My sense is that good progress was made in several sub-areas toward making some of this potential more accessible to teachers and learners.

I much enjoyed individual interactions with a number of participants and found several of the group activities to be enlightening and informative for future teaching and research. With respect to future possibilities I was particularly pleased to meet and be able to interact with Gene Klotz whose contributions to mathematics education I have long admired. I was surprised and pleased to learn in the pre-conference exchange of proposed projects by how many commonalities our ideas appeared to have. Following discussions at the workshop I look forward to collaborating with Dr. Klotz and others in the group on the development of a wiki for people interested in Mathematics and Art and the generation of an accessible and well-supported gallery of mathematical images.

I am pleased that the BIRS executive committee saw fit to support this workshop (a wise choice I would contend) and am grateful to have been able to be a participant. The workshop organizers have done a fine job in moving a spirited, knowledgeable and energetic (but not particularly 'herdable' - if that is a word) group of individuals several steps further in the direction of the generation of some significant resources for the field. Thank you.

Dirk Huylebrouck, Hogeschool Wetenschap en Kunst

It was an enriching experience: great people, great site. It is the second time I come here, and if possible, Ill return. This is one of the best environments for a mathematics and art conference.

The promise of working on real proposals was not entirely met. Most people thought more about their own forthcoming publication, their own project, than about the general promotion of the Math & Art idea. A few more workshops could help to encourage a climate of collaboration, so that individual concerns would prevail less on immediate individual aspirations. I am not finger pointing the organizing scientist here, who gave a lesson in democratic approach and encouragement of collaboration, nor the participants (I was one myself, so I was co-responsible!), no, it is a human feature, and through more workshops like the present one, something can be done about it.

I was glad to conclude, that, in the end, it was agreed upon some strategies for involving more closely other mathematics and art organizations, though in the beginning the exchanges about this idea were hesitating (to put it mildly). It will need more diplomatic efforts, and hopefully more participants from South-America, Europe, will participate in the 2009 meeting in Banff, so that the 2010 conference may turn out to become a conference where Bridges, Isama, ISIS, Katachi, Nexus, Math& Design and whatever other math and art groups may meet. An international gathering, say, every four years, of these different organizations could even attach as a math and art movement to the ICM meetings, organized every four years, and, thus, give it more scientific recognition. Maybe the good Banff center dinners and lunches contributed to this idea, as the dinner table served for the talks with the always open-minded Reza Sarhangi, (co)-organizer of the Bridges meeting.

I learned about the existence of two great books, and met (one of) their authors: - Experiences in geometry: Euclidean and non-Euclidean with History, David W. Henderson and Daina Taimina (ISBN 0131437488) - Mathematics and the Aesthetics: New Approaches to an Ancient Affinity (CMS Books in Mathematics) by Nathalie Sinclair, David Pimm, William Higginson, Editors (ISBN-10: 0387305262, ISBN-13: 978-0387305264) There are many books with similar titles and with enchanting reviews, so that meeting the authors, and going through some pages, is actually more convincing. I think Ill soon (=next academic year) teach a course based on the first, with the second as an indispensable reader. This is a down-to-earth simple consequence of this meeting, but a very useful one to my students and me.

Finally, here is the summary along the idea of Carlo Squin. If I were to meet Bill Gates in Banff and had one minute to talk to him, I would say: Hello, ho, youre Bill Gates! I am here in Banff for a meeting about mathematics and art. You know, I am often surprised architects very proudly assert everyone knows the Egyptian pyramids are the first great human realization, while in fact, mister Gates, it is YOUR field, the field of reasoning, logic and mathematics which started the odyssey of humanity. Yes, yes, in black Africa, the continent you want to help in particular, there, the first record of logically grouped was found, 22000 years ago. No joke, mister Gates, it has been confirmed in many scientific publications and in the media, and here at the meeting all find it an evident statement many civilizations contributed to mathematics. However, we need help to overcome colonial and neo-colonial unwillingness to reach the general public: some findings were only revealed on the dying bed of a discoverer. And in Africa, Western publications do not reach the audience. So, 50 years after its discovery in Congo, many Africans still think it was YOU who invented information recording, ha! (- to be followed by a demand for money).

Craig S. Kaplan, University of Waterloo

I arrived in Banff without a specific agenda, simply hoping to find some way to contribute my knowledge of mathematics and software development to the group. I wanted to suggest ways that other projects might be able to benefit from a computer-based component in general, and possibly my own computer graphics research in particular. The ideal situation would be to find a small group of collaborators with whom I could develop educational materials that simultaneously have a CS research component.

The experience has been enjoyable and inspiring. Through productive discussion with other participants, I came away with ideas for math education and for my own undergraduate computer science teaching. Naturally, some of the discussion also led to new ideas for research (which, being related to art-math, might always filter down to the educational level).

Everyone at this workshop believes that there is a role for art to play in the teaching of mathematics. Art provides a context for the math, motivates it by demonstrating an application, and makes it visible and tactile. Student projects based on art-math can engage them across multiple modalities, in a way that pencil-and-paper work never could.

I believe the most important aspect of using art to teach math is that the presenter not simply treat the math as being self-evident. It isn't sufficient to show an example and proclaim ”behold!”. To be educational, a piece of mathematical art must be the launching point for a deep presentation of the underlying concepts, and never an end in itself. I believe that this attitude is shared with many of the other participants, which makes me confident that the educational materials that result from this workshop will have real value.

My impression is that during the week, the group arrived at a concrete goal: a series of books containing lesson plans and project ideas that math educators could incorporate into their curricula. This is a worthwhile goal, and one to which every participant is able to contribute. These books could definitely help bring the beauty of art-math to a wider audience in an educational setting.

Gene Klotz, Swarthmore / Math Forum at Drexel

I had a very interesting and generally worthwhile experience, even as a newcomer to this community. My interests and focus are different from most people but I found numerous persons to talk with, and some interest in my ideas. I wish I'd had more time to present my ideas at first and more time to interact with people in a focused way at the end.

My interests are in web-based projects and this was addressed but not in the depth I would have liked. However, my Math Image project ideas were strengthened and I plan to submit a proposal to various funding agencies.

I've formed potentially useful contacts and expect to continue interacting with a couple people at least. I'll give the community the opportunity to comment on my proposal before submitting it to funding agencies and will encourage people to come up with wiki ideas for the Math & Art community as an associated project.

There are likely to be several projects that come out of the workshop, it would appear. Several groups seem to have coalesced with common interests and clear ideas. I also would not be surprised if this workshop were to have a lasting impact on the Math & Art community: spending a week together looking for common goals and projects was a good thing.

Very nice facilities and support. BIRS provided a fine context in which to grow ideas and to interact in productive ways.

Blake Mellor, Loyola Marymount University

This was a very valuable experience for me personally. I had the chance to meet with many other people interested in connections between mathematics and art, and in particular to meet some of the more established people in the field. I also was able to work on projects with my collaborator, Gwen Fisher, and to become involved in a larger collaboration (which I describe more below) which I believe will fit very naturally with my ongoing projects.

The facilities were excellent I look forward to returning for Bridges 2009!

The needs I feel are most important are two fold. The first is the need for easily accessible and ready-to-use materials and modules which integrate mathematics and art in a substantial way that teaches real mathematics and relates to real art. The second is the need to assess whether the integration of mathematics and art is successful in (1) improving the learning of the mathematical content and/or problem-solving skills and/or (2) improving the students attitudes towards mathematics and their ability to do mathematics.

I think the first need has been addressed at several levels, with proposals to create collections of materials for K-12, college courses for liberal arts students, and college courses for more mathematically advanced students. They are also supported by proposals to collect images and other materials on CD or a website.

The second need was raised, but was not addressed as centrally, though a few of the participants (specifically Gwen Fisher) are planning to do some research on this question.

Partnerships formed/Accomplishments/Projects Planned I formed a collaboration with Gwen Fisher, Kevin Hartshorn, Doris Schattschneider and Carolyn Yackel to edit a collection of activities/projects for Mathematics for Liberal Arts. We plan to create some sample projects by July 2007, along with detailed guidelines for the projects, and send out a call for proposals by the end of the summer. We hope to collect a range of projects, on different topics and of different lengths, to be a resource for teachers of college Math for Liberal Arts courses, and possibly also for high school teachers.

Long range impact Collecting resources in a form easier for novice teachers to use will greatly increase the use of art in mathematics education, and that will lead naturally to greater interest in the more research-level questions at the intersection of mathematics and art.

David Rappaport, School of Computing, Queen’s University

My mathematical teaching experience is constrained to university lecturing to computer science students. I have an ongoing interest in teaching mathematics and think that some of my ideas of using music to teach concepts in mathematics would be useful at other educational levels. The interaction this week has been a tremendous learning experience, where some of my impressions were confirmed and others reformed.

I think that it is important to engage mathematics students so that they can gain a deeper understanding. The hands-on workshops this week, for example George Hart’s workshop on building a truncated icosahedron using recycled compact discs, addressed this aspect of learning. In fact this aspect of deep learning seems to be a common trait of all of the presentations and ideas that were proposed this week.

I have had a very productive week. My focus on using music to teach mathematical concepts match well with several participants. We are in the process of organizing a paper to be written to articulate and expand some of the ideas that we demonstrated here at BIRS on Thursday morning. I will be continuing a long collaboration with Paco Gomez and Godfried Toussaint. I am very pleased that Susan Gerofsky and Reza Sarhangi will be a part of our collaboration. I am also very excited to have made a connection with Bill Higginson, and Stewart Craven. Bill is at Queen’s my home institution, and Stewart is down the road at the Toronto Board of Education. They have both shown interest in the ideas we presented. I feel that their knowledge and experience in the field of education will be an enormous help for us, and may lead to fruitful collaborations.

I would like to think that the work we are preparing will have a positive practical impact on how mathematics is taught at a variety of levels. At the very least I will embark on a new and stimulating area of study and research.

The facilities here are very conducive to group interaction and I have nothing but good things to say about my experience here. My only regret is that I could not interact with more of the people at our workshop. Furthermore it would have been interesting to be able to interact with other groups at the Banff Centre. However, I guess there is only so much that can be done in 5 days.

David A. Richter, Ph.D.

I wrote these notes about my personal outcomes. I don’t think they answer all the questions you asked above, and certainly you won’t find the answers in the order you requested. Nevertheless, the substance is here: For me, there were four identifiable outcomes. The first is that I got more people to think about holding a student mathematical art contest. The second is that I am more determined, perhaps inspired, to complete an individual project in mathematical art. The third is that I have joined a collaborative team to work on a project in mathematics instruction via the arts. Finally, I have expressed some interest in contributing to a wiki on mathematical art. Here follow some details about these outcomes. Initially I presented the idea of an international student mathematical art contest. I will continue to think about realizing this project. However, I do not have the resources to take on a project on the scale that I envisioned. Therefore I will continue to contemplate trying it on a local level, maybe at my university or my town. Throughout the conference, several participants expressed interest in this project and offered their advice. This workshop has provided incentive for me to continue on an individual project. My wife and I are working on a quilt which illustrates the conjugacy classes of the quilt group, the group of all rigid motions preserving the integer lattice in the plane. We are nearly finished with the entire quilt, but we have barely started the write-up. I plan to take some notes on the project which a quilter with a modest background in geometry can understand, and present the work either the Bridges or ISAMA conference. I call this an "individual" project because my wife did not attend this workshop and obviously, being married, we have the capacity to work very closely as single unit. Thus, I do not consider this to be a new collaboration. I have joined an editorial board to assemble mathematical art projects into a resource book. The other members who have expressed the most interest are Doug Burkholder, Pau Atela, and Bob Bosch, and it seems likely that we will include others. These projects will be designed to supplement standard core courses for students majoring in mathematics and/or computer science. The students will work on each project outside of class, and each project should require about 2-4 weeks to obtain a final product. We plan to assemble about 10-20 projects into a single book after some field experience. Given that we need at least one or two more semesters to field-test these projects, and given that we wish to invite others to contribute, I anticipate that at least a couple years will elapse before we have a finished product. Gene Klotz has shown a lot of interest in warehousing mathematical art images using some of the infrastructure of the Math Forum and also he likes the idea of a wiki on mathematical art. I have expressed interest to him about contributing to either of these projects. Due to the scale and complicity of this project, however, I cannot lead this project at this time. That's it. Thank-you for a very productive and enjoyable workshop!

Reza Sarhangi

Department of Mathematics, Towson University, Towson, Maryland, USA

My experience at Banff included the following important mathematics education: (a) I became familiar with several mathematics education internet resource that have connections with art and I was not aware of them, (b) meet with some individuals who are key people in developing art related mathematics activity materials and manipulative sets for K-12 teachers, (c) I also learned about new college level mathematics education materials that were related to the arts, (d) participated in some mathematics and music proposals for writing joint papers and producing workshop activities

We need to reach public and present mathematics in an appropriate way. I believe one appropriate and important approach for this action is to use art as a media for presentation or as a field for integration. Poincar says: The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful. By proper use of art, which is not the only approach but one to be indicated during our workshop, we hope that the new generations become introduced to the beauty of mathematics. In a traditional synthetic geometry course we are introduced to rigorous treatment of axiomatic systems. During which process we also learn historical and philosophical implications of various discoveries in Euclidean and non-Euclidean geometries. In addition, as a part of reasoning or as a mathematical challenge, we also learn how to make geometric constructions using compass and straightedge. Geometric constructions and the logic behind of the taken steps bring excitements to us while exercising our intelligence to justify the steps to reach to a conclusion. Geometric constructions have formed a substantial part of mathematics trainings of mathematicians throughout human life. Nevertheless, we are witnessing a lack of attention to the impertinence of geometry in shaping our understanding of mathematics in colleges and universities. One way of addressing this problem is to produce textbooks and workshop activities in connections of mathematics and the arts.

During the workshop we were introduced to some innovative and integrative techniques that promote interdisciplinary work in the fields of mathematics and art education. I found the following areas that I can contribute: (a) I will participate in writing chapters for three mathematics education via art textbooks that will be produced in near future. The textbooks will be produced for college students majoring in elementary, middle school, and secondary education, (b) I also will participate in writing chapters for liberal art college level textbooks in mathematics and art connections, (c) Another area that I will participate will be in creating mathematics workshop activity packages for teachers in different levels that use visual art or music.

I see that this workshop will impact our education system in producing interdisciplinary resource materials in mathematics education in areas such as textbooks, workshop activity books, and internet recourses.

Doris Schattschneider, Moravian College

My experience at this workshop has been an enriching one. I have met many new people and gotten to know many others better and have substantive discussions with them (I was at Bridges conferences with several of the participants in the past). At times, I had some frustration at the lack of pre-planned programming, but letting the connections and conversations happen and drive the schedule was a good idea. The presentations/workshops were very interesting and there were just enough of them, with very different flavors.

In a short period of time, many plans for materials on math/art have been discussed, and some detailed outlines for some projects have been formulated. I felt that I was able to contribute to many of the conversations, in formal groups and in informal settings. One unexpected connection and collaboration was with Philip Wagner, who came looking for input from conference participants on his proposed ”Fusion” project in San Francisco. Since I'm a consultant for a curriculum project and have curated some art/math exhibitions at art museums in the past, I was able to give several suggestions to him. I hope that this collaboration will continue.

The greatest need for integrating art(s) in the teaching of mathematics, in my opinion, are good resources, readily available, and mentoring of teachers in how to use these. At the elementary and secondary level, this is especially important, since these teachers have enormous constraints under which they teach and they often feel that there is no way they can go beyond the text, or integrate other material with the required material. The BIRS workshop addressed many aspects of these needs, and if the proposed projects are culminated, at least some of this need will be met.

I joined discussions with two groups– those preparing materials to integrate art into the teaching of math at the college level–both the liberal arts level and the math major level. I expect to be actively involved with both these projects. Also, mentioned above, I collaborated with Philip Wagner. My own proposed project, of preparing a CD with art images to be used in teaching math/art courses, was not pursued by any group, but several participants who are teaching such courses in liberal arts colleges told me they really would like to have such a resource. I hope to pursue this further.

I learned several new and interesting mathematical ideas/facts– how to fold a perfect 6-pointed star, how to dance a sword-star, how to clap a polygon. Most of all, I felt part of a group of educators who are very interested in promoting the use of art in math education. I'm sure that these partnerships will continue well beyond the workshop at Banff.

There is a real possibility that what started at this workshop will have long-range impact. No one will leave here without having learned many new ideas and everyone seems to have the commitment to carrying out the various proposed projects.

Carlo H. Séquin

EECS Computer Science Division, University of California, Berkeley, CA 94720-1776

Overall Experience General Impressions The Banff Center presents a wonderful environment for a productive workshop to make new contacts, to exchange ideas, and to plan new activities. The facilities (A/V, computers, conference rooms) are first-rate. The surroundings are spectacular and inspiring. Good ideas came from walks in this nice scenery.

Positive Results, Benefits A primary result of this workshop was to make new, lasting contacts and to become aware of various existing resources related to the connections between art and math in particular, web sites and publications. I also gathered useful ideas for new student projects, classroom activities, and works of geometrical art. Examples are: Class room activities: Rhythmic clapping exercises related to n-gons and periodic motions on a circle. Geometrical sculptures: Make models of manifold surfaces from CDs connected with cable ties. Student projects: Define sets of points or line segments in 3-space that yield different semantically meaningful shadows when projected in different directions.

Accomplishments I had a secondary personal goal for these five days in Banff. I hoped to find time to write the bulk of a paper for Bridges 2007 for which I was still missing a crucial results related to the challenge of mapping the Hurwitz group of order 504 onto a genus-7 surface in a nice and symmetrical way. I had hoped that in Banff I would meet some knowledgeable topologists or group theoreticians who could me help with this problem. Indeed, I did make several crucial contacts (e.g., David Richter) and got myself educated on some fundamental issues that had been puzzling me for quite some time. While at this point I do not yet have the final solution for the genus-7 embedding, I am definitely a couple of steps closer. I have been able to tackle a very similar, but less complex problem: Mapping a manifold of 24 pentagons onto a genus-4 surface. Insight gained in solving this simpler task will help me in my quest for the final solution.

Planned Follow-up Activities In three small group discussions and various lunch and breakfast conversations, we were able to outline some promising and potentially very exciting student activities and projects. Different group members will write up those ideas in more detail and put them in a shared place on the web. After the activities have been tested at least once with some actual students, the result from those trials will also be described and shared in that same place. Ultimately, the fleshed-out material will be carefully edited, so that the write-ups can become chapters in a book on college-level art-math projects. Here are two outlines of projects that I am committed to writing up in detail:

Graph Embeddings: Study knot and/or graph embeddings in 2-manifolds embedded in 3-space. What is the lowest-genus surface in which a given graph or knot can be embedded without crossings? What is the maximal symmetry that can be achieved? Make a large, sculpture-like model for long-term display in a public place.

Knot-spanning Surfaces: Learn about simple knots and about the 2-manifolds that use the path of the knot as their boundaries. Study different such knot-spanning surfaces and determine their genus and whether they are one-sided or two-sided. Deform some promising looking surfaces into artistically shaped forms and then build a real, tangible 3D artifact to represent this configuration. Possible realization techniques could be masking-tape surfaces attached to a knot made of copper tubing, or shapes formed from thick wires and from wire meshing.

Daina Taimina, Cornell University

BIRS is a wonderful place for work -everything that could possibly distract you or take your time away is taken care of and no excuses left not to be productive. I was finding people all around campus here very welcoming and helpful. It was a key to successful workshop. Even if we had sessions in certain times, there were so many other opportunities to get together informally and share our ideas and do brainstorming. I was finding it very useful to do some of brainstorming sessions in small groups and walking.

I liked very much beginning of the conference with introductions by all participants - it really gave a good overview what people are doing and what they are interested in, I think it helped later to make connections. There is a lot of need for informal math education. During our workshop issues in K-12 and college level math education were addressed but I think another direction to think is life long learning. It has been my experience that people are getting interested in math behind objects they see as art objects, and then they want to have an explanation and understanding. I am doing quite a lot internet correspondence with people later in their lives who now want to learn more mathematics. Talking about mathematics for general audience is different than teaching it in class. General audience is more interested to grasp concepts and do not need details. And that is really good opportunity for arts to come in. In our workshop there were many activities showed and developed that can be used for such audiences also but I think that can be continued.

At this point I can not give a strong partnership formed for particular project but I feel that I got an excellent opportunity to get to know many people I did not know before and learn about what they are doing. Concrete project is to connect with Carol Bier and Textile Society of America and write a proposal for a workshop. Also I will be writing a paper for Mathematical Intelligencer.

Thank you very much for the opportunity to be a part of this group and learn a lot!

Professor Godfried T. Toussaint

School of Computer Science and Centre for Interdisciplinary Research in Music Media and Technology Schulich School of Music McGill University, Montreal, Quebec, Canada

This workshop was for me a unique experience that provided me with connections to elementary and high school teachers of mathematics that would have been difficult to realize otherwise. The group and the surroundings at BIRS provided a most stimulating environment in which to do research. In short this was one of the most positive workshop experiences of my life.

Many young students today are turned off from mathematics because it is presented to them in a non-exploratory, non-creative dry manner. with little applications or connections to other exciting subjects such as art and music. The need to remedy this situation in education was considered in depth at this workshop through many discussions in both large and small energetic study groups.

My proposal for this workshop was to explore and outline which concepts in mathematics would be suitable for teaching students at all levels by using musical rhythm to act as an accessible spring board amd motivating factor. Several participants were excited about my proposal and joined me to form a group. These participants were: Susan Gerofsky, Francisco (Paco) Gomez, David Rappaport, and Reza Sarhangi. As a group the five of us met every afternoon to explore our ideas. We also met in pairs and threes in the mornings and evenings. We presented progress reports to the entire group every day, and received valuable suggestions from participants not in our group. In particular we benefitted from suggestions and useful references given to us by William Higginson and Dirk Huylebrouk. By the end of the workshop we had put together the skeleton of a paper we will write on this topic addressed to high school teachers. We also presented an example one-hour lesson to the entire group.

In general I think this workshop will in the long range increase awareness among teachers at all levels, that mathematics can be taught in a more interesting and fun manner for both students and teachers. On a personal level I have started a new collaboration with Susan Gerofsky who is in an education department. This collaboration will result in the publication of materials that will improve mathematics education in the future.

The workshop lasted four full days. Perhaps for this type of exploratory workshop seven days is more appropriate?

(1) Philip Wagner, Fusion Project, Integrating Mathematics and Art, San Francisco, CA.

(2) Excellent workshop with terrific people. I learned a great deal and everyone was anxious to share.

(3) I will collaborate with Stewart Craven and Nat Friedman and work with teaching facilitators to instruct teachers at 3-5 pilot schools in San Francisco. I will also work with art museum docents on these interdisciplinary projects that relate the museum collections with public mathematical education. In addition, I will coordinate and evaluate these programs.

(4) The long term goal is a mutual benefit for public education and art museums relating art and mathematics.

(5) Our short term goal is to improve understanding of MATH among middle school teachers and their students using ART. Working with the District we will identify those subject areas on standardized tests most likely to be improved using this approach. We must improve test scores. That is the reality.

Carolyn Yackel

Mercer University

My experience at BIRS was both intellectually invigorating and physically exhausting. I am proud of the work I accomplished with my team–Gwen Fisher, Kevin Hartshorn, Blake Mellor, and Doris Schattschneider– this week, but I am also apprehensive about the large amount of work to which we have committed ourselves over the next one to two years by agreeing to edit and write the liberal arts math and art book. In one sense, I am very happy about having partners in this endeavor that I felt sure I was destined to undertake during the next few years. In another sense, this conference has moved up the timeline of my individual plans for a book by several years. Again, having collaborators means we'll have more ideas and a much broader set of skills between us. However, it will also mean that I will have to compromise my goals and aspirations for the final project to fit with those of the group.

Aside from the identifiable project in which I have agreed to participate, I believe that this conference has also been a success for me in that I have made the acquaintance of and begun friendships with many colleagues who I greatly admire. I often find that these relationships blossom over time into collaborations. For example, I could imagine that in the future I would work with some of my new acquaintances to plan a conference, to serve on a committee, or even to write a paper in addition to the book we are already planning to edit.

My math education needs are for projects for my liberal arts math course taught through fiber arts. I suppose I gained some ideas for this course. Through discussions with my team, I certainly gained some ideas about the importance of not ignoring the artistic aspect of the course. In addition, I will be happy to have access to the book we will put out. Another big issue that faces education is the problem that in US culture not knowing math is perfectly acceptable. I think that we may be able to begin to make some in roads into this problem by using math and art to change attitudes towards mathematics. However, I think that this process will neither cause attitudes to change quickly, nor will it have an immediate impact on student scores. Furthermore, I believe that the math and art projects that experts choose to do with students during school time must be carefully chosen to fit with the curriculum students are already learning. Notice that this is not the case for special projects that take place outside of the classroom environment. Finally, I think that the mathematics contained in any activity or project is best if it is implicit in the project or activity rather than being superimposed. Implicit mathematics calls out to be discovered and investigated, whereas superimposed mathematics is artificial and tends not to engage curiosity.

As stated above, I plan to work on the Liberal Arts math and art book as an editor together with Kevin Hartshorn, Blake Mellor, and Doris Schattschneider as coeditors and Gwen Fisher as illustrator until she allows us to promote her to coeditor. My current obligation is to write a sample chapter about temari balls. It it due to the group by May 10th. We intend to have a proposal to a publisher by the end of June, and a call for chapters to potential authors by MathFest 2007. Other planned projects include digging back out a temari ball project/problem I was working on a year ago. Carlo Sequin re-inspired me with some new information which may enable me to either solve the problem or remember the solution I thought I had, but never had time to write up. Having Gwen Fisher teach me to bead led me to ask a beading question, which I may follow up on, if I have time. Gwen also wanted to work on a math ed project with me, which we may do, if she is still interested, and if I have time.

I think the long range outcome of this workshop will be at least two or three books in the next couple of years. I also think that a much more subtle result will be the collaboration of a younger group of mathematical artists or the incorporation of that younger group into the collaboration of the slightly older generation. The results of those collaborations may not be known for years. However, having had this time to meet, talk, and really try to work together has allowed us to know each other in a way that would usually take the time of several conferences.

List of Participants

Alagic, Mara (Wichita State University)
Atela, Pau (Smith College)
Bier, Carol (Mills College; also Research Associate, The Textile Museum)
Bosch, Robert (Oberlin College)
Burkholder, Doug (Lenoir-Rhyne College)
Craven, Stewart (Toronto District School Board)
de Vries, Gerda (University of Alberta)
Fisher, Gwen (Cal Poly)
Friedman, Nathaniel (University at Albany-SUNY)
Gerofsky, Susan (University of British Columbia)
Gomez, Francisco (Paco) (Polytechnic University of Madrid - McGill University)
Greenfield, Gary (University of Richmond)
Hart, George (Stony Brook University)
Hartshorn, Kevin (Moravian College)
Higginson, William (Queens University)
Huylebrouck, Dirk (Hogeschool Wetenschap en Kunst)
Kaplan, Craig (University of Waterloo)
Klotz, Gene (Swarthmore College & The Math Forum at Drexel)
Mellor, Blake (Loyola Marymount University)

Rappaport, David (Queen's University)
Richter, David A. (Western Michigan University)
Rimmington, Glyn (Wichita State University)
Sarhangi, Reza (Towson University)
Schattschneider, Doris (Moravian College)
Sequin, Carlo (University of California, Berkeley)
Taimina, Daina (Cornell University)
Toussaint, Godfried (McGill University)
Wagner, Philip (The Fusion Project)
Yackel, Carolyn (Mercer University)

Chapter 3

Nonholonomic Dynamics and Integrability (07w5029)

Jan 28 - Feb 02, 2007

Organizer(s): Boris Khesin (University of Toronto), Sergei Tabachnikov (Penn State University)

Overview and introduction

Nonholonomic mechanics describes the motion of systems subbordinated to non-holonomic constraints, i.e., systems whose restrictions on velocities do not arise from the constraints on the configuration space. The best known examples of such systems are a sliding skate and a rolling ball, as well as their numerous generalizations. These systems usually exhibit very peculiar, often counter-intuitive, behavior. For example, a golf ball rolls inside a vertical tube, while oscillating and seemingly defying gravity, and a rattleback top (Celtic stone) spins only in one direction and resists spinning in the opposite one. Non-holonomic mechanics is also a cornerstone of the control theory, where the non-holonomic property is pivotal in the descriptions of attainable configurations; see [2, 5, 6, 9, 14].

The integrability vs. chaos dichotomy in such systems is one of their main points of interest, which is yet to be better understood. Furthermore, a more profound understanding of the relation between several competing paradigms in nonholonomic mechanics, the applications to control theory, as well as the similarities with Hamiltonian systems would be very important for further progress in the theory. This workshop served as a unique opportunity to bring specialists in these domains together and foster interactions between researchers with diverse and often complimentary backgrounds in nonholonomic mechanics and in the adjacent areas, including sub-Riemannian geometry, Hamiltonian systems, billiard theory, sub-elliptic operators, motion planning in robotics, and others.

Below we outline several major topics which emerged in many talks at the workshop.

Nonholonomic mechanics, symmetries, control theory and the Hamilton-Jacobi equation

Connections between nonholonomic mechanics and control were described in the talk by A. Bloch (University of Michigan). Systems subject to nonholonomic constraints have natural links to nonlinear control systems as the constraints often induce good controllability properties. He discussed the important distinction between kinematic and dynamic nonholonomic systems and described the different optimal control problems that arise for these two classes of systems. A. Bloch discussed integrable systems that arise naturally in optimal control of nonholonomic systems and in nonholonomic systems themselves, as well as aspects of stability and stabilization of such systems. He also showed how one can get asymptotic stability in certain classes of nonholonomic systems even in the absence of external dissipation.

Numerous examples show that momentum dynamics of nonholonomic systems is remarkably different from that of holonomic/Hamiltonian systems. For example, symmetries do not always lead to spatial momentum conservation as in the classical Noether theorem. D. Zenkov (North Carolina State University) in the talk *Momentum conservation, integrability, and applications to control* discussed various examples, including nonholonomic momentum conservation relative to the body frame and their role in the theory of integrable nonholonomic systems. A new integrable nonholonomic system was introduced in the talk, and other applications of momentum dynamics to control of nonholonomic systems were discussed.

However, the theory of the Hamilton-Jacobi equation becomes subtle in the nonholonomic context. This was the subject of the talk *What happened to the Hamilton-Jacobi equation* by L. Bates (University of Calgary). By looking at some examples he attempted to explain why there is no Hamilton-Jacobi equation in nonholonomic dynamics and the implications this has for the solvability of completely integrable nonholonomic systems, [4].

This was in a nice contrast with the talk *Hamilton-Jacobi theory for nonholonomic mechanical systems* by M. de Leon (CSIC Real Academia de Ciencias), who developed in his talk a Hamilton-Jacobi theory for nonholonomic mechanical systems. The results were applied to a large class of nonholonomic mechanical systems called Chaplygin systems, [13].

J. Sniatycki (University of Calgary) gave the talk *Conservation laws, symmetry and reduction*. For a non-holonomically constrained mechanical system he described the distributional Hamiltonian formulation of its dynamics, formulated a non-holonomic analogue of Noether's theorem, and discussed the notion of symmetry of such a system. He also discussed various types of constants of motion and singular reduction of symmetries, [18].

Rolling problems: geometry, symmetries, control

A. Agrachev (SISSA) gave a talk on *Rolling balls and octonions*, in which he discussed hidden symmetries of the classical nonholonomic kinematic system (a ball rolling over another ball without slipping or twisting) and explained the geometric meaning of basic invariants of the corresponding vector distributions, see [1].

R. Montgomery (UC Santa Cruz) continued this topic in the talk G_2 *and the rolling distribution* based on his joint paper [7] with Gil Bor. The act of rolling one surface along another surface without slipping or spinning defines a rank 2 distribution on 5-manifold, the 5-manifold being a circle bundle over the product of the two surfaces. This distribution is manifestly invariant under the product K of the isometry groups of the two surfaces. When the two surfaces are spheres then K is the product of two rotation groups, one for each sphere. However, something miraculous happens when the ratio of radii of the spheres is 1:3: the local symmetry group of the rolling distribution becomes much larger. This local automorphism group becomes the first exceptional Lie group, namely, the split real form of the Lie group G_2. A proof of this fact was described using explicit constructions and relying heavily on the theory of roots and weights for the Lie algebra of G_2.

The control theory for the problem of rolling appeared in the talk of V. Jurdjevic (University of Toronto) *Rolling sphere problems on spaces of constant curvature* (joint work with J. Zimmerman) [11]. The setting of the rolling sphere problem on Euclidean space $\mathbb{E}^n$ for $n \geq 2$ consists of determining the path of minimal length traced by the point of contact of the unit sphere $\mathbb{S}^n$ on $\mathbb{E}^n$ as it rolls without slipping between two specified points of $\mathbb{E}^n$ and from a given initial rotational configuration to a prescribed terminal rotational configuration [10].

In this lecture Jurdjevic presented the results, in which the rolling sphere problem is extended to situations in which a sphere $\mathbb{S}^n_\rho$ of radius ρ rolls on a stationary sphere $\mathbb{S}^n_\sigma$ of radius σ, and to the hyperbolic analogue in which the spheres $\mathbb{S}^n_\rho$, $\mathbb{S}^n_\sigma$ are replaced by the hyperboloids $\mathbb{H}^n_\rho$, $\mathbb{H}^n_\sigma$ having hyperbolic radii ρ, σ with $\sigma \neq \rho$. The notion of rolling is taken in an isometric sense; the length of the path of the point of contact is measured by the metric of the stationary manifold and the orientations of the rolling manifold are expressed by the elements of its isometry group. This larger geometric perspective, that encompasses both the Euclidean and the hyperbolic geometries, also includes the unit hyperboloid $\mathbb{H}^n$ rolling isometrically on $\mathbb{E}^n$.

Billiards and sub-Riemannian geometry

One of the most well known open problems in the theory of mathematical billiards is to prove that the set of periodic billiard trajectories has zero measure. This conjecture is motivated by spectral theory (Weyl asymptotics for the spectrum of the Laplace operator). Recently a new approach to this problem was developed using ideas from sub-Riemannian geometry and the theory of exterior differential systems [3]

It is known that a planar Birkhoff billiard cannot have a 2-parameter family of 3-periodic orbits (this fact was proved by a number of authors in different ways), while spherical billiard domains can (for example, the spherical geodesic triangle with right angles). Yu. Baryshnikov (Bell Labs) explained in the talk *Spherical billiards with many 3-periodic orbits* this phenomenon and described spherical billiards having this property: the boundary of such a billiard must contain three segments of the sides of an equilateral right triangle.

V. Zharnitsky (University of Illinois) in the talk *Periodic orbits in outer billiards* (joint work with A. Tumanov) described an adjustment of the exterior differential systems method to the study of periodic trajectories of outer billiards. He proved that if the set of 4-period orbits in the outer billiard has non-empty interior then the table has four corners that form a parallelogram [19].

Nonholonomic systems and Lie groups

Nonholonomic mechanical systems are not Hamiltonian. L. Garcia-Naranjo (University of Arizona) in the talk *Almost Poisson bracket for nonholonomic systems on Lie groups* described the dynamics of nonholonomic systems in term of a bracket of functions that fails to satisfy the Jacobi identity. Now one speaks of an almost Poisson bracket. This approach avoids dealing with Lagrange multipliers, but, in practice, is difficult to implement because it involves heavy computations in coordinates.

He considered the so-called LL and LR systems where the configuration space is a Lie group and both the Hamiltonian and the constraints have invariance properties. These invariance properties allowed him to give a geometric construction of a bracket for the description of the system on a reduced space. This construction avoids computations in coordinates and provides relatively simple formulas for the bracket. The idea involved in the construction generalizes the theories of Lie-Poisson and semidirect product reduction to the nonholonomic setting. The constraint functions of the resulting bracket are Casimirs, so the constraints are satisfied automatically.

Discretization of integrable nonholonomic systems on Lie groups was described by Yu. Fedorov (Universitat Politechnica de Catalunya). Recently the formalism of variational integrators (discrete Lagrangian systems) was extended to systems with nonholonomic constraints. Fedorov briefy described this formalism and applied it to the case when the configuration space is a Lie group G and the discrete Lagrangian is left-invariant, while discrete constraints are left- or right-invariant with respect to the action of G. As examples, he constructed discretizations of several classical integrable nonholonomic systems with an invariant measure, in particular, the celebrated Chaplygin nonholonomic sphere problem. It appears that the resulting discrete dynamics is similar to that of the continuous models. He also proposed a method of choosing left-invariant discrete nonholonomic constraints that ensures preservation of the energy integral in the discretizations. The conservation of an invariant measure in the discrete systems was also discussed, [8].

P. Lee (University of Toronto) gave a talk *Infinite-dimensional geometry of optimal mass transport* on non-holonomic distributions in the infinite-dimensional context (joint work with B. Khesin) [12]. He considered the following nonholonomic version of the classical Moser theorem: given a bracket generating distribution on a manifold, two volume forms of equal total volume can be isotoped by the flow of a vector field tangent to this distribution. These results were discussed in the talk from the point of view of an infinite-dimensional non-holonomic distribution on the diffeomorphism groups. Furthermore, in the 60's Arnold showed that the Euler equation can be thought of as the geodesic flow on the group of volume-preserving diffeomorphisms. In a similar fashion, Otto showed that the mass transport problem can be consider as the geodesic problem on the Wasserstein space of all volume forms with the same total volume. In particular, the Wasserstein space can be regarded as the quotient of the group of all diffeomophisms by the subgroup of volume preserving ones, while the geodesic flow on the diffeomorphism group, given by the Burgers equation, is closely related to that on the Wasserstein space. It turns out that this relation between diffeomorphism group and the Wasserstein space can be understood via Hamiltonian reduction.

Applications of control theory and sub-Riemannian geometry

Yu. Sachkov (Program Systems Institute) gave a talk *Maxwell strata and conjugate points in Euler's elastic problem*. In 1744 Leonard Euler considered the following problem on stationary configurations of elastic rod. Given a planar elastic rod with fixed endpoints and tangents at the endpoints, it is required to find possible profiles of the rod with the given boundary conditions. Euler derived differential equations for stationary configurations of the rod, reduced them to quadratures, and described their possible qualitative types. Such configurations are called Euler elastic.

The question on stability of Euler elastic was solved only in some partial cases. In the talk, a full solution to the problem of stability of Euler elastic was described. In addition to this local problem, the corresponding global optimal control problem was also considered. Stability of Euler elastic corresponds to local optimality of extremals of a certain optimal control problem. It is known that extremals cannot be optimal after Maxwell points (where distinct extremal curves with the same length and cost functional meet one another) or after conjugate points (at the envelope to the family of extremal trajectories).

The group of discrete symmetries of the system of extremals is generated by the group of discrete symmetries of the equation of a pendulum. Maxwell points are described via the study of fixed points of the action of this symmetry group. Maxwell points for all types of elastic are found, [17].

Another application to the problem of optimal laser-induced population transfer in n-level quantum systems was discussed by W. Respondek (INSA de Rouen) in the talk titled *Integrability and non-integrability of sub-Riemannian problems* (joint work with A. Maciejewski). This problem can be represented as a sub-Riemanniann problem on $SO(n)$ and it is known that for $n = 3$ the Hamiltonian system associated with PMP (Pontryagin Maximum Principle) is integrable. In the first part of the talk, he showed that this changes completely for n larger than 3. Namely, the adjoint equation of PMP does not possess any first integral independent of the Hamiltonian on the leaves of the symplectic foliation. In proving non-integrability he used the Morales-Ramis theory.

In the second part he showed that the above-mentioned integrability of the adjoint equation for $SO(3)$ is a particular case of a more general result. Namely, he proved that the adjoint geodesic equation for a 3-dimensional homogeneous sub-Riemannian space possesses an additional quadratic first integral if and only if the space is symmetric, [15].

Yet another application was *A simple example of the Arnold diffusion* outlined in the talk by M. Levi (Penn State University) on his joint work with V. Kaloshin. They gave a simple geometrical explanation of the Arnold diffusion. The idea was illustrated for the case of a particle in a periodic potential in $\mathbf{R}^3$, and, in a slightly different setting, for the geodesic flow with time-periodic metric.

Slipping and rolling toys, bicycles, and nonholonomic engineering problems

T. Tokieda (University of Cambridge) talked about *Slipping and rolling toys and their integrability*. He discussed, both on the board and through toy demonstrations, a number of nonholonomic problems which look integrable—conserved quantities, quasi-periodicity, etc.—but seem awkward to fit into the current models of integrability. They exhibit other peculiarities, such as chirality and finite-time singularity, and he argued that these ought to be a generic part of physically realistic models of nonholonomic integrability.

The engineering perspective on nonholonomic problems was presented by A. Ruina (Cornell University) in his talk *Some mechanics perspectives on non-holonomic constraints*. He showed (with video) that despite common mythology, equations of motion for non-holonomic systems can sometimes be found by simple means; that the most common non-holonomic systems, by virtue of their symmetry, cannot have the most interesting of non-holonomic features, the asymptotic stability. He argued that the word “non-holonomic” might sensibly be replaced with “skates and wheels”. Furthermore, despite more than a century-long history, there are no established equations of motion for a reasonably-general non-holonomic bicycle (demonstrated with video), [16].

In addition to talks, numerous informal discussions took place. The overall volume of interaction between the participants was very high.

The bibliography below lists several books on non-holonomic systems, sub-Riemannian geometry and optimal control theory, as well as various papers related to the talks.

List of Participants

Agrachev, Andrei (International School for Advanced Studies)
Baryshnikov, Yuliy (Bell Laboratories)
Bates, Larry (University of Calgary)
Bloch, Anthony (University of Michigan)
de León, Manuel (Instituto de Matemáticas y Fsica Fundamental)
Fedorov, Yuri (Universitat Politechnica de Catalunya)
Garcia-Naranjo, Luis (University of Arizona)
Jurdjevic, Velimir (University of Toronto)
Khesin, Boris (University of Toronto)
Lee, Paul (University of Toronto)
Levi, Mark (PennState University)
Montgomery, Richard (University of California, Santa Cruz)
Respondek, Witold (INSA de Rouen)
Ruina, Andy (Cornell University)
Sachkov, Yuri (University of Pereslavl)
Sniatycki, Jedrzej (University of Calgary)
Tabachnikov, Sergei (Penn State University)
Tokieda, Tadashi (Cambridge University)
Zenkov, Dmitry (North Carolina State University)
Zharnitsky, Vadim (University of Illinois in Urbana-Champaign)

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Chapter 4

Numerical Analysis of Multiscale Computations (07w5069)

Jan 28 - Feb 02, 2007

Organizer(s): Bjorn Engquist (University of Texas at Austin), Olof Runborg (Royal Institute of Technology, Sweden), Steve Ruuth (Simon Fraser University), Richard Tsai (University of Texas at Austin)

Short Overview

With the efficiency of modern computers and the maturity of numerical methods for solving differential equations and linear systems, the focus of scientific computation has recently been shifting towards more difficult problems where classical single physics models are not accurate enough, and a coupling of multiple physics models needs to be considered. In particular, there is an emergence of methods that replace heuristics and empirical observations in coarse scale single physics models by direct numerical simulations of more accurate models defined on finer scales. This workshop addressed the numerical analysis of such multiscale approaches.

As an example, in simulation of complex fluids, such as polymers in a solvent, a coarse scale model is given by Navier-Stokes type PDEs for non-Newtonian fluids. In such a model, the total stress is typically obtained through empirical arguments based on physical insights. The accuracy of the model thus hinges critically on the quality of the heuristics, which is often not satisfactory. In the emerging methods, the total stress could instead be obtained by direct fine scale numerical simulations of the interaction of polymer particles and the background fluid. With these approaches, the resulting models typically require solving problems involving wide ranges in spatial and temporal scales. In fact, one of the main challenges for simulating fine scale models, e.g. molecular dynamics and kinetic Monte-Carlo, lies in computationally resolving details such as oscillations over the length scales, both spatial and temporal, where interesting phenomena take place. It is generally an impossible task to solve the fine scale equations accurately over the length scale of the interesting continuum quantities, and in the time dependent case, for a time scale comparable to that of interesting events.

The new class of numerical methods alluded to above typically tackle this difficulty by exploiting separations of scales or other scale structures such as self similarity in the governing physical model. There is a large class of problems that exhibit this separation of scales it implies that enough information about the fine scale influence on the coarse scale dynamics can be obtained by performing spatially localized simulations over short times, thus gaining efficiency. The numerical complexity of these methods is therefore expected to be much smaller than direct simulation of the full fine scale model. This sort of multiscale approach makes it feasible to treat problems that could not be handled previously and to obtain higher accuracy in the simulation of important physical phenomena. So far the main excitement has been driven by multiscale mathematical models in applied sciences problems including materials science, chemistry, fluid dynamics, and biology. In this workshop, the focus was on the analysis of the numerical methods motivated by these attempts.

Objectives of the workshop

Abstractly, the problems we are interested in can be described as follows. A fine scale model for the high-dimensional variables x is known. We want to compute a number of coarse variables, X, using a tentative governing equation. Parts of the data in the governing equation is, however, missing and must be constructed from the solutions X of localized simulations of the fine scale model. In those simulations, parameters such as boundary and initial conditions, depend implicitly on the coarse variables X.

A difficulty is how to choose a good coarse model when only the fine scale model is known. The models of each scale can be of different types: e.g. PDEs, ODEs, integral equations, stochastic processes. In many applications, stochastic modeling is more appropriate. The coarse and fine scale models are often not of the same type. For instance, molecular dynamics (ODEs) can be a fine scale model of fluids, while a coarse scale model could be Navier-Stokes (a PDE). Similarly, Monte-Carlo methods can be used either as a valid method for the coarse scale system or used in a fine scale to evaluate coarse scale quantities, while some other type of computational model is used in the complement scale.

There are also situations where there is no explicit equation for the coarse variables, and we are interested in developing direct numerical procedures to consistently drive the coarse scale evolution by using snapshots of fine scale solutions. As an example, the Fermi-Pasta-Ulam problem is to study the adiabatic energy transfer in a system of interlaced linear stiff springs and soft' nonlinear ones. The adiabatic energy transfer in the system is an important phenomenon that only professes itself when the energies of the right springs in the original system is explicitly computed. Each spring in the system oscillates at similar fast time scales, and it is not obvious what coarse variable should be used to consistently compute the energy transfer in the system. Hence, in this workshop we proposed to address the following specialized aspects related to the multiscale computational approaches alluded to above:

1. Under what conditions and in what sense do the multiscale approaches converge (while still having a significantly smaller complexity than that of direct simulation of the fine scale model)? What is the accuracy and the stability properties of the methods?

2. Is the coarse system properly closed, i.e. are the chosen X variables enough to describe the coarse dynamics that is consistent with the fine scale system? If not, what additional auxiliary coarse variables are needed to close the coarse system?

3. How can we find reasonable coarse models in a systematic way? Can numerical methods be used to automate the process?

These pose challenging computational and analytical problems. Some initial work has been done on Item 1 for some classes of methods, but many open questions remain. Item 2 and 3 have been addressed traditionally by applying empirical rules based on physics, theories developed from mathematical physics, such as statistical mechanics, or by rigorous analytical approaches such as homogenization in simplified settings.

Workshop Themes

Coarse variables

One generic difficulty for the new multiscale methods is to chose the coarse, macroscopic variables. The speakers followed different approaches to accomplish this. In the dynamical systems setting Sharp used time averages of the state variables of the given mechanical systems and analyzed how the dynamics of these time averages could possible form a closed system. In LeBriss method the slow variables correspond to the amplitudes of the oscillations. It should be noted also that by choosing a symplectic method the energy is approximately kept constant and can be considered as a coarse variable that is implicitly computed and used. Ariel proposed a systematic way to find slow variables by considering polynomial functions of the fast variables and show that a finite number of such slow variables could result in an effective system which can be considered closed. Ariel further demonstrated the effectively of such multiscale method comparing to some typical test problems such as the Fermi-Pasta-Ulam problem. Bold considered the distribution function of frequencies as a slow-moving variable. Vanden-Eijden and Szepessy used statistical properties of the fast variables.

Complexity

One of the main goals of the new multiscale methods is to significantly reduce the complexity compared to solving the full problem. Overall, we see the use of coarse variables and special sampling approaches to breach complexity limitation. This aspect was highlighted in several talks:

Vanden-Eijnden showed that for a class of stochastic systems for modeling chemical reactions with drastically different rates, one could efficiently simulate the slow reactions without fully resolving all the fast reactions. This is done using a multilevel sampling technique that resembles a recent Monte Carlo algorithm by Petzold ang Gilespy.

Gilbert and Iwen presented an efficient way to identify a few salient Fourier modes in a given signal. This algorithm is based on random sampling theory. For a given error tolerance and probability, the algorithm can return the correct solutions in sublinear time. The underlying mathematical principle is related to the theory of compressive sensing.

Ariels talk focused on the idea of searching for a set of functions (slow variables) that changes slowly when evaluated along the solutions of the given dynamical systems with fast oscillation. The time evolution of the slow variables can be computed at a cost almost independent of the fastest scale of the problem.

LeBris used the generating function for a given Hamiltonian system to integrate out the effects of fast harmonic oscillations. The technique suggested provides a way to construct efficient symplectic integrators for the remaining slow dynamics. This is an approach that relies on specific knowledge and theory about the fast oscillations of special systems for breach the limitation of formal complexity.

Luskin analyzed the quasi-continuum model. The efficiency of this algorithm comes from using coarse-grained representative atoms and simplified interaction rule (Cauchy-Born) in the bulk of a material (solid) under simulation. The algorithm requires a coupling between the coarse grained bulk and detailed atomistic simulations.

Coupling of models

The challenges of coupling of different models in a multiscale computation are typically about (1) the identification of suitable macroscale or microscale variables; (2) making sure that the mathematical models used in either scales are compatible; and (3) identification of subdomains in which macroscale model are not sufficient and should be replaced by a fine scale model. In particular, the compatibility of the models used at different scales typically involves finding the suitable boundary conditions, for the fine scale domains, that correctly reflect macroscale information, finding the suitable boundary conditions for the macroscale simulations that correctly reflect the effect from fine scale simulations, and setting up the states of the chosen microscale variables which are consistent with the macroscale information.

If some sort of homogeneity in the macro domain is assumed, periodic boundary conditions for the fine scale simulations make sense. Generally speaking, microscale simulations feedback to the macro model through Dircihlet or Neumann type conditions. Initializing the micro variables may be an important issue to resolve.

Ren analyzed different couplings of continuum and atomistic models of fluids by exchanging possible combinations of flux and velocity information in thinly overlapped domains. He analyzed the instability of certain specific coupling methods.

Yi Sun couples a general interface tracking algorithms with Monte Carlo simulations. The MC simulations concentrate on small domains distributed along the interface. Periodic boundary condition is assumed tangential to the interface, and flux type boundary conditions need to be enforced according to macro variables.

Quasicontinuum method uses linear interpolation to couple macro and micro models.

Convergence

Souganidis presented general framework relying on comparison principle for establishing convergence in a numerical method. Luskin proved convergence of the quasicontinuum method in one dimension. Ariel showed that if sufficient number of slow variables is found, then one can construct efficiently convergent numerical approximations of the slowly changing properties of the fast dynamics. Szepessy discussed a hierarchy of stochastics models related to a Langevin dynamics. He proved convergence of one model to the other under certain appropriate scaling. In summary, the residual of the heavy particle's Hamiltonian path in the Kolmogorov equation for Langevins equation yields a weak error representation. To estimate terms in this representation, the correspondence to the essential martingale property for Ito integrals present in the Langevin dynamics comes here in the Hamiltonian dynamics in more subtly form by Fourier representations based on the first variation of the particle paths with respect to the noisy initial data.

Open Problems and Future Directions

Here, we list a couple of questions that have been raised and discussed during the workshop.

What are the main challenges in multiscale modeling, computation, and analysis?

Is it possible and sensible to develop multiscale approach for phenomena that have many or even a continuum of scales?

For oscillatory problems with time-dependent and crossing eigenvalues, what could be done to overcome the possible singularities that may arise?

In many problems, one should consider even longer time scales ($1/eps^2$). What are the additional challenges for such considerations? What may be the additional problems for ”stacking” the two-scale algorithms?

In deriving macro- models from micro- models that include stochastics, finding good ways of using noise helps in this coarse-graining process.

A hard open problem is to find mathematical methods to derive efficient pseudo-potentials in molecular dynamics. Now, this problem is studied with chemists' and physicists' lifelong experience. It would be helpful to find simpler mathematical ways.

Some discussions on the spectrum of the KdV model took place. The issue of whether using only a few salient Fourier modes in this type of problems is sufficient was raised. More generally, for what type of equation/operator does this approach make sense?

Multiscale problems are also multi-disciplinary, so we need to have more input from experts in other areas. Experts on micro in physics, biology and chemistry should help to give input on our work and show problems where new math is needed.

More problems starting from the Schrdinger equation would be good to have, since this is the model where everything is precisely defined, taking steps into coarser scales from there is interesting.

From purely analytical view point, the passage from molecular dynamics regime to continuum regimes are still unclear.

Excerpts from the talks (grouped by topics)

Dynamical systems

Speaker: Ariel, Gil (University of Texas, Austin)
Title: Applications of heterogeneous multiscale methods to stiff ordinary differential equations

This talk focused on a class of ordinary differential equations (ODEs) whose dynamics exhibits oscillatory behavior in two well separated timescales. Ariel proposed a numerical algorithm that consists of two steps. The first identifies a set of variables that characterize the slow dynamics. If a sufficient number of independent variables are found, they serve as a coordinate system for the slow modes of the dynamical system, and consequently, the application of an averaging theorem provides closure to the dynamics. He showed that for a class of equations, including the Fermi-Pasta-Ulam problem, a closed set of slow variables can be constructed by forming the appropriate linear combinations of the monomials of the original state variables.

Speaker: Bold, Katy (Princeton University)
Title: Coarse-graining the dynamics of coupled oscillator models

Bold presented a multiscale computational approach to study the collective dynamics of Kuramoto model and a model of glycolytic oscillations in yeast cells these models are used to study coupled transient dynamics and synchronization. The two models of coupled, heterogeneous oscillators we consider are the Kuramoto model and a model of glycolytic oscillations in yeast cells. She pointed out that for these two problems, it is essential to use some appropriate distributions of the state variables on the macroscopic level and to constraint the microscopic simulations. She also discussed the existence and stability of fully and partially synchronized states in these models. Bold also described her approach to a multiscale modeling for the graph problems.

Speaker: Le Bris, Claude (CERMICS-ENPC, Paris)
Title: Construction of symplectic schemes for highly oscillatory hamiltonian systems

Le Bris described a systematic Hamilton-Jacobi approach to construct symplectic integrators for highly oscillatory systems. The approach is based upon a two-scale expansion of the generating function. Some non-symplectic integrators are also constructed, slightly modifying the above strategy. He pointed out that in the context of multiscale applications, symplecticity of a numerical scheme is not the ultimate goals, but the efficiency of this algorithm is. He proposed the idea of partial symplecticity for the slow variables. In parallel to symplecticity, symmetry preservation is also a desired properties. Symmetry in a scheme implies conservation of invariants in the system. He discussed his reflection on whether replacing symplecticity with symmetry in the algorithm would be appropriate for non-integrable systems.

Speaker: Sharp, Richard (University of Texas at Austin)
Title: Systematic reinitialization for heterogeneous multiscale methods

Sharp used a series of examples, including the problem of Kapitzas pendulum and a chain of molecules in 3D with stiff bonding to illustrate the necessary considerations and approach for designing multiscale algorithms that use the time averages of the state variables as part of the macroscopic variables. Sharp went on to describe how to find the initial data for the given stiff system so that the averages of the solution using these initial data are consistent with the macroscopic variables. His computations suggested that the preservation of the total energy in these mechanical systems have tremendous consequence for the multiscale simulations. Some questions and discussions followed on his numerical simulations of the Kaptizas inverted pendulum.

Stochastics systems

Speaker: Luskin, Mitchell (University of Minnesota)
Title: Mathematical results and challenges for the quasicontinuum approximation

Luskin derived and compared several quasicontinuum approximations to a one-dimensional system of atoms that interact by a classical atomistic potential. He proved that the equilibrium equations have a unique solution under suitable restrictions on the loads, and we give a convergence rate for an iterative method to solve the equilibrium equations.

Speaker: Szepessy, Anders (Royal Institute of Technology)
Title: Stochastic dynamics in atomistic and continuum models

Stochastic Langevin dynamics is used for simulating molecular dynamics at constant temperature. More than thirty years ago, Robert Zwanzig showed that a Hamiltonian system of harmonic heat bath interactions with particles, for stochastic initial data, can be modeled by a Langevin equation. It is also well known that Langevin dynamics can be approximated by simpler Smoluchowski dynamics (Brownian dynamics) on long diffusion time scales as the molecular friction increase. Szepessy presented some simple ideas and error estimates for coarse-graining: heat baths to Langevin, Langevin to Smoluchowski, and Smoluchowski to continuum stochastic PDE for phase change dynamics.

Speaker: Vanden-Eijnden, Eric (New York University)
Title: Numerical techniques for multiscale stochastic dynamical systems

Vanden-Eijnden pointed out that traditional implicit ODE schemes become ineffective when the dynamics of the system is stochastic in some way, e.g. if it is governed by a stochastic differential equation, a Markov chain as in kinetic Monte Carlo methods, or even by a large set of ODEs whose solutions are chaotic. He spoke about a type numerical methods that are built on asymptotic techniques and limit theorems for singularly perturbed Markov processes, originally developed in the 70s by Khasminskii, Kurtz, Papanicolaou, etc. These limit theorems provide one with closed effective equations for the slow variables in the system, and the coefficients in these equations are given by expectations over the statistics of the fast variables conditional on the value of the slow variables. In general, these expectations cannot be computed analytically, but it is possible to estimate them on-the-fly when needed via short runs of the fast variables. Once this is done, the slow variables can be evolved using the effective equations by one macro-time-step, and the procedure can be repeated. Vanden-Eijnden carefully compared the differences between the Heterogeneous Multiscale Methods and the Equation-Free approaches. In this approach, the initial condition for each microscopic run does not seems to be important. This is in contrast to many other problems that were presented in this workshop. There were discussions on how one can set up a model stochastic problem with correlated noise so that certain auxiliary macroscopic variables are indispensable for a consistent multiscale approximation.

Solids and fluids

Speaker: Ren, Weiqing (New York University)
Title: Numerical test of multiscale methods for fluids

Ren proposed a numerical approach that couples the continuum equations and molecular dynamics over an interface that separate the two models. The boundary condition for the continuum equations at the interface is provided by molecular dynamics, and vice versa. Two crucial problems need to be resolved: 1) what information (macroscopic) needs to be exchanged between the two descriptions? 2) How to impose BCs that are prescribed macroscopically on the particle systems? The second question typically leads to artificial disturbance being reflected from the coupling interface. There was a brief discussion about the approach of overlapping MD and continuum domains for tacking this problem. Ren concentrated on the first question in this talk. In the existing methods, coupling between (a) field functions (e.g. velocity); (b) flux of conserved quantities; and (3) mixture of the above have been proposed, while flux(continuum)-flux(particle) and velocity(continuum)-velocity(particle) are the most popular ones. Ren presented his finding in the performance of these different schemes. He concluded that flux-velocity coupling yields faster convergences than other combinations, flux-flux coupling is marginally unstable, and velocity-flux coupling yields errors that increase geometrically in time.

Speaker: Sun, Yi (New York University)
Title: The heterogeneous multiscale methods for interface tracking

This talk proposed two integrated multiscale techniques to the simulation of combustion fronts and epitaxial growth. For the application to the combustion fronts the main task is to overcome the numerical difficulties caused by different time scales on the transport part and the reactive part in the model. Sun showed numerical results related to Majda's model and reactive Euler equations in one and two dimensions show substantially improved efficiency over traditional methods. For the epitaxial growth, a hybrid method coupling the kinetic Monte-Carlo simulation on the microscale with the level set method for island dynamics on the macroscale is introduced. Sun further demonstrated the efficiency of his approaches by presenting relevant numerical computations for island growth and step edge in the epitaxial growth model. In both applications, the macroscopic variables include an interface that separates different materials as well as some statistical quantities, such as density of the particles, related to the microscopic state variables. A discussion on the evaluation of the interface dynamics from the microscale simulations, which is typically measured by fluxes of the materials near the interface. Again it is very important to correctly handle the boundary conditions at each of the microscopic domains. Sun described his approach that set up microscopic domains that are in alignment with the interface normal in order to better derive and enforce the boundary conditions.

Kinetics

Speaker: Gamba, Irene (University of Texas, Austin)
Title: Non-equilibrium statistics and the need of multi scale analysis in heterogeneous modeling for charged transport in nano structures

Gamba discussed issues of semi-classical and classical kinetic models for hot electron transport in nano channels and their macroscopic limits. Such models depend on the relative effective forces. Gamba also discussed the applicable regimes of the Boltzmann-Poisson and Wigner-Poisson models, which depend on the scales and their macroscopic approximation equations for their kinetic moments. An important issue for the characterization of the multi-scale is the boundary conditions. In addition, she presented some recent deterministic solvers for kinetic equations and benchmark simulations against Monte Carlo solvers for

Boltzmann-Poisson systems.

Speaker: Tharkabhushanam, Sri Harsha (University of Texas, Austin)
Title: Deterministic solvers for non-linear Boltzmann equation and computations of non-equilibrium statistical state

This speaker presented a new deterministic solver for the non-linear Boltzmann Transport Equation (conservative and non-conservative) for Variable Hard Potential (VHP) model. The speaker pointed out that in the Boltzmann collision kernel, the inter-dependence of the pre- and post- collisional velocities and the distribution posses the major computational challenge. Using a Fourier transform technique, the complexity in computing the collision integral is isolated to a separate integral over the unit sphere. The resulting numerical algorithm requires solving a constrained minimization problem in order to achieve conservation of mass. The high-velocity tail behaviour (power-like tails) of the distribution function is captured by the computation of high-order moments of the distribution function.

Novel applications

Speaker: Gilbert, Anna (University of Michigan)
Title: Sublinear algorithms for (sparse) signal representation

Gilbert gave an expository overview of several sublinear algorithms for sparse signal representation, including the Fourier sampling algorithm of Gilbert, Muthukrishnan, and Strauss. These algorithms are exponentially faster than traditional algorithms and use exponentially less space; some reading only a small fraction of the signal. In exchange for this exponential decrease in resources used, the algorithms are randomized approximation algorithms. She discussed in detail a combinatorial technique, called group testing, which is an essential step in the new Fourier methods. There are discussions about when these type of random algorithms yield deterministic result.

Speaker: Iwen, Mark (University of Michigan)
Title: Sub-linear time approximate DFT algorithm for spectral methods

The fast approximate in ”Improved time bounds for near-optimal sparse Fourier representation via sampling” will be briefly reviewed. Iwen presented some empirical results on the speed, accuracy, and noise tolerance from the DFT algorithm proposed by A.C. Gilbert, S. Muthukrishnan, and M. J. Strauss. Iwen then focussed on using the algorithm for sub-linear time/space spectral methods for the KdV equations.

Analysis

Speaker: Souganidis, Panagiotis (University of Texas, Austin)
Title: Error estimates for finite difference numerical approximations to solutions of fully nonlinear first and second order PDE

Souganidis first reviewed the basic convergence theory for monotone finite difference approximations to viscosity solutions and then presented new results about error estimates. The second part of his talk focused on the homogenization of Hamilton-Jacobi equations and degenerate parabolic or elliptic equations in a class of random media. This type of equations is highly related to flow in heterogeneous media in which the heterogeneity is larger than the pore scales. To date, there are very limited computational strategies for this type of problems. Tsai is now interacting with Souganidis and his student on these problems.

Interactions:

A. Discussions and exchanges of ideas:

Runborg, Iwen, and Gilbert discussed a joint project on developing spectral methods for PDEs using the new fast Fourier Transform algorithm.

Sun and Tsai discussed the multiscale modeling and computation for a class of neural networks that involve stiff Dirac-delta forcings.

Engquist, Runborg, and Tsai discussed new numerical approach for high frequency wave propagation using Gaussian beams.

Vanden-Eijnden, Ariel, and Tsai discussed the possibility of improving the convergence of time averaging.

Several discussions on the range of validity of the Cauchy-Born rule following Luskins talk took place.

LeBris and Sharp discussed the issue of conservation of total energy in his algorithm. There were discussions on the possibility of adding suitable macroscopic variables could improve the conservations.

LeBris and Szepessy started thinking on symplectic methods for optimal control point of view.

Discussions about general approaches in finding coarse variables took place after Bolds talk. Comments about the convergence of the presented graph problem were exchanged.

Ariel, Szepessy, and Vanden Eijnden discussed the issues of strong versus weak convergence of the Zwanzig model to the Langevin equation.

Some discussions on the spectrum of the KdV model presented in Iwens talk took place. The issue of whether using only a few salient Fourier modes in this type of problems is sufficient was raised. More generally, for what type of equation/operator does this approach make sense?

B. Mutual visits:

Tsai visited Luskin at Minnesota after the workshop.

Vanden-Eijnden invited Tsai to a workshop in the summer at St. Malo.

Gilbert and Runborg met in Cambridge and further developed their collaboration.

List of Participants

Ariel, Gil (University of Texas)
Bold, Katy (Princeton University)
Engquist, Bjorn (University of Texas at Austin)
Gamba, Irene M. (University of Texas, Austin)
Gilbert, Anna (University of Michigan)
Iwen, Mark (University of Michigan)
LeBris, Claude (ENPC)
Luskin, Mitchell (University of Minnesota)
Ren, Weiqing (New York University)
Runborg, Olof (Royal Institute of Technology, Sweden)
Ruuth, Steve (Simon Fraser University)
Sharp, Richard (University of Texas at Austin)
Souganidis, Panagiotis (University of Chicago)

Sun, Yi (New York University)
Szepessy, Anders (Royal Institute of Technology KTH)
Tharkabhushanam, Sri Harsha (University of Texas)
Tsai, Richard (University of Texas at Austin)
Vanden-Eijnden, Eric (Courant Institute)

Statistics:

Tenured professors: 9
Assistant professors: 3
Post-docs:3
Graduate students: 3
Men: 15
Women:3

Chapter 5

Explicit methods for rational points on curves (07w5063)

February 4-9, 2007

Organizer(s): Nils Bruin (Simon Fraser University), Bjorn Poonen (University of California, Berkeley)

Introduction to the field

One of the “big problems” of number theory is to understand the set of rational points on a variety, or equivalently, the rational solutions to a system of polynomial equations. Despite thousands of years of research, we are very far from having a general method for solving all such problems. There is even some evidence that deciding the existence of a rational solution is an undecidable problem (the corresponding problem for integers, “Hilbert’s Tenth Problem”, was proved to be undecidable by Martin Davis, Hilary Putnam, Julia Robinson [6]. and Yuri Matijasevič [12]. Therefore many researchers have tried to solve special cases of the general problem.

One way to subdivide the task is to classify varieties by their dimension, which can be defined as the dimension of the complex analytic space whose underlying set is the set of complex number solutions to the system. This space will be a complex manifold if the equations satisfy the differential criterion for smoothness. The study of this complex analytic space is useful for more than just classification: it was discovered in the 20th century that the geometry of this space has a profound influence on the set of rational points.

The rational points on 0-dimensional varieties are easy to understand. By suitable projection, one reduces to the problem of understanding the rational roots of a polynomial in one variable with rational coefficients, and there are elementary methods for understanding these.

Rational points on curves (1-dimensional varieties X) are already much harder: there is still no general algorithm for determining the set of rational points that has been proved to determine the rational points in every case. One can reduce to the case where the curve is smooth, projective, and geometrically integral, or equivalently, where the corresponding complex analytic space is a compact Riemann surface; from now on we will assume this. Then one can subdivide the problem further, according to the topological genus g of the compact Riemann surface. This nonnegative integer g can also be defined algebraically as the dimension of the space of regular differentials, or the dimension of the sheaf cohomology space $H^1(X, \mathcal{O}_X)$.

Major number-theoretic breakthroughs of the 20th century have given us a qualitative understanding of the set $X(\mathbb{Q})$ of rational points on a (smooth, projective, geometrically integral) curve as above. Many of these results generalize to the case where the field $\mathbb{Q}$ of rational numbers is replaced by a finite extension, or even some other types of fields, but for simplicity we will discuss the case of $\mathbb{Q}$.

In the case $g = 0$, the problem of deciding whether $X(\mathbb{Q})$ is nonempty is equivalent to the problem of deciding whether a quadratic form in three variables represents 0, and a criterion for this in terms of congruences goes back to work of Legendre. Moreover, if a rational point exists, then X is isomorphic to the projective line $\mathbb{P}^1$ over $\mathbb{Q}$, and hence the rational solutions may be parametrized. For instance, the special case of (the projective closure of) the curve $x^2 + y^2 = 1$ yields the familiar parametrization of Pythagorean triples.

In the case $g = 1$, it is still not known how to decide whether $X(\mathbb{Q})$ is nonempty. Suppose that $X(\mathbb{Q})$ is nonempty. Then the choice of a point in $X(\mathbb{Q})$ leads to a group structure on the variety X, so $X(\mathbb{Q})$ acquires the structure of an abelian group. The famous Mordell-Weil theorem (due to Mordell in the special case we are considering), proved in the 1920s, states that this group $X(\mathbb{Q})$ is finitely generated. The proof combines a generalization of Fermat's method of infinite descent with a study of the sizes of numerators and denominators of the coordinates of rational points. The Mordell-Weil theorem remains a qualitative result, however, in the sense that there is no algorithm that has been proved to construct generators for this group, or even to calculate the rank of this group. More precisely, researchers have developed algorithms to solve these problems, but these algorithms terminate in general only if for every elliptic curve E over $\mathbb{Q}$, there exists a prime p such that the p-primary part of a torsion abelian group called the Shafarevich-Tate group is finite, as has been conjectured.

In the case $g \geq 2$, Gerd Faltings [4] proved Mordell's 1922 conjecture that $X(\mathbb{Q})$ is finite, and a few years later Paul Vojta [83] gave a completely different proof. But these proofs are ineffective, even in principle: given an explicit curve, the proofs do not give a procedure for listing the rational points: they only (with extra work) give an upper bound for the number of rational points, depending on the input curve. These upper bounds typically appear to be ridiculously large.

There are other techniques that were developed to solve these problems:

1. In some cases, one can determine the rational points on a curve X by finding a non-constant morphism from it to an abelian variety A whose group of rational points is finite; here one often uses the Jacobian J of X, since J is the universal abelian variety through which all morphisms from X to abelian varieties map. If one succeeds in finding such a morphism $X \to A$, one can hope to determine all rational points on A and then examine their preimages in X.

2. If one finds a morphism from X to an abelian variety A such that $A(\mathbb{Q})$ is infinite, but satisfies $\operatorname{rank} A(\mathbb{Q}) < \dim A$, then there is a p-adic analytic method due to Chabauty [12] that provides an upper bound on the number of rational points on X. Moreover, this upper bound is usually reasonable, and often is even sharp, in which case it can be used to determine the set $X(\mathbb{Q})$. The method operates by first computing $A(\mathbb{Q})$ (in fact, one can usually get by with knowledge of a subgroup of finite index therein), and then looking at the intersection of the image of the 1-dimensional p-adic manifold $X(\mathbb{Q}_p)$, with the p-adic closure of $A(\mathbb{Q})$ in $A(\mathbb{Q}_p)$: the latter closure can be shown to be a p-adic analytic submanifold of dimension at most $\operatorname{rank} A(\mathbb{Q})$, so dimension counting suggests that the intersection above is 0-dimensional; Chabauty proved that it was finite, and Robert Coleman [12] showed how to obtain a very explicit upper bound on $\#X(\mathbb{Q})$ via this method.

3. If for every abelian variety quotient A of the Jacobian of X, the inequality $\operatorname{rank} A(\mathbb{Q}) < \dim A$ is violated, then one can try instead ideas originating in work of Chevalley-Weil [4], again generalizing Fermat's infinite descent. One can replace the problem of finding rational points on a given curve X of genus at least 2 with the problem of determining the rational points for a finite set of unramified covers of the given curve. This is often helpful, and it may be that in principle combining this method with Chabauty's method always succeeds in determining the rational points (see [12] and [13], for instance), but in practice, the fact that the covering curves have higher genus than X often makes the computation too time-consuming to carry out to completion. Also, it seems very difficult to prove that this combination of methods would always succeed in principle.

Recent developments and open problems

In [8], Minhyong Kim introduced a new idea for studying rational and integral points on curves. Loosely speaking, the Jacobian of a curve classifies geometrically abelian covers of the curve, and Chabauty's method can be understood as applying descent to pass to the tower of geometrically abelian unramified covers of p-power degree. Kim's idea was instead to use the tower of covers coming from the pro-p nilpotent quotient of the algebraic fundamental group of the curve. In direct analogy with Chabauty's method, he defines a "unipotent Albanese map" from $X(\mathbb{Q}_p)$ not to $J(\mathbb{Q}_p)$ but to the p-adic points of a pro-unipotent algebraic group $\pi_{1,DR}(X,x)$. Using this, he gave a new proof of Siegel's theorem on the finiteness of the set of solutions to S-unit equations, for S a finite set of places of $\mathbb{Q}$. Although this particular result can also be obtained by the more elementary approach of applying Chabauty's method to unramified covers of $\mathbb{P}^1 - \{0,1,\infty\}$, it seemed possible that Kim's method might be applicable to other situations for which it is not clear that applying Chabauty's method to unramified covers would work. As evidence for this, Kim showed that various conjectures (about Galois cohomology or Galois representations) would imply that his technique would prove the finiteness of the set of integral points on any hyperbolic curve over $\mathbb{Q}$, and in particular the finiteness of the set of rational points on any smooth projective curve of genus at least 2 over $\mathbb{Q}$.

This raises several questions:

1. Is it actually the case that Kim's approach is equivalent to Chabauty's approach applied to unramified covers? If not, is one approach stronger than the other?

2. Does Kim's approach lead to a new proof of Faltings' theorem in general?

3. Does Kim's approach suggest an algorithm, along the lines of the algorithm that implements Chabauty's ideas?

One of the main goals of the workshop was to bring people together to try to gain insight on these difficult questions.

Workshop presentations

Expository talks

Tim Dokchitser – *Analytic ranks of Jacobians of curves*

This talk concentrated on a conjectural but, if ever proven, very powerful way of computing the free rank of abelian varieties over the rational numbers.

One associates to any Abelian variety over the rationals an analytic object, its L*-series*. This is an analytic function in, say, s, defined by a convergent series for $\mathrm{Re}(s) > 3/2$. According to a conjecture by Birch and Swinnerton-Dyer, this function extends to a meromorphic function on the entire complex plane, and the order of vanishing at $s = 1$ should correspond to the free rank of the group of rational points on the abelian variety. Furthermore, the lowest order derivative that does not vanish should take a value at $s = 1$ which is a combination of virtually all interesting arithmetic geometric quantities associated to the abelian variety. In particular, the conjectural order of the Shafarevich-Tate group can be read off from that value.

In practice, even getting a complete description of the L-series can be troublesome, because some of the relevant arithmetic information that makes up the L series is hard to compute. Using even further conjectures, one can often make an educated guess about this information. In this talk, the speaker showed how to apply these ideas in practice . He demonstrated how his newly developed software in the computer algebra system MAGMA can be used and showed some impressive examples. One of the highlights was a genus 3 curve, for which he conjectured that the Jacobian should be of rank 5.

An interesting question raised by his talk is whether algebraic methods (e.g., 2-descent on the Jacobian of this genus 3 curve) can obtain this result. Some of the participants thought about this for a while, and could not see an easy way to do it. So at least for the time being, it seems as the analytic approach and the algebraic approach complement each other, each able to contribute information that might be inaccessible via the other approach.

William McCallum – *Introduction to explicit Chabauty methods*

Given that one of the main themes of this workshop was *non-Abelian Chabauty* - a generalisation of Chabauty's original method to obtain a partial proof of Mordell's conjecture, the organizers invited one of the experts on the method to give a lecture series on the introduction into the original idea.

Let C be a complete, irreducible, nonsingular algebraic curve over the field of rational numbers, of genus at least 2. Suppose we have a degree 1 divisor class on C. We can use that to consider C as a subvariety of the Jacobian J of C. Hence, the rational points of C can be considered a subset of the rational points of J.

We consider $J(\mathbb{Q}) \subset J(\mathbb{Q}_p)$. for some prime p. A nice property of p-adic analytic commutative lie-groups is that a finitely generated subgroup of rank r is contained in an analytic submanifold of dimension at most r. Hence, if $J(\mathbb{Q})$ is of rank strictly lower than the dimension of J, then it is contained in a proper submanifold $\overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)$.

We can then find a bound on $\#C(\mathbb{Q})$ via the inclusion $C(\mathbb{Q}) \subset C(\mathbb{Q}_p) \cap J(\mathbb{Q})$. The latter is an intersection of a 1-dimensional $\mathbb{Q}_p$-analytic algebraic variety with a proper p-adic analytic submanifold of the ambient space. One would expect a 0-dimensional (in fact finite) intersection and one can show that this is indeed the case. The cardinality of this analytic intersection provides an upper bound on $C(\mathbb{Q})$.

In this talk, it is explained how all analytic computations can be formulated in terms of p-adic integration on the curve and several well-known examples from the literature are explained and demonstrated.

Several modifications of the method, in particular the use of covers and replacing J (if possible) with a computationally more accessible Weil-restriction of an elliptic curve are also mentioned.

Edward Schaefer – *Bounding the Mordell-Weil rank of the Jacobian of a curve*

A crucial ingredient for the application of Chabauty's method to a curve C over $\mathbb{Q}$ with Jacobian J, is a detailed knowledge of $J(\mathbb{Q})$, the Mordell-Weil group. In particular, one needs to know the free rank of this group.

One can read off this rank from a quotient $J(\mathbb{Q})/pJ(\mathbb{Q})$. The method of *descent* tries to approximate this group by a group that is quaranteed to contain the given group, the p-Selmer group of J. The cardinality of the latter thus provides an upper bound on the cardinality of the former, and thus implies a bound on the Mordell-Weil rank of J.

The talk explains in detail how the general Galois-cohomological framework one can use to describe the required objects can be translated into explicitly computable objects.

As a particular example, a famous historical computation is repeated, thus providing the required information to complete the argument given in McCallum's talk.

Michael Stoll – *Local-global obstructions, coverings, and Mordell-Weil sieving*

If Chabauty's method applies, i.e., if $J(\mathbb{Q})$ is of smaller rank than the dimension of J, then it provides a proof that $C(\mathbb{Q}) \to J(\mathbb{Q})/NJ(\mathbb{Q})$ is injective for some explicit N. It then remains to determine which classes in $J(\mathbb{Q})/NJ(\mathbb{Q})$ do contain a rational point.

As it turns out, assuming we can consider C as a subvariety of J, considering the intersection of the image of $J(\mathbb{Q})$ in $\prod_{p\in S} J(\mathbb{F}_p)$ with $\prod_{p\in S} C(\mathbb{F}_p)$ for some suitably chosen set of primes S provides quite strong congruence information on the image of $C(\mathbb{Q})$ in $J(\mathbb{Q})$. In fact, heuristically (see [13]), one expects that once should be able to get accurate information for any N. The procedure of obtaining such information is now known as *Mordell-Weil sieving*.

In particular, the same heuristics predict that using this procedure for a curve C that does not have rational points, one should be able to show this using a suitably chosen set S.

This talk explains this procedure and its link with the idea of using coverings as in [4] and the Brauer-Manin obstruction in general.

Talks on non-abelian Chabauty

Richard Hain – *Higher Albanese Manifolds*

This talk explains the construction of *higher albanese manifolds* in the complex analytic situation. The same construction is used in a p-adic setting for Kim's non-abelian Chabauty.

The usual complex analytic Albanese variety of a curve C may be defined by integrating holomorphic 1-forms along paths on the complex Riemann surface corresponding to the curve. These provide functions on the path space of C that are defined on classes of paths modulo the commutator of the fundamental group, and equip this quotient with a manifold structure.

For higher albanese varieties, one replaces the integrals by *iterated* integrals as studied by Chen. These are defined only on path classes modulo terms from the lower central series of the fundamental group. This talk gave an overview of this construction, including Chen's theorem relating the pro-unipotent completion of the fundamental group with the Hopf algebra of homotopy functionals.

Kiran Kedlaya – *p-adic Hodge Theory*

This talk provided an introduction into p-adic Hodge theory. Given a smooth proper scheme X over $\mathbb{Z}_p$, one has the p-adic étale cohomology of its base extension to $\overline{\mathbb{Q}}_p$ and its de Rham cohomology with $\mathbb{Q}_p$-coefficients. Comparison theorems proved by Faltings and Tsuji describe how one can recover either of these cohomology spaces from the other by using Fontaine's "big ring" B_{crys}. They allow one to compare the p-adic integration map from a curve to the Lie algebra of its Jacobian over $\mathbb{Q}_p$ with a cohomologically defined map. The results described so far belong to abelian p-adic Hodge theory. The talk ended with a brief sketch of some non-abelian generalizations.

Minhyong Kim – Non-abelian Chabauty

Chabauty's original partial proof of Mordell's Conjecture (now Faltings' Theorem) on finiteness of the number of rational points on algebraic curves of genus at least 2 is based on considering the curve as a subvariety of an abelian variety. Since the Albanese variety is universal with respect to that property, no generality is lost by considering the curve as a subvariety of the Albanese. Chabauty's argument is based on the assumption that the rational points of the Albanese lie in a proper p-adic submanifold. This assumption does not hold in general.

One can try to use larger group varieties — non-abelian ones. The higher Albanese varieties are some of the next simplest examples, being unipotent. The hope is that even in the case where the rational points in the classical Albanese variety lie p-adically dense, we can find a higher Albanese where the rational points do lie in a proper submanifold.

For non-complete hyperbolic curves — in particular $\mathbb{P}^1$ over $\mathbb{Z}$ minus 3 points — Kim was able to prove that this is indeed the case. He was thus able to recover Siegel's result on finiteness of the number of solutions to S-unit equations [8]. He has also shown that various "motivic conjectures," such as the Bloch-Kato conjecture on surjectivity of p-adic Chern class maps or the Fontaine-Mazur conjecture on representations of geometric origin, would imply that his method reproves the theorems of Faltings and Siegel for hyperbolic curves over $\mathbb{Q}$ [9].

In his series of talks, Kim explained the construction he used, with the express purpose of looking whether this method can be made explicit and perhaps be used to produce actual bounds on the number of solutions. The progress made during the workshop shows that this may indeed be the case.

After giving some motivating examples, he explained the map (arising in Grothendieck's section conjecture) from the set of rational points on a variety to the Galois cohomology H^1 of its fundamental group over $\overline{\mathbb{Q}}$, and then he discussed the version of this for the pro-unipotent completion of the fundamental group, and finally connected this with the de Rham picture in which p-adic iterated integrals play a key role.

Research talks

Abstracts of all but one of the research talks are given in Appendix B below. We provide a summary of the talk for which no abstract was provided.

Iftikhar Burhanuddin – *Brauer-Siegel Analogue for Elliptic Curves over the Rationals*

See abstract.

Jordan Ellenberg – *Obstructions to rational points on curves coming from the nilpotent geometric fundamental group*

In this talk another interesting obstruction arising from the nilpotent fundamental group is discussed. It is torsion, so this obstruction is invisible for the uniponent Albanese construction used by Kim, who tensors with $\mathbb{Q}_p$. As a special example, for $\mathbb{P}^1$ minus three points, the ordinary quadratic Hilbert symbol was recovered.

The talk was a report on ongoing research. No explicit new results could be reported on yet.

Florian Hess – *Explicit generating sets of Jacobians of curves over finite fields, using some class field theory*

See abstract.

Catherine O'Neil – *Trilinear forms and elliptic curves*
See abstract.

Samir Siksek – *Chabauty for Symmetric Powers of Curves*
See abstract.

William Stein – *Explicitly computing information about Shafarevich-Tate groups of elliptic curves using L-functions, Euler systems, and Iwasawa theory*
See abstract.

Michael Stoll – *Rational points on small curves of genus 2 - an experiment*
See abstract and [2].

Ronald van Luijk – *Cubic points on cubic curves and the Brauer-Manin obstruction for K3 surfaces*
See abstract.

The impact of the workshop

Here we describe some new collaborations, projects, and success stories that came into being because of our workshop. Some of these took place during the workshop itself; in other cases, participants told us a few weeks later about progress that had ensued.

- Jordan Ellenberg, Richard Hain, Minhyong Kim, and Kirsten Wickelgren (a graduate student) began a new collaboration in order to compute the "Selmer varieties", which are the analogue of the p-adic closure of the Mordell-Weil group, in Chabauty's method. The computation of these varieties seems to be the most difficult part of Kim's approach, the main obstacle to making the approach a viable method. Yes, on Thursday night of the workshop, they made significant progress! This group of researchers also hopes to study a characteristic-zero function field analogue.

- Kirsten Wickelgren writes also that during the workshop Minhyong Kim made an observation about a map that she now uses to compute examples for her Ph. D. thesis.

- Richard Hain, a topologist who is an expert in the theory of iterated integrals and pro-unipotent fundamental groups in the classical case (as opposed to the p-adic case used in Kim's work) wrote that during the course of the workshop he went from having little understanding of Kim's program to having a good grasp of it. As a result, he and Kim are going to write a updated exposition of some of the key topological ideas, but with an eye towards applications to Kim's program. In particular, they will treat iterated integrals on algebraic curves, iterated Coleman p-adic integration, computing the Hodge filtration via the pole integration. Some of these topics have not been fully developed even in the research literature, so their new exposition will be very welcome.

- Iftikhar Burhanuddin (a graduate student) wrote that he received valuable feedback in response to his workshop talk on an elliptic curve analogue of the Brauer-Siegel theorem, and that this feedback is guiding the computational data that he will collect for his Ph. D. thesis.

- At the workshop, a small subset of the participants met to discuss the issue of implementing the computation of p-adic iterated integrals, a necessary step in making Kim's approach practical. Kiran Kedlaya, who was scheduled to deliver a lecture series for graduate students and lead them in a project at the Arizona Winter School this year, wrote that the workshop gave him the idea of involving the students in a project along these lines. At the workshop, we were discussing only the case of good reduction, but Kedlaya has been led to begin investigating the more general case of semistable reduction as well.

- Nils Bruin, Bjorn Poonen, and Michael Stoll stayed one extra day at BIRS; during that day, they worked together on developing explicit 2-descent for general curves of genus 3. Significant progress was made, showing that the computations required could be reduced to the point of almost being doable with current computing power and class group algorithms (modulo the Generalized Riemann Hypothesis). In fact, a few weeks later, we had our first success along these lines, albeit for a curve with very small discriminant.

In addition, many participants, both those studying computational number theory and those involved in more theoretical aspects, wrote to us expressing their thanks for the opportunity to learn about the new ideas that were in the process of being developed. Despite some of the subject matter being highly technical, having experts present who could explain things in a friendly learning environment led to many people leaving with a good sense of the issues involved.

For more information

A more extensive account of the presentations given in this workshop can be found on the website

`http://www.cecm.sfu.ca/~nbruin/banff2007`

For nearly all talks, either copies of the slides used or extensive notes taken by Bjorn Poonen are available. Links to preprints and further reading are also accessible.

Appendix B: Schedule of the workshop

Monday, February 5, 2007

9:00 – 9:10 Introduction to BIRS

9:10 – 10:00 William McCallum – *Introduction to explicit Chabauty methods I*

10:30 – 11:20 William McCallum – *Introduction to explicit Chabauty methods II*

2:30 – 3:20 Richard Hain – *Higher Albanese Manifolds*

4:00 – 4:50 Kiran Kedlaya – *p-adic Hodge Theory*

At the request of the organizers, I will introduce/review some constructions from p-adic Hodge theory that intervene in the usual Chabauty method; these include the comparison isomorphism between the de Rham and etale cohomology groups of a curve, and the Bloch-Kato exponential map. I will focus on the case of good ordinary reduction, where these constructions can be made reasonably explicit. The goal is to analogize the explicit descriptions to the higher unipotent de Rham and etale fundamental groups, in a manner useful for doing nonabelian Chabauty; as time and my abilities permit, I will start doing this (again only in the good reduction case) using some work of Martin Olsson.

Tuesday, February 6, 2007

9:10 – 10:00 Minhyong Kim, I

10:30 – 11:20 Minhyong Kim, II

2:30 – 3:20 Edward Schaefer – *Bounding the Mordell-Weil rank of the Jacobian of a curve*

We use a Chabauty computation to determine the set of rational points on a curve of higher genus. The input for a Chabauty computation includes the Mordell-Weil rank of the associated Jacobian. Traditionally we bound, and hope to determine, the Mordell-Weil rank using a Selmer group. In this talk, we will survey the methods for computing a Selmer group of a Jacobian using functions on the curve. We will review both major methods. The first is quite general, but is inefficient for cyclic covers of the projective line (like hyperelliptic curves). The second method addresses such covers.

4:00 – 4:50 Michael Stoll – *Local-global obstructions, coverings, and Mordell-Weil sieving*

We will discuss how one can obtain information on rational points by combining coverings with local information. We will focus on the case of abelian coverings and explain the relationship with the Brauer-Manin obstruction. If explicit generators of the Mordell-Weil group are known, this can be implemented efficiently, leading to a procedure known as the Mordell-Weil sieve. We will formulate a conjecture that, if valid for a given curve, implies that we can effectively decide whether a given coset of N times the Mordell-Weil group meets the image of the curve or not. If we know that each such coset contains at most one point coming from the curve, this means that we can determine the set of rational points on the curve.

5:00 – 5:50 Jordan Ellenberg – *Obstructions to rational points on curves coming from the nilpotent geometric fundamental group*

Wednesday, February 7, 2007

8:40 – 9:30 Minhyong Kim, III

9:40 – 10:30 Samir Siksek – *Chabauty for Symmetric Powers of Curves*

Let C be a curve of genus $g \geq 3$ and let $C^{(d)}$ denote its d-th symmetric power. We explain an adaptation of Chabauty which allows us in many cases to compute $C^{(d)}(\mathbb{Q})$ provided the rank of the Mordell-Weil group is at most $g - d$. Cases for which our method should work include:

(i) $d < \gamma$ where γ is the gonality of C and the jacobian is simple (here $C^{(d)}(\mathbb{Q})$ is finite).

(ii) C is hyperelliptic and $d = 2$ (here $C^{(d)}(\mathbb{Q})$ is infinite).

(iii) C is bielliptic and $d = 2$ (here $C^{(d)}(\mathbb{Q})$ can be infinite). Our adaptation of Chabauty differs from the classical Chabauty in that we combine Chabauty type information given by several primes.

Example. Let C be the genus 3 hyperelliptic curve

$$C : y^2 = x(x^2 + 2)(x^2 + 43)(x^2 + 8x - 6) \tag{5.1}$$

with Jacobian having rank 1. Let $\pi : C \to P^1$ be the x-coordinate map. We show that $C^{(2)}(\mathbb{Q})$ consists of $\pi^{-1}P^1(\mathbb{Q})$ plus 10 other points which we write down explicitly. Here we needed to combine the Chabauty information at primes $p = 5, 7, 13$. It is noteworthy that $C^{(2)}$ in this example is a surface of general type.

Thursday, February 8, 2007

9:10 – 10:00 Tim Dokchitser – *Analytic ranks of Jacobians of curves*

10:30 – 11:20 Ronald van Luijk – *Cubic points on cubic curves and the Brauer-Manin obstruction for K3 surfaces*

It is well-known that not all varieties over $\mathbb{Q}$ satisfy the Hasse principle. The famous Selmer curve given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$, for instance, indeed has points over every completion of $\mathbb{Q}$, but no points over $\mathbb{Q}$ itself. Though it is trivial to find points over some cubic field, it is a priori not obvious whether there are points over a cubic field that is galois. We will see that such points do exist. K3 surfaces do not satisfy the Hasse principle either, which in some cases can be explained by the so called Brauer-Manin obstruction. It is not known whether this obstruction is the only obstruction to the existence of rational points on K3 surfaces. We relate the two problems by sketching a proof of the following fact. If there exists a smooth curve over $\mathbb{Q}$ given by $ax^3 + by^3 + cz^3 = 0$ that is locally solvable everywhere, that has no points over any cubic galois extension of $\mathbb{Q}$, and whose Jacobian has trivial Mordell-Weil group, then the algebraic part of the Brauer-Manin obstruction is not the only one for K3 surfaces. No knowledge about K3 surfaces or Brauer-Manin obstructions will be assumed as known.

2:30 – 3:20 Catherine O'Neil – *Trilinear forms and elliptic curves*

We explain a correspondence between triliear forms and triples of genus one curves with a fixed Jacobian and some added structure. We generalize the addition law on elliptic curves to addition on certain "cubes" of numbers. We explain how this works for arbitrary rings, and we give a natural construction of points on elliptic curves to other points on other elliptic curves which generalizes a known construction from class field theory.

4:00 – 4:50 William Stein – *Explicilty computing information about Shafarevich-Tate groups of elliptic curves using L-functions, Euler Systems, and Iwasawa theory*

I will discuss theoretical and computational results toward the following problem: given a specific elliptic curve over $\mathbb{Q}$, compute the exact order and structure of its Shafarevich-Tate group in practice. I view this problem as a motivating question for organizing both theoretical and algorithmic investigations into the arithmetic of elliptic curves and the Birch and Swinnerton-Dyer conjecture.

5:00 – 5:30 Iftikhar Burhanuddin – *Brauer-Siegel Analogue for Elliptic Curves over the Rationals*

The height of a rational point on an elliptic curve measures the size of the point. The enormous gap between the lower and upper bound (Lang's conjectures) of the height of such a point, prompted the comparison of the elliptic curve scenario with that of the multiplicative group, the Brauer-Siegel theorem. In this talk, a conjectural Brauer-Siegel theorem for elliptic curves over the rationals will be discussed and interesting questions which arise in this context motivated by computation will be presented.

Friday, February 9, 2007

9:10 – 10:00 Florian Hess – *Explicit generating sets of Jacobians of curves over finite fields, using some class field theory*

10:30 – 11:20 Michael Stoll – *Rational points on small curves of genus 2 - an experiment*

We considered all genus 2 curves $y^2 = f(x)$ where f has integral coefficients of absolute value at most 3; there are about 200,000 isomorphism classes of such curves. Using various methods (point search, local solubility, 2-descent, Mordell-Weil sieve), we attempted to decide for each curve whether it possesses rational points. In all but 42 cases, we were successful; in the remaining cases, our result is conditional on the Birch and Swinnerton-Dyer conjecture. In the talk, we will explain the methods we used and the improvements we came up with, and discuss the results.

List of Participants

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Bright, Martin (University of Bristol)
Broker, Reinier (Fields Institute)
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Ellenberg, Jordan (University of Wisconsin)
Hain, Richard (Duke University)
Hess, Florian (Technische UniversitŁt Berlin)

Kedlaya, Kiran (Massachusetts Institute of Technology)
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Logan, Adam (University of Waterloo)
McCallum, William (University of Arizona)
O'Neil, Catherine (Barnard College, Columbia University)
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Siksek, Samir (University of Warwick)
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Stoll, Michael (Bayreuth)
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Watkins, Mark (University of Bristol)
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Chapter 6

Operator Structures in Quantum Information Theory (07w5119)

Feb 11 - Feb 16, 2007

Organizer(s): David Kribs (University of Guelph), Mary Beth Ruskai (Tufts University)

Overview of the Field

This workshop brought together two distinct communities, mathematicians working on operator structures and mathematically oriented scientists in several disciplines working in quantum information theory (QIT). Based on feedback and observations during the workshop, it was extremely successful. There was no perceptible decline in attendance at talks or variation with subtopic.

It seems clear that this workshop will lead to an increased role for operator spaces in quantum information theory, and may be a landmark event. With a few exceptions, their role in QIT has been limited to the implicit use of a few concepts, such as a type of completely bounded norm, or a particular operator algebra, without recognition of the larger mathematical structure. The workshop has unequivocally changed this view. A few days after it ended, several participants posted the paper [64] "Unbounded violation of tripartite Bell inequalities" demonstrating the value of tools from operator spaces and tensor norms. In addition to obtaining a striking new result with important physical implications, the authors reformulate a question about commutative Banach algebras, which has been open for over 30 years, as a question about the types of states which give unbounded violations of Bell inequalities. At the August BIRS workshop 07w5013 Operator Spaces and Group Algebras, Ed Effros began with a talk on "Quantized functional analysis and quantum information theory".

Because of the varied backgrounds of the participants, the first three mornings were devoted to expository talks on operator spaces, quantum error correction and quantum Shannon theory. These were extremely valuable, with considerable demand for copies of the speakers notes and slides. The remaining time was divided between longer invited talks on important aspects of current research and short reports on recent results. We also had two sessions on open problems and time for discussion and relaxation.

Two important new results were announced at the workshop, in the talks by Junge [40] and Klappenecker [45], which are described in Sections 6 and 6, respectively. Several new collaborations were started during the workshop; one of these [47] uncovers a new connection between quantum cryptography and error correction based on complementary channels. A new result [55] was obtained on one of the open problems during the workshop. (See item D of Section 6.)

In July 2007, counter-examples were found [32, 84] to the so-called multiplicativity conjecture for tensor products of quantum channels, which had been open for over 5 years. This has significant implications for another important group of four equivalent "additivity" conjectures involving tensor products; these conjectures have been open for about 15 years and the different forms were shown to be globally equivalent by Shor in 2003 [74] As explicitly acknowledged in the announcements [32, 84], these breakthroughs were the result of the open problem sessions at the workshop and subsequent discussions among some of the participants. For additional information, see Section 5.6 of [70].

Summary of Scientific Developments Reported

Operator spaces

Edward Effros and Vern Paulsen gave a pair of well received introductory lectures on the abstract theory of operator spaces. Effros discussed some of the historical development of the subject, pointed out various subdisciplines of mathematics in which operator spaces arise, and prepared the audience for some of the technical complexities which arise in Pisier's L_p version.

Paulsen focussed on concrete operator spaces. He described a technique he had developed and presented elsewhere, but not written up, to compute completely bounded (CB) norms. In his approach, the CB norm can also be computed in a different way using the commutant. Although operator algebraists have not been enthusiastic about this reformulation, it seems likely that it will be useful in QIT where the commutant describes the algebra associated with the environment. One group [47] has already begun to explore this approach. Paulsen's talk was met with an exceptionally high level of interest from researchers in quantum information, and numerous inquiries from participants motivated Paulsen to write up his notes [62]. This talk was the genesis for the paper [37].

Ruskai (in discussions and her subsequent short talk) pointed out that in the usual models of quantum information, the algebra of observables for the environment is the commutant of those for the quantum information processing system. This suggests a new interpretation of the physical significance of the commutant in more abstract settings, and allows one to define concepts like the complement of a channel in these settings. Moreover, because the interplay between system and environment is important, it is often useful to have an alternative description in terms of the environment's algebra, as Paulsen has done for CB norms.[1]

Both Effros and Paulsen emphasized the important role of the Haagerup tensor product, which was new to most of the audience and has yet to be exploited in quantum information theory. An underlying theme of Effros and Paulsen was the notion that any good mathematical concept can be quantized. von Neumann algebras have long been regarded as quantizations of (single variable) integration theory; operator spaces can be considered as quantizations of Banach space theory which yield a non-commutative version of vector-valued integration. Of course, it is not always clear when a quantized version of a mathematical concept models a physical quantum system, as demonstrated in Neufang's talk.

Marius Junge then spoke on the L_p version of operator spaces. He announced a counter-example (with Q. Xu) [40] to a convexity conjecture of Lieb and Carlen [3], which was open for over ten years. Their conjecture can be interpreted as an attempt at a non-commutative generalization of Minkowski's inequality to three spaces on which one has a Schatten L_p norm with $1 < p < 2$. The counter-example establishes that in non-commutative situations which require mixed L_p spaces, the naive formula $\left(\mathrm{Tr}_1(\mathrm{Tr}_2 A_{12}^p)^{q/p}\right)^{1/q}$ does not yield a norm. Thus, one is inevitably led to operator spaces, in particular, the non-commutative vector-valued L_p spaces for which Pisier defined norms using complex interpolation [39, 65, 66, 67].

Junge also described the CB-entropy, a form of conditional information which arises by differentiating a suitable CB-norm at $p = 1$ and can be used to define the minimal CB entropy of a quantum channel. The minimal CB entropy of the identity channel on M_d is $-\log d$ which is consistent with the fact that it preserves maximal entanglement. When one interprets the minimal CB entropy as a measure of optimal entanglement preservation, the seemingly anomalous fact that the CB norm of the identity map is $d^{1/p} \neq 1$ becomes quite natural.

Matthias Neufang gave a talk on quantum groups and an associated class of multiplier algebras, based on recent joint work with Junge, Ruan, and Spronk. He noted that these algebras yield natural "quantum" channels for which the minimal CB entropy can be computed explicitly and shown to be positive. Thus, these channels are "classical" in the sense that they do not preserve enough entanglement to yield even one EPR

[1] Earlier Jencova [10] had simplified the proof of additivity of CB entropy by showing that a particular CB norm could be expressed as the norm of the channel lifted to the commutant. It seems worth asking whether this is a special case of Paulsen's result or if both are special cases of a more general relationship.

pair. Nevertheless, they may have useful properties for quantum error correction, as they appear to have an abundance of noiseless subsystems, decoherence-free subspaces, and unitarily correctable codes. This may be less perplexing in view of the fact that some of the most useful quantum codes are derived from known classical codes.

Quantum Shannon Theory

The concept of a completely positive map on an operator algebra predates operator spaces and plays an important role in quantum information theory. When the system and environment are initially disentangled, the effect of noise, which comes from interactions with the environment is described mathematically by a completely positive, trace preserving (CPT) map on the algebra of the system. The term "quantum channel" is often used when a CPT map represents noise. Shannon developed the mathematical theory for dealing with noise in classical communication. Extending this to quantum systems is challenging and much richer because of the many different types of quantum information processing protocols.

Patrick Hayden gave an introductory overview of the current state of the art in the quantum information analogue of Shannon's communication theory. He described Schumacher's typical subspace theorem and the recent progress [4] made in quantum distributed compression of correlated sources (quantum Slepian-Wolf theorem and fully quantum version thereof, aka "mother protocol"), which greatly unify the theory of entanglement distillation and quantum channel capacities. The information-theoretic significance of these results lies in their giving operational significance to quantities like the quantum conditional entropy.

Graeme Smith discussed the notorious problem of the non-additivity of the coherent information (with the implication that the quantum channel capacity is not known except for so-called degradable channels, on which Mary Beth Ruskai gave an overview with open problems). To approach this problem, he introduced a single-letter upper bound [62], whose good properties allowed the derivation of the best upper bounds on the capacity of the depolarising channel to-date.

Ruedi Seiler presented the work of his group in Berlin [8, 9] on the quantum version of Sanov's theorem, which is about hypothesis testing of a set of alternatives against a single null hypothesis. As in the classical case, the optimal rate of discrimination is given by a (minimal) relative entropy; strangely, however, the optimum is not achieved by a universal strategy. Related to this topic, Arleta Szkola and Koenraad Aundeaert explained their complementing work on the quantum Chernoff bound, described in Section 6. i.e. error-symmetric distrimination between two hypotheses in the asymptotic regime. The resolution of this problem relies on a beautiful new trace inequality discovered to evaluate the asymptotics of the optimal test.

Andreas Winter discussed at length an open question originating from joint work with John Smolin and Frank Verstraete, which they call the "asymptotic quantum Birkhoff conjecture". (See Conjecture 13 in [78] and Problem 30 at [83].) It is well-known that bistochastic quantum channels have more extremal points than the unitary conjugations. However, there are indications that in the limit of high tensor powers of a bistochastic channel it becomes asymptotically well approximated by mixtures of unitary conjugations. Winter's talk was followed by a vigorous discussion about alternative conjectures. One issue is whether it is reasonable to expect that unitary conjugations will suffice when the known sets of extreme points include partial isometries.

Quantum error correction

The workshop included a series of talks that focussed on mathematical aspects of error correction in quantum computing. David Kribs started with an introductory lecture on the basic framework for quantum error correction. In the standard noise model, the state is corrupted by a quantum channel represented by a CPT map $\mathcal{E}$. The goal of quantum error correction is to find a subspace $\mathcal{C} \subset \mathcal{H}$ and a recovery operation, $\mathcal{R}$, also a CPT map such that $(\mathcal{R} \circ \mathcal{E})$ acts like the identity on the convex set of density matrices associated with the code subspace $\mathcal{C}$. If P denotes the projection onto $\mathcal{C}$, this can be stated formally as

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \qquad \forall\, \rho \,\in\, P\mathcal{B}(\mathcal{H})P \tag{6.1}$$

When the CPT map is written in the Choi-Kraus operator sum form $\mathcal{E}(\rho) = \sum_a E_a \rho E_a^*$, it is well-known that Eq. (6.1) is equivalent to the condition

$$PE_a^* E_b P = \lambda_{ab} P \tag{6.2}$$

for some complex set of numbers λ_{ab} and $\mathcal{C} = P\mathcal{H}$. The intuition behind this equivalence is best seen by considering the special case in which $E_a = t_a U_a$ with U_a unitary and t_a^2 interpreted as the probability that the error $U_a : \psi \mapsto U_a\psi$ occurs. Then $\lambda_{ab} = x_a \delta_{ab}$ says that the different unitary errors map vectors in $\mathcal{C}$ into orthogonal subspaces. The recovery operation can then be thought of as the two-step process

i) identify the orthogonal subspace, $U_a\mathcal{C}$, and

ii) apply the inverse operation U_a^*.

The sufficiency of the condition Eq. (6.2) can then be regarded as arising from the fact that the operators E_a are only determined up to a unitary transformation which can be used to "diagonalize the error operators" so that $\lambda_{ab} = x_a \delta_{ab}$.

Ruskai initiated a vigorous discussion which clarified some of the limitations of this viewpoint. For example, there are non-unital qubit channels (of which only the amplitude-damping channel has been studied extensively) with exactly two (non-unitary) error operations E_1, E_2 for which no change of basis can make one of them the identity. Thus, one can not associate a probability with particular error operator, and the channel (almost) always changes the state. Nevertheless, any code and recovery process which corrects all unitary one qubit errors, will correct all one qubit errors, including these non-unital ones. In this case, one can correct errors that can not even be detected! This is a consequence of the quantum measurement process which has the effect of either correcting the error or converting it to a detectable one; mathematically, this can be viewed as a consequence of the fact that the set of correctable errors froms a vector space. (The error detection condition is given by Eq. (6.2) with $E_a = I$.)

In most discussions of quantum error code construction, one deals with error models which are even more specialized than the random unitary model mentioned above. One considers only errors which are tensor products of the identity and the three Pauli matrices. An Abelian subgroup of the group generated by such errors is called a stabilizer and the simultaneous eigenspace is called a stabilizer code. Although examples of codes which do not arise from the stabilizer formalism exist [69, 76] most work has concentrated on stabilizer codes.

Kribs also discussed the important new framework of "operator quantum error correction" [48, 49]. This is a unified framework for quantum error correction that includes the standard active framework discussed above, as well as the fundamental passive techniques for error correction – decoherence-free subspaces and noiseless subsystems. Technically speaking, a quantum system A is a *subsystem* of a Hilbert space $\mathcal{H}$ if there is a subsystem B such that $C = A \otimes B$ is a subspace of $\mathcal{H}$. Then A is correctable for the action of $\mathcal{E}$ if there is a CPT map $\mathcal{R}$ such that

$$\forall \rho^A \, \forall \rho^B \, \exists \, \tau^B \, : \, (\mathcal{R} \circ \mathcal{E})(\rho^A \otimes \rho^B) = \rho^A \otimes \tau^B. \tag{6.3}$$

Intuitively, quantum information is encoded into the A subsystem, and the recovery operation $\mathcal{R}$ is not concerned with noise acting on the ancilla subsystem B. For this reason, these codes have been popularly called "subsystem codes". Simply put, the standard active case corresponds to $\dim A = 1$ with general $\mathcal{E}$, $\mathcal{R}$, and decoherence-free subspaces (respectively noiseless subsystems) are captured when no recovery is required ($\mathcal{R}$ = identity map) and $\dim A = 1$ (respectively $\dim A > 1$). Other cases correspond to situations not recognized in any formal way previously within quantum error correction.

Andreas Klappenecker gave an expository talk on the state of the art for subsystem code constructions. Because the Pauli matrcies form a basis for M_2, any code which can correct all one-qubit Pauli errors can correct arbitrary one qubit errors, including some very strange ones, as described above. In view of this, Knill defined a general notion of "nice error bases" in higher dimensions. Klappenecker reviewed his work with Rötteler [43] in which they found new codes based on these nice error bases, only to find that most of them could also be realized as stabilizer codes. Then, he described the light at the end of the tunnel in which the group representation theory which had led to so much frustration finally found a natural place under the umbrella of subsystem codes [3, 44, 45]. He concluded with the announcement of two new results. First he gave a negative answer to Poulin's question [68] of whether a subsystem code might require fewer syndrome measurements than an optimal stabilizer code. Then he described the construction of a subsystem code which can beat classical Hamming bound [45].

The expository error correction talks concluded with a discussion by Andrew Cross on the basics of fault tolerant quantum computing. Cross also presented joint work with Aliferis [1] showing how the use of a particular subsystem code, the so-called "Bacon-Shor codes" appears to significantly improve threshold estimates by reducing the number of measurements required in recovery operations. The implications of subsystem codes for quantum computing are still very much being explored. This was particularly evident at a Perimeter Institute workshop on fault tolerant quantum computing held in June, 2007, a few months after the BIRS workshop, in which subsystem codes were a predominant theme in discussions.

Fault tolerant computation seems to require codes which, at least in principle, permit perfect recovery. However, in other types of quantum information processing, as in classical information theory, there is a role for codes which lead to optimal recovery in certain situations [23, 89]. Bernhard Bodmann described work with Kribs and Paulsen [10] which finds optimal encoding schemes when one subsystem is essentially noiseless and the other is subject to a phase-damping channel.

Cedric Beny spoke about his recent work with Kempf and Kribs [7] that further generalizes the basic quantum error correction framework to operator algebras by considering the Heisenberg picture, and how the new approach provides a formalism for the correction of hybrid classical and quantum information [50].

Holbrook and Zyczkowski described joint work [15, 16, 17, 18] with Choi and Kribs motivated by the fundamental equation Eq. (6.2). For a fixed operator T they seek solutions of the matrix equation $PTP = xP$ with P a projection and $x \in \mathbf{C}$. When P is rank one, the set of x is precisely the numerical range of U and is the convex hull of the eigenvalues of T. Therefore, they call the set of x for which P has rank k the "rank-k numerical range of T" and conjecture that for normal T this is equal to the intersection of the convex hulls obtained from all possible choices of $N - k + 1$ eigenvalues. They can verify the conjecture when T is self-adjoint and reduce the general normal case to T unitary, for which significant partial results were reported. (See the open problems section for an update.) The long range goal of this work would be to construct new codes from the compatibility of the allowable projectors P_a for different T_a. Ruskai pointed out some interesting open problems described in [38] and [75] on which this approach to code construction could be tested.

Lattice spin models

The mathematical study of quantum spin models has had close interactions with developments in the theory of operator algebras for many decades. Many results of physical interest have been rigorously derived in the operator algebra framework. It is an instance of an almost perfect match between mathematics and physics where the pursuit of rigor does not require a sacrifice of physical interest. More recently, a new dimension has been added to this fruitful interaction. The exciting developments in quantum information theory, especially the study of entanglement, has provided new insight in the structure of physical states of quantum spin models, particularly their ground states.

Frank Verstraete gave a comprehensive review of new techniques he and others [71, 72, 80, 81] have created for the computational study of quantum spin models based on new ideas from quantum information theory. These developments are especially timely because of the impressive progress by experimental physicists in their ability to create strongly correlated and entangled states exhibiting a wide variety of quantum phase transitions, Bose-Einstein condensation and other exotic states of matter. Matrix Product States (MPS), also known as Finitely Correlated States (FCS) and their higher-dimensional generalizations called Projected Entangled Pair States (PEPS) were particularly highlighted. These are special states that have good computational properties and which form the basis for a host of efficient and very successful algorithms collectively known as Density Matrix Renormalization Group (DMRG) methods.

Bruno Nachtergaele gave an overview of Lieb-Robinson bounds and applications to quantum information theory. The original work by Lieb and Robinson [53], dating back to the early seventies, provided a proof that the Heisenberg dynamics of a translation invariant quantum spin system on a lattice has a bounded group velocity. The key step is a commutator estimate. Recently, Nachtergaele and Sims [58] found an improved commutator estimate which enable one to extend Lieb and Robinson's result to a rather general class of systems defined on a metric graph, which covers almost any conceivable architecture for a quantum information processing device. One example of a recent application that was presented is a lower bound for the time required to establish significant correlations between two regions in space using any local dynamics. The lower bound is linear in the distance between the regions [12, 21, 60]. Another application is a Lieb-Schultz-Mattis Theorem in arbitrary dimensions [28, 59], which shows that under certain conditions low-energy excitations will occur even in the absence of continuous symmetry breaking.

When a spin chain is in a pure state, the entanglement entropy S_n of a block of length n in a pure state is the entropy of the reduced density matrix of this block. It is called the entanglement entropy because it corresponds to the standard measure of the entanglement between this block and the rest of the chain. In many situations, as the length of the chain becomes infinite, $\frac{1}{n}S_n \to 0$ which implies that S_n grows sublinearly with n. Numerical experiments by Vidal,et al [82] suggested logarithmic growth $S_n \sim \log n$ at a critical point. This was subsequently proved rigorously by Jin and Korepin [35] for the ground state of the XX model. Although Korepin was originally scheduled to talk about this work, personal circumstances forced him to cancel. Therefore, Milan Mosonyi included an overview of entanglement entropy for spin chains in his talk. Mosonyi, then described related work with Fannes and Haegeman [22] in which they showed that one can construct quasi-free states for which $S_n \sim n^\alpha$ for any $0 < \alpha < 1$.

For lattices, it has been conjectured that away from a critical point the entanglement entropy obeys scales like the ”area” of the boundary. This implies a uniform bound away from the critical point in the case of a spin chain. Subsequently, it was realized that the circumstances under which an area law does or does not hold depends on additional properties [11] of the lattice model. This topic was touched upon briefly in both Mosonyi's and Verstraete's lecture. Since the workshop, there has been considerable progress [29, 87] on this topic, exploiting the improved Lieb-Robinson bounds described in Nachtergaele's lecture.

In quantum statistical mechanics, the use of quasi-local algebras plays an important role in showing that the infinite volume thermodynamic limit exists for certain types of lattice systems. In generalizing the concept of cellular automata to quantum systems, one challenge is to allow an initially finite lattice system to grow without bound in any direction. R. Werner described his approach [8] to quantum cellular automata based on quasi-local algebras.

Quantum state discrimination

Two speakers, Szkola and Audenaert, gave talks about their recent work on state discrimination. Taken together their results determine the quantum Chernoff bound, thereby settling a problem which has been open for several years. The problem is to find the best measurement for distinguishing two (known) quantum states ρ and σ. In general this cannot be done with full certainty (unless the states are orthogonal), and so the goal is to find the minimum error $\mathrm{MinErr}(\rho,\sigma)$. Assuming that multiple copies of the states are available, this leads to the question of finding the best way to distinguish $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, and hence to find the asymptotic rate $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{MinErr}(\rho^{\otimes n}, \sigma^{\otimes n})$. The corresponding rate in the classical problem was found by Chernoff, and there has been a search for the quantum version in recent years. In a major breakthrough in 2006, Szkola and Nussbaum showed that the rate is lower bounded by the quantity $\log\big(\min_{0\le s\le 1} \mathrm{Tr}\rho^s\sigma^{1-s}\big)$, which is almost a direct translation of the classical Chernoff bound into quantum language. Szkola described this result in her talk and discussed its relation to the notion of the quantum Hellinger arc which interpolates between the states ρ and σ.

At the end of 2006, in another breakthrough, Audenaert and his collaborators proved that this quantity is also an upper bound for the rate, thereby establishing equality. In his talk Audenaert emphasized the properties of the quantum Chernoff bound as a measure of the distance between two states, and showed how it induces a metric on the state space. He explained how the Chernoff bound follows from a new matrix inequality which states that for positive matrices A, B and all $0 \le s \le 1$

$$\tfrac{1}{2}(\mathrm{Tr}(A+B) - \mathrm{Tr}|A-B|) \le \mathrm{Tr}A^s B^{1-s}. \tag{6.4}$$

This inequality, which has other applications, allows one to relate the trace-norm distance of two states to their Renyi relative entropy.

In a related talk on state discrimination, Anna Jencova described her work [5] with Guta on quantum statistics, which is concerned with using results of measurements to infer properties of quantum states and systems. In particular she described results about local asymptotic normality. This property applies to a sequence of two dimensional random variables whose distributions depend on an (unknown) parameter. When localized in the neighborhood of some fixed point of the parameter space, the property implies convergence to a family of Gaussian distributions. Jencova talked about the theory of quantum statistical experiments, and showed how this leads to a quantum version of local asymptotic normality. For both classical and qubit systems, it has been shown that weak and strong convergence are equivalent. Whether or not this holds for general quantum systems is still open. However, Guta and Jencova are able to prove weak convergence for general quantum systems.

Other Topics Covered

Robert Alicki gave a provocative talk based on the assertion that a scalable quantum computer would be a "perpetuum mobile of the second kind". The key assumption is that the system is initially in a metastable state which is almost a KMS state. The talk was punctuated by a vigorous discussion of the validity of these assumptions. In the end, many participants felt what was presented was a proof by "reductio ad absurdum" from dubious assumption and that what was needed was an argument based on rigorous bounds.

Jen Eisert reported on some recent work [25, 41] regarding the so-called cluster states, which have high multi-article entanglement in the sense that entanglement within some subsystems persists after repeated measurements. These states have important applications in the development of the so-called "one-way" quantum computer. Eisert described modified protocols which can tolerant some noise or imperfect clusters.

Wolf reported on recent work with Cirac [86] on the question of whether or not a quantum channel can be written non-trivially as the composition of two other channels. It may be surprising that there are situations in which this cannot be done; for qubits, these are precisely the channels with exactly three Kraus operators. More generally, one can ask when a channel T can be written as $T = S^n$. Even for T very close to the identity channel, this need not always be possible because T completely positive need not imply that $T^{1/n}$ is completely positive. After giving an overview of the various phenomena which can occur in different situations, Wolf described some of the open questions which remain.

Dennis Kretschmann presented recent work (with Dirk Schlingemann and Reinhard Werner) [46] concerning a continuity theorem for the Stinespring representation of a quantum channel. Stinespring's Theorem guarantees that every quantum channel can be represented (non-uniquely) in the Heisenberg picture by an isometric embedding into a larger space. This suggests that two channels which are 'close' in some sense should be representable by isometries which are also close. As Kretschmann and co-authors showed, the correct notion of closeness for channels in this context is the completely bounded or 'cb' norm, which is a regularized version of the operator norm. The authors prove inequalities comparing the cb norm of the difference of two channels with the minimal operator norm of the difference of the isometries in their Stinespring representations. These inequalities provide dimension-independent bounds for the information-disturbance tradeoff inherent in any measurement of a system. The inequalities also allow a continuity estimate for the no-broadcasting theorem, and a strengthened proof of impossibility for quantum bit commitment.

Open problems

One of the highlights of the workshop were two sessions reserved for discussion of open problems, to which the participants responded enthusiastically. Ruskai started things off by distributing a preliminary draft of [70]. Many participants described open problems during their talks, as well as in the dedicated sessions. Participants were asked to write up these problems and send them to R. Werner for inclusion on his open problem web site. A list of the major problems follows, with comments on recent progress.

List of open problems

A. The first 6 sections in Ruskai's list [70] were discussed in the workshop. Some contain several related problems. The 7-th section was outside the scope of the workshop and added later. The main sections are

   1. Extreme points of CPT maps: As a consequence of the workshop, Ruskai showed that what are sometimes called "quasi-extreme" points which have Choi rank d but are not true extreme points of the convex set are in the closure of the set of extreme points. This allows a clearer statement of the open problems.
   2. Convex decompositions of CPT maps or A block matrix generalization of Horn's lemma (with K. Audenaert).

3. Generalized depolarized channels: The original emphasis was on the generalized Werner-Holevo channel. However, the developments in [84] suggested a further generalization using any channel that is very noisy to replace the completely noisy one.

   Progress: Michalakis [56] has solved Problem 6 for p = 2. It seems likely that his methods can be extended to provide some results for Problem 7 and for Problem 12.

4. Random sub-unitary channels

5. Additivity and multiplicativity conjectures: Very few changes were made to earlier versions of this section despite the recent breakthroughs. It seemed better to leave things in the original form to emphasize the impact of that work. One minor modification was the extension of Shor's Theorem 3 in Section 5.3 to $p < 1$ by the use of the general form of Klein's inequality, which then gives a single simple argument for all $p > 0$. This may be useful in doing numerical work to move from existence theorems to explicit counter-examples.

   Progress: Section 5.6 describes the current status of counter-examples [31, 32, 84] to multiplicativity conjectures and the implications for additivity. It also contains weaker versions of multiplicativity conjectures. It should again be emphasized that these counter-examples were stimulated by the open problem session and discussion at the workshop.

6. Coherent information and degradability: This problem was presented in Ruskai's talk at the workshop based on work in [19] which may contain some related open questions.

7. Local invariants for N-representability

B. Entropic uncertainty relations for more than two observables by D. Leung, S. Wehner, and A. Winter

C. Quantum Birkhoff Conjecture. This is conjecture 13 in [78] and Problem 30 at [83]. After Winter's talk there was a vigorous discussion as to whether or not it might be necessary to extend the conjecture to include partial isometries, such as the random sub-unitary maps described in Section 4 of [70]. A 1958 paper [54] entitled "The convex hull of sub-permutation matrices" and related work from that era might be useful to those who favor including random sub-unitary maps.

D. Best Constant in Norm bounds on Commutators: This problem was posed in [7] and presented to workshop participants by K. Audenaert along with a summary of known results.

   Progress: During the workshop, S. Michalakis [55] solved this problem for the commutator $[X, X^*]$ by showing that $\sqrt{2}$ is sharp in $\| \, [X, X^*] \, \|_2 \leq \sqrt{2} \|X\|_2^2$.

E. Structure of the nth matrix range of an operator, by V.I. Paulsen.

F. Structure of higher rank numerical ranges, by M.-D. Choi, J.A. Holbrook, D.W. Kribs, K. Zyczkowski, as outlined above.

   Progress: Shortly after this workshop, a flurry of work came to light from mathematicians working on higher rank numerical ranges. Most importantly, a related convexity conjecture was settled in the affirmative by Woerdeman [85], and then by Li and Sze [52] using different techniques which they were able to apply to the normal case. Thus, the door has been further opened for potential applications in quantum error correction.

Appendices attached as pdf files

A. Some open problems in quantum information theory by M.B. Ruskai

B. Entropic uncertainty relations for more than two observables by, D. Leung, S. Wehner, and A. Winter

C. Structure of the nth matrix range of an operator, by V.I. Paulsen.

D. Best Constant in Norm bounds on Commutators by K. Audenaert

E. Abstract of E. Effros for BIRS workshop 07w5013 on Operator Spaces and Group Algebras

List of Participants

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Eisert, Jens (Imperial College)
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Chapter 7

Mathematical Methods in Philosophy (07w5060)

Feb 18 - Feb 23, 2007

Organizer(s): Aldo Antonelli (University of California-Irvine), Alasdair Urquhart (University of Toronto), Richard Zach (University of Calgary)

Mathematics and philosophy have historically enjoyed a mutually beneficial and productive relationship, as a brief review of the work of mathematician-philosophers such as Descartes, Leibniz, Bolzano, Dedekind, Frege, Brouwer, Hilbert, Gödel, and Weyl easily confirms. In the last century, it was especially mathematical logic and research in the foundations of mathematics which, to a significant extent, have been driven by philosophical motivations and carried out by technically-minded philosophers. Mathematical logic continues to play an important role in contemporary philosophy, and mathematically trained philosophers continue to contribute to the literature in logic. For instance, modal logics were first investigated by philosophers, and now have important applications in computer science and mathematical linguistics. The theory and metatheory of formal systems was pioneered by philosophers and philosophically-minded mathematicians (Frege, Russell, Hilbert, Gödel, Tarski, among many others), and philosophers have continued to be significantly involved in the technical development of proof theory, and to a certain degree also in the development of model theory and set theory. On the other hand, philosophers use formal models to test the implications of their theories in tractable cases. Philosophical inquiry can also uncover new mathematical structures and problems, as with recent work on paradoxes about truth. Areas outside mathematical logic have also been important in recent philosophical work, e.g., probability and game theory in inductive logic, epistemology, and the philosophy of science. In fact, it seems that technical mathematical work is currently enjoying something of a renaissance in philosophy.

The workshop on "Mathematical Methods in Philosophy" brought together eminent and emerging researchers who apply mathematical methods to current issues in philosophy. These mathematical methods come mainly from the fields of mathematical logic and probability theory, and the areas of application include philosophical logic, metaphysics, epistemology, the philosophy of mathematics, and the philosophy of science.

Overview of the Field

Philosophical logic

Philosophical logic includes logical systems such as logics of possibility and necessity (alethic modal logic), of time (temporal logic), of knowledge and belief (epistemic and doxastic logic), of permission and obligation (deontic logic). This area is unified by its methods (e.g., relational semantics, first introduced by philosophers Saul Kripke and Jaakko Hintikka in the 1950s, algebraic methods, proof theory), but it has diverse applications in philosophy. For instance, logics of possibility and time are mainly useful in metaphysics whereas logics of knowledge and belief are of interest to epistemology. However, the methods employed in the study of these logics is very similar. Other related logics which have important applications in philosophy are many-valued logics, intuitionistic logic, paraconsistent and relevance logics.

Foundations and philosophy of mathematics and computation

One of the main advances at the intersection of mathematics and philosophy is the development of foundations of mathematics in the early 20th century (type theory and set theory). Intra-mathematical considerations of course played a very important role in motivating such developments, but philosophical concerns shaped and drove much of this development. The same is true for the subsequent development of disciplines of mathmatical logic such as proof theory and computability theory.

Formal theories of truth and paradox

The nature of truth is a central topic in metaphysics and philosophy of logic, and work on truth is closely connected to epistemology and philosophy of language. Significant advances have been achieved over the last 30 years in formal theories of truth, and there are close connections between philosophical work on truth and model theory (especially of arithmetic). One of the most important approaches to truth are the revision theories of truth first introduced by Saul Kripke and Solomon Feferman.

Formal epistemology

Formal epistemology is an emerging field of research in philosophy, encompassing formal approaches to ampliative inference (including inductive logic), game theory, decision theory, computational learning theory, and the foundations of probability theory.

Set theory and topology in metaphysics

Set theory has always had a close connection with mereology, the theory of parts and wholes, and topology has also been fruitfully applied in metaphysics.

Presentation Highlights

Philosophical Logic

Steve Awodey and **Kohei Kishida** (Carnegie Mellon University), *Topological semantics for first-order modal logic*

Awodey presented a new theorem, which extends Tarski's classical topological completeness result from propositional to first-order S4 modal logic.

Dave DeVidi (University of Waterloo), *Non-constructive uses of constructive logics*

While intuitionistic and other constructive logics have their first home in foundations of mathematics, their appearance in, for instance, metaphysical debates apart from mathematics is familiar, thanks to the work of philosophers such as Michael Dummett. In such cases, the reasons offered for supposing that constructive logic is correct are recognizably akin to those offered by mathematical constructivists. In recent times, though, it has become increasingly common to see versions of constructive logic advocated for philosophical purpose—as part of a solution to a paradox, for instance—when no appeal to constructivist motivations is offered and no plausible one seems possible. It is not uncommon to see such proposals rejected on the grounds of incompatibility with constructivism ("no intuitionist could consistently say *that*"). This objection is beside the point if the appeal to constructive logic has some suitable non-constructivist motivation—for then the name "intuitionistic logic" (e.g.) becomes a historical curiousity, instead of an indication of who may appeal to that logic as the correct one. This response is often claimed, but seldom defended. DeVidi described some cases of this sort, and considered the prospects for giving a non-constructivists but philosophically satisfactory defense of the claim that some-or-other constructive logic is the correct one for certain purposes.

Eric Pacuit (University of Amsterdam) and Horacio Arlo-Costa (Carnegie-Mellon University), *Quantified Classical Modal Logic and Applications*

Pacuit introduced and motivated the study of classical systems of first-order modal logic. In particular, he focussed on the study of neighborhood frames with constant domains and offer a series of new completeness results for salient classical systems of first order modal logic. He discussed general first-order neighborhood and offer a general completeness result for all classical systems of first-order modal logic. Finally, he showed how to extend this analysis to freely quantified classical modal logic.

Graham Priest (University of Melbourne and University of St. Andrews), *Many-valued modal logic*

In standard modal logics, the worlds are two-valued. There is no reason why this has to be the case, however: the worlds could be many-valued. In this talk, Priest looked at many-valued modal logics. He started with the general structure of such logics. To illustrate this, he considered modal logic based on Łukasiewicz's continuum-valued logic. Priest then considered one many-valued modal logic in more detail: modal First Degree Entailment (FDE). Tableaux for this and its special cases (K_3 and LP) were provided. Modal many-valued logics engage with a number of philosophical issues. The final part of the talk illustrated with respect to one such: the issue of future contingents.

Timothy Williamson (University of Oxford), *Adding probabilities to epistemic logic*

Williamson used a case study to illustrate the philosophical interest of adding epistemic probabilities to standard possible world models of epistemic logic. It is familiar that the non-transitivity of the accessibility relation between worlds corresponds to the failure of the KK principle—if you know, you know that you know. How far can we turn the screw with counterexamples to the KK principle? That is, how low can your epistemic probability that you know p go at a world at which you do in fact know p? Answer: As close to 0 as you like. Some of the relevant models can be instantiated in quite realistic settings. Willimason considered implications for debates about the standard of epistemic warrant required for assertion and about apparent counterexamples to otherwise plausible closure principles for knowledge.

Theories of truth and paradox

JC Beall (University of Connecticut) and **Michael Glanzberg** (University of California, Davis), *Truth and paradox*

Beall and Glanzberg aimed to give a big-picture sketch of truth and paradox – chiefly, the Liar (but also related truth-theoretic paradoxes).

Approaches to the Liar that they mentioned are all marked by the ways they navigate between completeness and consistency. Some key examples of these approaches include those which:

- Reconsider logic:
  1. Paraconsistent: the Liar teaches us that EFQ fails, that some sentences are true and false, but our language is nonetheless non-trivial (i.e., some sentences are 'just true').
  2. Paracomplete: the Liar teaches us that LEM fails, in some way that avoids a variant Liar which reinstates the paradox.
- Reconsider the semantics:
  1. Contextual: the Liar teaches us that truth is contextually sensitive, shifting the extension of 'true' from context to context.
  2. Revision Theory: the Liar teaches us that 'true' is governed by *rules of revision*.

Each of these options seeks to reject some portion of consistency or completeness, and yet present a coherent and appealing environment in which logic and semantics can coherently proceed.

Solomon Feferman (Stanford University), *A nicer formal theory of non-hierarchical truth*

A new formal theory S of truth extending PA is introduced, whose language is that of PA together with one new unary predicate symbol $T(x)$, for truth applied to Gdel numbers of suitable sentences in the extended language. Falsity of x, $F(x)$ is defined as truth of the negation of x; then the formula $D(x)$ expressing that x is a determinate meaningful sentence is defined as the disjunction of $T(x)$ and $F(x)$. The axioms of S are those of PA extended by (I) full induction, (II) strong compositionality axioms for D, and (III) the recursive defining axioms for T relative to D. By (II) is meant that a sentence satisfies D if and only if all its parts satisfy D; this holds in a slightly modified form for conditional sentences. The main result is that S has a standard model. As an improvement over earlier systems developed by Feferman, S meets a number of leading criteria for formal theories of truth that have been proposed in the recent literature, and comes closer to realizing the informal view that the domain of the truth predicate consists exactly of the determinate meaningful sentences.

Volker Halbach (University of Oxford), *The Kripke-Feferman theory of truth*

Feferman proposed to axiomatize Kripke's theory of truth in classical logic. The resulting theory is called the Kripke-Feferman (KF) theory of truth. I argue that this theory introduces some unwanted features because it relies on classical logic, and that Kripke's theory should be axiomatised in partial logic.

It has been argued by Reinhardt that nevertheless KF may be taken as a tool for generating theorems of a theory of truth in partial logic by focusing on those sentences A that can be proved to be true in KF. Halbach argued that this justification of KF fails, as a natural axiomatisation of Kripke's theory in partial logic is proof-theoretically much weaker than the theory generated by KF.

Jeff Ketland (University of Edinburgh), *Truth and reflection*

Say that a truth theory is deflationary if, when added to any of a suitable class of base theories, the result is a conservative extension. Say that a truth theory is reflective if, when added to any of a suitable class of base theories, the reflection principles for that theory become theorems. Reflective truth theories are desirable; for, just as when we accept a statement A, we should accept "A is true," similarly, when we accept a theory T, we should accept "All axioms of T are true." Stewart Shapiro (1998) and Ketland (1999) noted that these conditions are incompatible: reflective truth theories are non-conservative, and thus non-deflationary. In particular, Tarski's compositional theory of truth is reflective (as are more sophisticated "self-applicative" truth theories, such as the Kripke-Feferman theory), and thus non-deflationary. It seems correct to conclude then that deflationism about truth is incompatible with results in mathematical logic. Several authors (Field, Azzouni, Halbach and Tennant) have presented responses to this argument against deflationism. Ketland surveyed these responses and offered some replies.

Greg Restall (University of Melbourne), *Modal models for Bradwardine's theory of truth*

Restall introduced Stephen Read's reconstruction of Bradwardine's theory of truth, and provided it with a simple model theory. This model theory can be used to provide a fixed-point construction to extend any classical theory with a Bradwardine truth predicate which diverges from Tarskian truth only on ungrounded sentences.

Foundations and philosophy of mathematics and computability

Grigori Mints (Stanford University), *Effective content of non-effective proofs*

Methods of proof theory allow to extract effective bounds from some non-effective proofs and point out possibilities of obtaining sharper bounds from mathematical proofs, sometimes depending of fewer parameters. We survey several such applications and illustrate the approach for a proof of Herbrand's theorem using compactness. A new cut elimination method (in particular a new proof of Herbrand's Theorem) is obtained here by "proof mining" (unwinding) from the familiar non-effective proof. That proof begins with extracting an infinite branch when the canonical search tree for a given formula of first order logic is not closed. Our reduction of a cut does not introduce new cuts of smaller complexity preserving instead only one of the branches.

Wilfried Sieg (Carnegie Mellon University), *Church without dogma: axioms for computability*

Church's and Turing's theses dogmatically assert that an informal notion of computability is captured by a particular mathematical concept. Sieg presented an analysis of computability that leads to precise concepts, but dispenses with theses.

To investigate computability is to analyze processes that can in principle be carried out by calculators. Drawing on this lesson we owe to Turing and recasting work of Gandy, Sieg formulated finiteness and locality conditions for two types of calculators, human computing agents and mechanical computing devices; the distinctive feature of the latter is that they can operate in parallel.

The analysis leads to axioms for discrete dynamical systems (representing human and machine computations) and allows the reduction of models of these axioms to Turing machines. Cellular automata and a variety of artificial neural nets can be shown to satisfy the axioms for machine computations.

Mathematics and logic in metaphysics

Gabriel Uzquiano (University of Oxford) and Stewart Shapiro (The Ohio State University), *Ineffability and reflection*

We know that not all concepts have extensions associated with them. In this contribution, Uzquiano and Shapiro explored the hypothesis that a concept F lacks an extension if and only if F is ineffable, by which they mean, roughly, that no concept at least as large as F is describable by logical vocabulary alone. They are interested in this hypothesis largely because it seems to them to give partial expression to the inchoate thought that the universe is ineffable. A first approximation to this thought takes the form of a second-order reflection schema on which, given a concept H at least as large as a concept F to which no extension corresponds, a sentence of pure second-order logic is true when relativized to the instances of H only if it is true when relativized to strictly fewer objects. We

One may be able to express the thought behind this reflection schema in finite compass by a sentence of a third-order language. However, once we allow ourselves the resources to do this, we find ourselves in a position to describe what is for a concept to be ineffable by the vocabulary of pure third-order logic, which betrays the very thought with which we started. This situation generalizes and, in their contribution, Uzquiano and Shapiro looked at the tension between, on the one hand, the drive to express the ineffability of the universe and, on the other, the constraint to remain faithful to it.

Harvey Friedman (The Ohio State University), *Concept calculus*

Friedman's Concept Calculus provides an unexpected exact correspondence between ordinary everyday thinking about ordinary everyday things and abstract mathematics.

As an example, Friedman identified a large range of principles involving just the two informal binary relations "better than" and "much better than," which give rise to a variety of formal systems which are mutually interpretable with a variety of standard formal systems from logic whose strengths range from weak arithmetics to various large cardinals.

It appears that an enormous range of informal concepts lend themselves to closely related investigations. For example, we have developed a kind of naive physics based on informal notions of time and space and point mass, which also corresponds, by mutual interpretation, to these same formal systems from logic.

The hope is that concept calculus can serve as a tool for organizing and analyzing metaphysical concepts that is in rough analogy with the way that the Newton/Leibniz calculus serves as a tool for organizing and analyzing physical concepts.

Philosophical issues and logic

Delia Graff Fara (Princeton University), *Relative identity and de re modality*

Fara defended the materialist thesis that material things are identical to the matter that composes them, by appealing to the semantic view that names are predicates; and by proposing and investigating a version of David Lewis's counterpart theory that appeals, rather than to Lewis's own modfied similarity relation, to relations of *relative* identity in the analysis of *de re* temporal and modal claims. This was carried out in the context of a metaphysics that's both actualist and three-dimensionalist.

Hannes Leitgeb (University of Bristol), *Applications of mathematics in philosophy: four case studies*

As we all know, mathematical methods are of crucial importance in science. Many believe that mathematics will play a similar role for philosophy once philosophical theories have reached a sufficient degree of complexity; to some degree, this has already happened. Leitgeb tried to support this thesis by stating four examples which are chosen (conveniently) from his own work:

1. Similarity, Properties, and Hypergraphs
2. Nonmonotonic Logic and Dynamical Systems
3. Belief Revision for Conditionals and Arrow's Theorem
4. Semantic Paradoxes and Non-sigma-Additive Probability Measures

Yiannis Moschovakis (UCLA), *Synonymy*
Moschovakis discussed some historical approaches to synonymy, and focussed on the mathematical problems of decidability which arise when senses are modelled rigorously in formalized fragments of language. About half of the talk was be dedicated to an exposition of the theory of referential intensions, by which (in slogan form) *the sense of a term is the natural algorithm which determines its denotation*. This modelling of meanings leads to both theorems and difficult open problems in the logic of synonymy.

Gillian Russell (Washington University, St. Louis), *One true logic?*
In their 2006 book *Logical Pluralism*, Beall and Restall argue that there is more than one correct logic. Russell examined that claim and present a different argument for a similar view.

Kai Wehmeier (University of California, Irvine), *Identity is not a relation*
Frege, Russell, and the early Wittgenstein all struggled with the notion of a binary relation that every object bears only to itself. In the *Tractatus* we even find an outright rejection of the notion, together with some gestures as to its eliminability from predicate logic. In the talk, Wehmeier sketched what seems to be the most promising argument against the existence of a binary relation of numerical identity, and discuss a few related logical issues.

Byeong-Uk Yi (University of Toronto), *Is logic axiomatizable?*
Yi defended the negative answer to the question in the title, "Is logic axiomatizable?," by considering sentences that involve plural constructions. He also compared his argument for the non-axiomatizability of logic with the usual argument for the non-axiomatizability of second-order logic and with Tarski's ω-consequence example in the beginning of his paper "On the concept of logical consequence," and how it relates to David Kaplan's proof of inexpressibility of certain sentences in elementary languages.

Outcomes of the Meeting

Surveys

One particular aim of the workshop was to provide the participants with a sense of the range of topics, the state of current research, the interconnections, and the important trends are in philosophical logic and related areas. To this end, the organizers invited three survey talks. These surveys provided an overview of the development of the field in the last 20–50 years, of the current state of the art, of the main open problems, and of anticipated future trends and developments:

Branden Fitelson (University of California, Berkeley), *Survey on formal epistemology: Some propaganda and an example*
Fitelson discussed various threads of "formal philosophy," as he prefers to call the field of formal epistemology. He gave a survey of the development of confirmation theory and the uses of probability theory in it, and ended with an illustrative application of confirmation theory to the problem of induction.

Markus Kracht (UCLA), *The certain past and possible future of modal logic*
The origins of modal logic are somewhere in philosophy. However, for more than fifty years there is also a more "technocratic'; approach to the field that applies mathematical methods. Over time, it has created its own terminology and, inevitably, its own problems that it likes to deal with. Other areas of application have also been found, for example computer science. While the techniques and results for propositional modal logic are by now fairly widely known even outside the circle of mathematicians, in the domain of modal predicate logic there still is some lack of knowledge transfer between philosophers and mathematicians.

Kracht outlined the past developments of modal logic with special attention to modal predicate logic, where, he argued, the greatest promise for 'technocratic' modal logic is still to be found.

Stewart Shapiro (The Ohio State University), *Life on the ship of Neurath: mathematics in the philosophy of mathematics*

Shapiro gave an "idiosyncratic" survey of the use of mathematics to support or otherwise assess programs in the philosophy of mathematics. It covered the "big three" views that dominated thinking in the early decades of the twentieth century: formalism, intuitionism, and logicism, and then moveed onto contemporary descendants of these views: *ante rem* structuralism, Scottish neo-logicism, fictionalism, and various reconstructive nominalisms.

Proceedings

A proceedings volume collecting selected papers from the workshop is planned. It will apear as a special issue of the *Journal of Philosophical Logic*.

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Feferman, Solomon (Stanford University)
Fitelson, Branden (University of California-Berkeley)
Friedman, Harvey (Ohio State University)
Glanzberg, Michael (University of California-Davis)
Halbach, Volker (New College, Oxford)
Jennings, Ray (Simon Fraser University)
Kalyvianaki, Eleni (University of Athens)
Ketland, Jeffrey (University of Edinburgh)
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Manne, Kate (Massachusetts Institute of Technology)
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Moschovakis, Yiannis (University of California, Los Angeles)
Moschovakis, Joan Rand (University of California-Los Angeles)
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Priest, Graham (University of Melbourne)
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Restall, Greg (University of Melbourne)
Roush, Sherrilyn (University of California-Berkeley)
Russell, Gillian (Washington University St. Louis)
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Chapter 8

Topology (07w5070)

Feb 25 - Mar 02, 2007

Organizer(s): Ian Hambleton (McMaster University), Matthias Kreck (University of Heidelberg), Ronald Stern (University of California, Irvine)

Introduction

The geometry and topology of manifolds is a large research area, making connection with many flourishing specialties such as algebraic topology, symplectic geometry, gauge theory, knot and links, and differential geometry. The purpose of this meeting was to bring together a broad selection of researchers from many flourishing areas of current work in topology, in order to promote awareness of new developments across the whole field.

This meeting was a sequel to our highly successful meeting ”Topology” 05w5067 held at BIRS (August 27 - Sept. 1, 2005), which had similar objectives and scope. The strongly positive comments we received from the participants at that time encouraged us to think that a meeting with this broader scope was a valuable service to the mathematics research community.

The format of the meeting was designed to promote interaction and discussion, as well as exposure of all the participants to certain themes of broad interest. These included the recent work of Perelman on Ricci flow and the classification of 3-manifolds, as well as topics in geometric group theory, coarse geometry and topology, the Novikov conjecture, and elliptic cohomology.

The proof of the Poincaré conjecture played a central role in the conference. John Morgan from Columbia gave three lectures with an excellent overview of the proof. This led to numerous lively and fruitful discussions between the participants. In general the atmosphere was very creative, and also those who did not give a lecture had the chance to explain their ideas in numerous discussions in smaller groups. As organizers, we were very pleased with the high scientific level of the talks, and with the energy and enthusiasm of all the participants.

We limited the talks to 5 per day and 45 minutes each, allowing ample time for informal interactions. The speakers were asked to address a broad audience and most of them did this very successfully. A good number of the talks were given by younger mathematicians.

Abstracts

Adem, Alejandro, University of British Columbia: Commuting Elements and Spaces of Homomorphisms Consider the space Hom(Q,G) of homomorphisms between a discrete group Q and a Lie group G.

This talk described basic properties of these spaces and how for certain discrete groups their contribution to bundle theory can be quantified using the cohomology of Q. We will also discuss the cohomology of some of these spaces, with particular attention to the case when Q is a free abelian group. A stable splitting for the space of commuting elements was described. This is joint work with Fred Cohen.

Baird, Tom, University of Toronto: Moduli spaces of flat connections on nonorientable surfaces

This talk presented recent work studying the topology of moduli spaces of flat connections on nonorientable surfaces and described some relationships with their counterparts for orientable surfaces. The main tool used was equivariant cohomology.

Bartels, Arthur, Universität Münster: The Farrell-Jones Conjecture in algebraic K-theory for hyperbolic groups.

This is joint work with Wolfgang Lück and Holger Reich. This talk presented a proof of the Farrell-Jones Conjecture in algebraic K-theory for hyperbolic groups in the sense of Gromov. This means that the algebraic K-theory $K_*(RG)$ of RG, for a ring R and a hyperbolic group G, can be computed in terms of $K_*(RV)$, where V varies over the familiy of virtually cyclic subgroups. This result has (among others) applications to Whitehead groups, the Bass conjectures and the Kaplansky conjecture.

Behrstock, Jason, University of Utah: Dimension and rank of mapping class groups.

We discussed recent work with Yair Minsky towards understanding the large scale geometry of the mapping class group. In particular, it was explained how to obtain various topological properties of the asymptotic cone of the mapping class group including a computation of its dimension. An application of this analysis is an affirmative solution to Brock-Farb's Rank Conjecture which asserts that MCG has quasi-flats of dimension N if and only if it has a rank N free abelian subgroup. This talk was of interest to a broad audience of topologists since it contained geometric group theory, low dimensional topology, and some classical dimension theory.

Bryan, Jim, University of British Columbia: The Quantum McKay Correspondence.

Let G be a finite subgroup of $SU(2)$ or $SO(3)$. The classical McKay correspondence describes the cohomology of the resolution of the orbifold C^2/G or C^3/G in terms of the representation theory of G. We give a quantum version of this. We described the quantum cohomology (and, more generally, all the Gromov-Witten invariants) of the resolution in terms of the ADE root system associated to G.

Davis, Jim, Indiana University: Mapping tori of self-homotopy equivalences of lens spaces (or - there are no exotic beasts in Hillman's zoo).

This is joint work with Shmuel Weinberger. We conjecture that the mapping torus of a self-homotopy equivalence of three-dimensional lens spaces is homotopy equivalent to a closed manifold. This talk presented a proof of this conjecture in the case where the lens space has prime order fundamental group. A feature of the proof is that it uses Gauss' Lemma on quadratic residues. This answers a question of Jonathan Hillman.

Brent Doran, Institute for Advanced Study, Princeton: Unipotent groups, contractible varieties, and some classical questions in affine geometry.

The study of contractible topological spaces began in earnest in 1935 with J.H.C. Whitehead's construction of the Whitehead space–a counter-example to his proof of the 3-dimensional Poincaré conjecture. In the 1960s geometric topologists studied properties of contractible topological spaces in detail as a testing ground for general theory. This talked investigated algebraic varieties which are contractible from the standpoint of algebraic geometry, formalized using the A^1-homotopy theory of Morel and Voevodsky. The class of such varieties is surprisingly rich including many smooth examples beyond affine spaces and, in many ways, the theory is analogous to the theory developed for contractible topological spaces. Over C or R, many of these are diffeomorphic to C^n or R^n. We then discussed a general construction of such varieties using a version of geometric invariant theory for unipotent groups and show how they relate to, and provide a testing ground for, various long-standing general conjectures in algebraic geometry. A basic conclusion: many of our most sophisticated invariants miss an enormous amount of structure in algebraic geometry. Optimistic corollary: we should look to methods of topology, adapted to algebraic geometry via A^1-homotopy theory, for aid in formulating classification theorems. Joint work with Aravind Asok.

Other current research interests: non-reductive geometric invariant theory and applications; moduli problems, especially at the moment moduli of bundles on curves and sheaves on surfaces; A^-1-homotopy, motives, and motivic cohomology; intersection cohomology of compactifications of locally symmetric spaces were discussed as well as some interesting interrelations among these topics.

Ebert, Johannes, Muenster: Spin structures on surface bundles.

A spin structure on a surface bundle $\pi : E \to B$ with connected compact orieted fiber F is a spin structure on the vertical tangent bundle T_vE. We adress the question of necessary and sufficient conditions on the existence of a spin structure. First of all, there must exist a spin structure σ on F which is invariant under the image of the monodromy homomorphism $\pi_1(B) \to \pi_0(Diff(F))$. But this is not sufficient. For any spin structure σ on F, there exists a class in $H^2(BDiff(F;\sigma);\mathbb{Z}/2)$ which is an obstruction to the existence of a spin structure on a surface bundle whose monodromy fixes σ. We show that this obstruction class is nonzero for any spin structure on any surface.

As necessary conditions for the existence of spin structures, we have divisibility relations for the Mumford classes $\kappa_n(\pi) \in H^{2n}(B;\mathbb{Z})$. We showed that the previously known divisibility relations without the assumption of a spin structure is strengthened by the factor 2^{n+1}. For even n, we show that this relation is optimal in the stable range, i.e. if the genus g of F is large compared to n.

Galatius, Soren Stanford University: The homotopy type of the cobordism category.

The d-dimensional cobordism category C_d has closed $(d-1)$-dimensional manifolds as objects and compact d-dimensional cobordisms as morphisms. Thom's theorem determines π_0 of the classifying space BC_d. This talk discussed joint work with Madsen, Tillmann and Weiss, in which we determine the homotopy type of BC_d. As a corollary we presented a new proof of Madsen-Weiss' theorem.

Ghiggini, Paolo, Universit du Qubec Montral: Contact structures, Heegaard Floer homology, and fibred knots.

Recently I proposed a strategy to prove that knot Floer homology detects fibred knots using taut foliations and contact structures. This strategy was implemented by myself in the particular case of genus-one knots, and by Yi Ni in the general case.

In the talk an outline the strategy was presented, as well as some hints about the proof in the case of genus-one fibred knots. It was also pointed out the difficulties that Yi Ni had to overcome in order to arrive to a complete proof.

Grodal, Jesper, Chicago/Copenhagen: Local-to-global principles for classifying spaces

This showed how one can sometimes "uncomplete" the p- completed classifying space of a finite group, to obtain the original (non- completed) classifying space, and hence the original finite group. This "uncompletion" process is closely related to well-known local-to-global questions in group theory, such as the classification of finite simple groups. The approach goes via the theory of p-local finite groups, more precisely a certain fundamental group. This talk was a report on joint work with Bob Oliver.

Hanke, Bernhard, University of Munich: Enlargearbility, coarse geometry and the Baum-Connes map

Enlargeability was introduced by Gromov and Lawson as an obstruction to the existence of positive scalar curvature metrics on closed spin manifolds M. Rosenberg introduced another "universal" index theoretic obstruction living in the K-theory of the reduced or maximal group C^*-algebra of the fundamental group $\pi_1(M)$. We reported on recent work of Kotschick, Roe, Schick and myself proving nonvanishing of this index obstruction for enlargeable manifolds. Our approach is independent from injectivity of the Baum-Connes assembly map. The discussions of the reduced and maximal C^*-algebra use quite different methods: The former one has a strong coarse theoretic flavour, whereas the latter one rests on the construction of a flat

Hilbert space bundle (as twisting bundle for the Dirac operator on M) out of a sequence of asymptotically flat bundles.

This construction can also be used to prove injectivity of the restriction of the Baum-Connes assembly map (with values in the K-theory of the maximal C^*-algebra) to K- homology classes dual to classes of cohomological degree 2. This verifies the strong Novikov conjecture for these classes and implies a result of Mathai and Connes-Gromov Moscovici on the invariance of higher signatures associated to cohomology classes of degree 2.

Hausmann, Jean-Claude, L'Université de Genève: The topology and geometry of Polygon spaces.

The study of polygon spaces in R^d started two decades ago with the thesis of K. Walker (for $d = 2$). They occur in connection with statistical shape theory and robotic. For $d = 3$, they became also a chapter of Hamiltonian geometry, as a rich source of examples, closely related to toric manifolds. This talk was a survey of these various aspects of polygon spaces, their classification and recent results.

Hedden, Matthew, Massachusetts Institute of Technology: On knot Floer homology and algebraic curves

It is well known that each torus knot arises as the intersection of an algebraic curve in C^2 with isolated singularity at the origin with the standard three-dimensional sphere. Indeed, the class of knots which arise in this way from algebraic curves with an isolated singularity is well understood. However, by deforming the sphere or relaxing the restriction on the curve's singular locus a much wider class of knots and links is obtained. This talk discussed the question of which knots arise from algebraic curves in the above sense, focusing our attention on some results indicating connections with the Ozsvath-Szabo Floer homology invariants. More precisely, Ozsvath and Szabo introduced an invariant, denoted $\tau(K)$, to knots in the three-sphere (this invariant was independently discovered by Rasmussen). We first showed that $\tau(K)$ provides an obstruction to knots arising from complex curves in the above sense. Restricting attention to fibered knots, we then proved the more surprising theorem that $\tau(K)$ detects when a fibered knot arises from a complex curve with a certain genus constraint. Coupled with work of Ozsvath and Szabo and recent independent work of Ghiggini, Ni, and Juhasz, an immediate corollary is that any knot which admits a lens space surgery can be realized as the intersection of a complex curve with the three-sphere.

Ji, Lizhen, University of Michigan: Large scale geometry and topology of subgroups of Lie groups and mapping class groups

For a discrete group, a natural problem concerns different versions of the Novikov conjecture in surgery theory, algebraic K-theory and $C*$-algebras. The original Novikov conjecture on homotopy invariance of higher signatures is equivalent to the rational Novikov conjecture in surgery theory, and the integral Novikov conjecture in surgery theory implies the stable Borel conjecture.

One approach to the Novikov conjecture uses the asymptotic dimension of the group endowed with a word metric, and another approach uses suitable compactifications of cofinite universal spaces for proper actions. We will study the validity of the integral Novikov conjecture and the existence of cofinite universal space for proper actions for the following closely related classes of groups: 1. Arithmetic groups such as $SL(n, Z)$, and more generally lattice subgroups of Lie groups 2. S-arithmetic subgroups of semisimple algebraic groups such as $SL(n, Z[1/p])$, which are usually not discrete subgroups of Lie groups. 3. Finitely generated subgroups of $GL(n, Q)$. 4. Mapping class groups $Mod_{g,n}$ of surfaces of genus g with n punctures.

Symmetric spaces, Bruhat-Tits buildings, Teichmuller spaces and their compactifications were be used together with basic tools such as the reduction theory for arithmetic subgroups. This talk also brought out similarity of these objects.

Kreck, Matthias, University of Heidelberg: Equivariant (co)homology and Poincare duality

Equivariant (co)homology defined via the Borel construction does not fulfill Poincare duality. To motivate what I am working on consider a closed oriented free smooth m-dimensional G-manifold (G a compact Lie group of dimension d). Then the equivariant homology of M is the homology $H_k(M/G)$ of M/G. By ordinary Poincare duality this is isomorphic to $H^{m-d-k}(M/G)$. Question: Is there an equivariant multiplicative cohomology theory $h^r_G(M)$ such that if M is as above a closed free G manifold, then $h^r_G(M) = H^r(M/G)$? If yes we then consider the shifted equivariant homology $h_k^G(M) := H^G_{k+d}(M)$. Poincaré duality for the theory h then holds for closed free G-manifolods and one can ask if this is the case for arbitrary closed G-manifolds.

I have constructed such a theory $h^k_G(M)$ as bordism classes of free G stratifolds of dimension $m-k$ together with a proper equivariant map to M. The corresponding homology theory $h_k^G(M)$ of bordism classes of free compact G-stratifolds with an equivariant map to M is canonically isomorphic to $H^G_{k-d}(M)$. Thus we obtain a geometric description of equivariant homology. Presently I'm investigating this new cohomology theory further and construct corresponding Bredon type equivariant (co)homology theories.

Lueck, Wolfgang, Universität Münster: Topological rigidity for non-aspherical manifolds (with M. Kreck).

The Borel Conjecture predicts that closed aspherical manifolds are topological rigid. We want to investigate when a non-aspherical oriented connected closed manifold M is topological rigid in the following sense. If $f : N \to M$ is an orientation preserving homotopy equivalence with a closed oriented manifold as target, then there is an orientation preserving homeomorphism $h : N \to M$ such that h and f induce up to conjugation the same maps on the fundamental groups. We call such manifolds Borel manifolds. We gave partial answers to this questions for $S^k x S^d$, for sphere bundles over aspherical closed manifolds of dimension less or equal to 3 and for 3-manifolds with torsionfree fundamental groups. We showed that this rigidity is inherited under connected sums in dimensions greater or equal to 5. We also classified manifolds of dimension 5 or 6 whose fundamental group is the one of a surface and whose second homotopy group is trivial.

Equivariant Chern characters

We first recall Dolds rational computation of a generalized homology theory in terms of singular homology. Essentially Dold shows that the Atiyah-Hirzebruch spectral sequence rationally collapses. The aim of this talk was to generalize it to the equivariant setting. We introduce the notion of an equivariant homology theory. We explained how under certain assumptions on the coefficients such as a Mackey structure it can be computed in terms of Bredon homology. This has many applications in connection with the Farrell-Jones Conjecture, and the Baum-Connes Conjecture and leads to a rational computation of the topological K-theory of BG for a discrete group G which has a finite model for its classifying space of proper G-actions.

Lurie, Jacob, Harvard University: Equivariant Cohomology Theories and Algebraic Groups.

I sketched a construction which produces an equivariant cohomology theory starting with an algebraic group (in a suitable setting). I then explained how this construction can be used to produce equivariant elliptic cohomology. This talk demonstrated Interactions between homotopy theory and algebraic geometry, elliptic cohomology, geometric representation theory.

Morel, Fabien, Ludwig-Maximilians-Universität Munich: Towards a surgerical approach to the classification of smooth projective varieties over a field.

This talk sketched a new approach to the study of smooth projective A^1-connected varieties over a field inspired by the classical surgery approach in differential topology. This approach relies on recent progress in the A^1-homotopy theory of smooth varieties. We explained basic facts concerning the A^1-fundamental group and illustrated the slogan that it should play a major role in this approach, as in classical differential topology.

Morgan, John, Columbia: Overview of Perelman's proof of the Poincare Conjecture and the Geometrization Conjecture.

Starting with the Ricci flow introduced by Hamilton, Perelman showed how to control the finite-time singularities in 3-dimensional flows and consequently extend such a flow to a Ricci with surgery defined for all positive time. The surgeries analytically necessary to deal with the finite-time singularities in fact perform the topological operation of connected sum decomposition necessary in order to simplify 3-manifolds into prime pieces. With the existence result result for Ricci flow with surgery defined for all positive times, and a complete understanding of the topological change at the surgery times, to prove the geometrization conjecture for a compact 3- manifold it suffices to prove it for any of the 3-manifolds that appear in the Ricci flow with surgery at any later time. The proof of the Poincaré Conjecture is completed by showing that if the initial 3-manifold is a homotopy sphere then after some finite time the the 3-manifold that appears in the resulting Ricci flow with surgery is empty (and hence satisfies the Geometrization Conjecture). To prove the general geometrization conjecture requires studying the limits as time goes to infinity in a general 3-dimensional Ricci flow with surgery. Here is where the incompressible tori appear and according the pieces that reqsult from cutting the manifold open along these tori are either hyperbolic or are collapsed. Perelman then states a result showing that the collapsed pieces are graph manifolds. This allows one to prove the full geometrization conjecture. These three lectures gave an overview of the ideas and techniques that go into these arguments and give an evaluation of the current state of confidence that these arguments are complete and correct.

Olbermann, Martin, University of Heidelberg: Conjugations on six-manifolds.

When we are trying to find simply-connected asymmetric manifolds, i.e. manifolds not admitting any non-trivial finite group action (for example among spin 6-manifolds), V. Puppe's method shows that in some cases, the only possible action would have to be a "conjugation". Conjugation spaces are spaces with involution such that the fixed point set of the involution has $\mathbb{Z}_2$-cohomology isomorphic to the $\mathbb{Z}_2$- cohomology of the space itself, with the little difference that all degrees are divided by two (e.g. $\mathbb{C}P^n$ with the complex conjugation). One also requires that a certain conjugation equation is fulfilled. This talk applied a new characterization of conjugation spaces to realize conjugation 6-manifolds. The main result is that for every closed oriented 3-manifold M there exists a simply connected spin conjugation 6-manifold with fixed point set M.

Pedersen, Erik, SUNY Binghamton

A few years ago T. Bauer, N. Kitchloo, D. Notbohm and I proved that if X is a loop space and the homology of X is finitely generated as an abelian group then X is homotopy equivalent to a compact, smooth, parallelisable manifold. It is likely this result holds without assuming that X is a loop space only assuming X is an H-space. This is not even known in the simply connected case because of our very poor understanding of the Arf invariant. So this talk discussed how to get a better understanding of the Arf invariant and noted that this is different from trying to prove the so-called Arf invariant problem.

Ranicki, Andrew, University of Edinburgh: Survey of codimension one splitting.

Much of high-dimensional manifold topology depends on codimension 1 splitting techniques, using algebraic K- and L-theory to decide if a homotopy equivalence of manifolds can be split along a codimension 1 submanifold. The talk surveyed the obstruction theory involved, and some of the applications.

The geometric Hopf invariant

This is a joint project with Michael Crabb. The geometric Hopf invariant of a stable map $F : \Sigma^k X \to \Sigma^k Y$ is a Z_2-equivariant map $h_{R^k}(F)$ which "counts the double points" of F. The homotopy class of $h_{R^k}(F)$ is the primary obstruction to F being homotopic to the k- fold suspension $\Sigma^k F_0$ of an unstable map $F_0 : X \to Y$. The geometric Hopf invariant has applications to double points of immersions of manifolds, and to surgery obstruction theory, including the non-simply connected cases.

Rosenthal, David, St. Johns University: On the K-theory of groups with finite asymptotic dimension.

In this work it is proved that the assembly maps in algebraic K- and L-theory with respect to the family of finite subgroups is injective for groups with finite asymptotic dimension that admit a finite model for the classifying space for proper actions. The result also applies to certain groups that admit only a finite dimensional model for this space. In particular, it applies to discrete subgroups of virtually connected Lie groups. This is joint work with Arthur Bartels.

Sauer, Roman, University of Chicago: On and around proportionality of the simplicial volume of finite volume manifolds.

The simplicial volume of compact and non-compact manifolds can behave quite differently. We gave a criterion saying in which cases the proportionality principle for Riemanian finite volume manifolds holds. In contrast to that, the well-known proportionality theorem for closed manifolds holds in general. Furthermore, we explained some related results about the relation between the locally finite and the relative simplicial volume and the relation to L^2-Betti numbers. This is joint work with Clara Loeh Schommer-Pries, Chris Berkeley

Stern, Ronald, University of California, Irvine

This talk presented joint work with Ron Fintushel developing techniques that demonstrate how to change smooth structures on a given smooth 4-manifold and how to create infinitiely many distinct smooth structures. This was then used to motivate the conjecture that any two smooth 4-manifolds are homeomorphic iff they are obtained by a sequence of these operations.

Symington, Margaret, Mercer University: Applications of toric geometry to more general manifolds.

This talk was an advertisement for the use of techniques motivated by toric geometry in dimension four to study the topology four- manifolds. Toric four-manifolds are quite tame, consisting exclusively of S^2xS^2 and blowups of CP^2. However, if one is willing to consider a toric structure on only part of the manifold, one can exploit the "local toric structure" to prove that a smooth surgery (rational blowdowns) preserves symplectic structures.

More recently, David Gay and I relaxed the symplectic condition on a toric manifold to characterize "toric near-symplectic manifolds". Doing so provides both examples and tools to understand and calculate (in terms of graphs in moment map images) emerging Gromov-Witten type invariants due to Taubes.

Taylor, Laurence Notre Dame: Homology with local coefficients.

Farrell and Hsiang noticed that the action of conjugation on Wall groups implies that the geometric surgery groups defined in Wall Chapter 9 do not have the naturality Wall claims for them. They fixed the problem. The observation here is that the definition of geometric Wall groups involves homology with local coefficients and these also lack Wall's claimed naturality. One would hope that a geometric bordism theory involving non-orientable manifolds would enjoy the same naturality as that enjoyed by homology with local Z coefficients. A setting for this naturality entirely in terms of local Z coefficients is presented in this paper.

Applying this theory to the example of non-orientable Wall groups restores much of the elegance of Wall's original approach. Even manifolds: A $4k$-dimensional, oriented manifold is even if the intersection form on the integral homology has all squares even. There is a condition on the tangent bundle which is equivalent to even and following Lashof we can study the resulting structures on bundles. Several corollaries will be given including computing the resulting bordism groups in terms of more classical ones. The 4-manifold case is especially interesting. Pin structures on surfaces This note records some results about pin^- structures on surfaces that probably should have been included in Kirby-Taylor. The action of the symplectic group is described using quadratic enhancements. The quadratic enhancement vanishes on the Lagrangian determined on a boundary is proved as well as a bit more. The quadratic enhancement on a dual to w_2 in an oriented 4-manifold vanishes on the image of H^1 is proved.

Unlu, Ozgun, McMaster University: Free actions of extraspecial p-groups on products of spheres.

Let p be an odd regular prime, we showed that the extraspecial p-group of order p^3 and exponent p acts freely and smoothly on two equidimensional spheres. We also discussed the problem for p-groups of larger order and give some partial results. (Joint work with Ian Hambleton.)

Vogtmann, Karen, Cornell University: Outer Spaces of Right-angled Artin groups.

Right-angled Artin groups form a bridge between free groups and free abelian groups, and hence their outer automorphism groups can be thought of as interpolating between $Out(F_n)$ and $GL(n, Z)$. The group $Out(F_n)$ is the group of symmetries of Outer space, a space of actions of F_n on trees, and $GL(n, Z)$ is a group of symmetries of the homogeneous space $GL(n, R)/O(n, R)$, which can be described as a space of actions of Z^n on R^n. We define an outer space for the outer automorphism group of a right-angled Artin groups G, in the case when the associated graph is connected and has no triangles, as a space of actions of G on appropriate objects. We proved that this space is finite-dimensional and contractible, that the action is proper, and we give upper and lower bounds on the virtual cohomological dimension of the outer automorphism group.

Wahl, Nathalie, University of Copenhagen: Stabilizing mapping class groups of 3-manifolds.

(joint work with Allen Hatcher) Let M be a compact, connected 3-manifold with a fixed boundary sphere $\partial_0 M$. For each prime manifold P, we consider the mapping class group of the manifold M_n^P obtained from M by taking a connected sum with n copies of P. We prove that the$i$th homology of this mapping class group is independent of n in the range $n > 2i + 1$. Our theorem moreover applies to certain subgroups of the mapping class group and include, as special cases, homological stability for the automorphism groups of free groups and of other free products, for the symmetric groups and for wreath products with symmetric groups.

Williams, Bruce, University of Notre Dame: A Parametrized Signature Theorem with Converse.

(Joint work with Michael Weiss) Suppose X^n is an oriented n-dim Poincare complex. If $4|n,$ the signature of $X, \sigma(X) \in$ **Z** is defined using symmetric structure on $H_{\frac{n}{2}}(X)$. If X is a manifold, then Hirzebruch showed $\sigma(X)$ has a "local description" in terms of Pontrjagin classes. This follows from the index theorem applied to the signature operator. By using the symmetric structure on $C(\tilde{X})$, the cellular chain complex of the universal cover of X, Ranicki defined the (visible) symmetric signature of X, $\sigma_V(X)$ which is a refinement of $\sigma(X)$. He proved that when $n > 4$, X is homotopy equivalent to a topological manifold if and only if $\sigma_V(X)$ has a local description in terms of a symmetric L-theory fundamental class for X. If $p\colon E \to B$ is a fibration with fibers n-dim Poincare complexes, then p has a parametrized (visible) symmetric signature, $\sigma_V(p)$. If p is a topological fiber bundle with closed n-dim fibers, then $\sigma_V(p)$ satisfies a certain fiberwise index theorem. In this talk I'll describe a further refinement $\sigma_{VA}(p)$ of $\sigma_V(p)$. We again get a family index theorem, but we also get a converse when $\dim B < n/3$, B is path connected, and $p^{-1}(b)$ is homotopy equivalent to a smooth manifold for some $b \in B$. Then the fibration p satisfies our signature family index theorem if and only if p is fiber homotopy equivalent to a fiber bundle with fibers closed n-dim manifolds.

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Chapter 9

North American Workshop on Tropical Geometry (07w5055)

Mar 04 - Mar 09, 2007

Organizer(s): Ilia Itenberg (University of Strasbourg), Grigory Mikhalkin (University of Toronto), Yan Soibelman (Kansas State University)

Overview of the Field

Tropical Geometry is a branch of Geometry that has appeared just recently. Formally, it can be viewed as a sort of Algebraic Geometry with the underlying algebra based on the so-called tropical numbers. The tropical numbers (the term ”tropical” comes from Computer Science and commemorates Brazil, in particular a contribution of the Brazilian school to the language recognition problem) are the real numbers enhanced with negative infinity and equipped with two arithmetic operations called tropical addition and tropical multiplication. The tropical addition is the operation of taking the maximum. The tropical multiplication is the conventional addition. These operations are commutative, associative and satisfy the distribution law.

It turns out that such tropical algebra describes some meaningful geometric objects, namely, the Tropical Varieties. From the topological point of view the tropical varieties are piecewise-linear polyhedral complexes equipped with a particular geometric structure coming from tropical algebra. From the point of view of Complex Geometry this geometric structure is the worst possible degeneration of complex structure on a manifold. From the point of view of Symplectic Geometry the tropical variety is the result of the Lagrangian collapse of a symplectic manifold (along a singular fibration by Lagrangian tori).

The easiest to describe are tropical varieties in dimension 1, i.e. tropical curves. These are the so-called ”metric graphs”, i.e. finite graphs equipped with an inner metric such that all ”leaves”, i.e. the edges adjacent to 1-valent vertices have infinite length. There is a finite-dimensional moduli space of tropical curves once we fix the number of cycles in the graph (this is the tropical counterpart of the genus) and the number of leaves (this is the tropical counterpart of the number of punctures). Such a moduli space is itself a tropical orbifold and there is a certain intersection theory on it.

From the point of view of Toric Geometry tropical varieties are limiting shapes of the amoebas of algebraic varieties under the deformation degenerating the argument torus. Such degeneration can be described by varying the base of the logarithm in the amoeba map to infinity. In toric geometry such construction is known as ”the patchworking”, it was introduced by O. Viro in 1979 for the needs of real algebraic geometry to give a way to construct real forms of complex algebraic varieties with controlled topology. Historically this was perhaps the first time of implicit appearance of tropical geometry. Since this appearance there were several discoveries and proposals that stirred the research related to the area, most notably the introduction of amoebas by I. M. Gelfand, M. Kapranov and A. Zelevinski (and, in particular, the introduction of non-Archimedean amoebas by Kapranov), the proposal to use tropical curves in the context of Mirror Symmetry (particularly, for computation of the Gromov-Witten invariants) by M. Kontsevich, the introduction of the Morse category by Fukaya and the introduction of tropical formalism to Computational Algebraic Geometry by B. Sturmfels. By now there exist distinct points of view on Tropical Geometry from different areas of Mathematics.

Recent Developments and Open Problems

Tropical Geometry already proved to be useful in quite distinct areas of Mathematics. By now it has applications in Real Algebraic Geometry, Enumerative Geometry, Mirror Symmetry, Symplectic Geometry and Computational/Combinatorial Geometry. The list of its applications keeps growing. E.g. most recently, Tropical Geometry had a brand-new appearance in the Statistical Physic work of R. Kenyon and A. Okounkov where they studied mathematical model for dimers accumulation. Currently there are several research groups around the globe that are doing active research in Tropical Geometry from somewhat different points of view.

The main developments so far concern applications of tropical curves. This is currently the most well-understood case of tropical varieties. Many open problems concern higher-dimensional case, in particular, tropical surfaces.

Many new developments in Tropical Geometry were presented during the workshop. These developments are described in the following section.

Presentation Highlights

Hannah Markwig gave a talk entitled ”*The j-invariant of a plane tropical cubic*” (joint work with Eric Katz and Thomas Markwig). Several results relate the j-invariant of an elliptic curve to the cycle length of a tropical elliptic curve. E. Katz, H. Markwig, and T. Markwig proved the following theorem which can be seen as one of the justifications of the fact that the cycle length is the tropical counterpart of the j-invariant. Given a plane cubic over the field of Puiseux series such that the tropicalization of the cubic has a cycle (and is dual to a triangulation), the cycle length is equal to the negative of the valuation of the j-invariant. As a corollary, one obtains that the tropicalization of a cubic whose j-invariant has positive (negative in another widely-used convention) valuation does not have a cycle. Possible generalizations of the theorem (for example, for smooth elliptic curves in other toric surfaces) and connections with bad reduction of elliptic curves over discrete valuation rings are subjects to study.

Takeo Nishinou gave a talk entitled ”*Counting problems in tropical geometry*”. The talk was devoted to a tropical count of holomorphic discs with certain Lagrangian boundary condition in a toric variety. This work is presented in [11] and can be viewed as a relative version of the count of tropical closed curves. In particular the Lagrangian was assumed to be presented by a fiber in the toric fibration. Accordingly, the corresponding tropical curve “with boundary” was allowed to have a 1-valent vertex at a finite distance (i.e. of sedentarity 0).

Eric Katz gave a talk entitled ”*Equivariant cohomology and localization in tropical geometry*” (joint work with Sam Payne, see [7]). E. Katz and S. Payne use localization to describe the restriction map from equivariant Chow cohomology to ordinary Chow cohomology for complete toric varieties in terms of piecewise polynomial functions and Minkowski weights. They computed examples showing that this map is not surjective in general, and that its kernel is not always generated in degree one. They prove a localization formula for mixed volumes of lattice polytopes and, more generally, a Bott residue formula for toric vector bundles.

Askold Khovanskii gave a talk entitled ”*Elimination theory and Newton polyhedra*” (joint work with A. Esterov, see [3]). The goal of elimination theory is to describe, for an algebraic variety $X \subset \mathbf{C}^n$ and a projection $\pi : \mathbf{C}^n \to \mathbf{C}^m$, the defining equations of $\pi(X)$ in terms of the equations of X. Let a variety $X \subset (\mathbf{C}^*)^n$ be defined by equations $f_1 = \ldots f_k = 0$ with given Newton polytopes and generic coefficients. Assume that $\pi(X) \subset (\mathbf{C}^*)^m$ is a hypersurface given by an equation $g = 0$. A. Esterov and A. Khovanskii describe the Newton polytope and the leading coefficients (that is, the coefficients of monomials which are on the boundary of the Newton polytope) of the Laurent polynomial g in terms of the Newton polytopes and the leading coefficients of the Laurent polynomials $f_1, \ldots, f_k$. Several problems related to Newton polytopes and tropical geometry are particular cases of this version of elimination theory. This work is directly related to the subject of the paper [21].

Eugenii Shustin gave a talk entitled ”*Recursive formulas for Welschinger invariants of real Del Pezzo surfaces*”. The Welschinger invariants are designed to bound from below the number of real rational curves passing through a given generic real collection of points in a real variety. In some cases these invariants can be calculated using Mikhalkin's approach which deals with a corresponding count of tropical curves. As is known, in certain situations (for example, in the case of generic collections of real points on a toric Del Pezzo surface equipped with the tautological real structure, see [6], or in the case of generic collections of points in the three dimensional real projective space, see [2]), there is a logarithmic equivalence between the Welschinger invariants and the corresponding genus zero Gromov-Witten invariants. I. Itenberg, V. Kharlamov, and E. Shustin consider generic collections of real points on the projective plane blown up at 4 real points in general position and, using appropriate tropical Caporaso-Harris type formulas, prove that the logarithmic equivalence of the Welschinger and Gromov-Witten invariants holds in this situation as well. More precisely, they prove the following statement. *Let D be an ample divisor on the projective plane blown up at 4 real points in general position; then the Welschinger invariant W_D is positive, and*

$$\log W_{nD} = \log GW_{nD} + O(n),$$

where GW_{nD} stand for genus zero Gromov-Witten invariants. The proof is based on a new version of the correspondence theorem.

The talk of Valery Alexeev was of a survey nature and dealt with several classical algebro-geometric points of view on tropical geometry. These ways include passing to one-parametric deformations as well as the log-geometry. A special emphasis was made on the case of moduli spaces of curves and Abelian varieties. Interestingly enough, in the recently constructed compactification of the moduli space of Abeian varieties (due to the speaker [1]) tropical Abelian varieties appear naturally as the boundary strata. This comes as a special case of a more general principle that exhibits tropical geometry as the boundary of complex geometry. Here the boundary can be interpreted either as the limit in the one-parametric families or as the log-geometry boundary.

Richard Kenyon gave an elegant talk presenting an application of tropical geometry to the geometry of statistical models, in particular, of the dimer model. The talk is based on his joint work with Andrei Okounkov [9]. They considered the dimer model on a planar hexagonal lattice with the edges weighted by a periodic (more precisely, doubly periodic according to the lattice) function. For each (finite) size of the mesh of the lattice we have several configurations with different probabilities. However, in the limit when this size goes to zero there is a unique configuration with 100% probability.

In the absence of boundary conditions this configuration is parameterized by a certain Harnack curve ([8]). (More precisely its so-called height function coincides with the so-called Ronkin function of the corresponding amoeba). The degree of the curve is determined by the period of the weights while the coefficients are determined by the weights themselves.

In the presence of the boundary condition the situation is much more interesting, particularly in the case when the boundary is a broken line with the three possible slopes in the real plane. The limiting configuration has the so-called frozen boundary which comes as the log-front of the Harnack curve responsible for the periodic weights and another curve, responsible for the boundary conditions, see [9]. The degree of the second curve depends on the number of chains in the broken line boundary. The geometry of the second curve is in a sense antipodal with respect to the geometry of the first curve.

There is also a “temprerature” parameter in this statistical model. Ironically, tropical configurations in this terminology correspond to the zero temperature (the most regular case). The limiting configuration is easy to find here and then one can trace the change of the situation when we increase the temperature.

An interesting application to classical geometry concerns finding a rational curve of degree d inscribed in a $3d$-gon in the plane that comes as a projection of a closed broken line in the 3-space. Here we assume that the sides of this $3d$-god are parallel to particular 3 directions in the plane that are projections of the coordinate axes with the kernel of the projection parallel to the vector $(1, 1, 1)$. By a projective duality consideration there are as many curves tangent to the lines extending the sides of the 3-gon as the number of rational curves of degree d passing via $3d - 1$ points, which is a rather large number. Nevertheless, there is a unique curve that is geometrically inscribed to the polygon in the case when the polygon is *tilable*, in other words if there exists a corresponding tropical inscribed curve.

The talk of Kristian Kennaway was devoted to the aspects of Tropical Geometry motivated by Physics, see [4]. The mathematical construction presented starts from the consideration of the so-called *alga* of a planar curve, i.e. its image in the argument torus under the argument map. It can be shown that in many cases the resulting set has a certain 3-valent graph as its deformation retract. The complement of this graph has d^2 hexagons and the graph itself has $2d^2$ vertices and $3d^2$ edges. One can do a twist along a ribbon in the neighborhood of each edge. This transformation produces a surface (usually of positive genus) with $3d$ boundary components. If we glue the boundary components with disks we get a surface of genus $\frac{(d-1)(d-2)}{2}$ that may be considered as an adjoint curve. Its geometry is intrinsically related to the geometry of the original curve.

The talk of Nikolai Mnev was devoted to combinatorial description of the Grassmannian variety $G_{n,k}$ [15], particularly to the relatively little-understood case $k > 2$. If $k = 2$ the case is well-understood thanks to the relation (up to a torus action or, more precisely, the Chow quotient in the sense of Kapranov) of $G_{n,2}$ to the moduli space $\overline{\mathcal{M}}_{0,n}$. In the general case one needs much stronger combinatorial tools.

Brett Parker presented his theory of "Exploded Fibration" [17]. This theory may be considered as a partial tropicalization. Namely, in this theory we do not have to tropicalize a variety completely, but may leave a part (or several parts) of it as is. We still have to tropicalize the junctions between such parts and this tropicalization works as usual and allows one to compute the curves in the initial variety if we know the curves in the non-tropicalized parts.

Mikael Passare gave a survey of the theory of *co-amoebas* (or *algae*) [20], i.e. the argument projections of complex algebraic and analytic varieties. It shares many features with the theory of amoebae, but it gives an essentially new point of view on complex varieties. As it was shown in the talk an elegant application of this theory gives a new computation of the value of the Riemann ζ-function at 2, see [18]. Namely, as it was shown in the work of Passare and Rullgård [19] the area of the amoeba of a plane curve can be computed in terms of the so-called *Monge-Ampère measure*. The latter can be computed by means of the area of the Newton polygon of the curve (the proportionality coefficient is π^2, where the square is responsible for dimension 2 of the ambient plane). The $\zeta(2)$ computation comes from integration of the Taylor expansion of the function that gives one of the three arcs in the boundary of the amoeba of the line. The area under (or raher over as the corresponding function is negative) this curve is one third of the total area of the amoeba of the line or $\frac{\pi^2}{6}$.

The area of the algae coincides with the area of the amoeba, so we have the same estimate for the area of alga. This area accumulates the easiest in the case of Harnack curves, see [10], [14], [8], [13] as this is the case when the argument map is birationally (from the point of view of real geometry) a covering of degree d^2. In the talk many other interesting examples were considered, particularly the amoebae and algae of hypergeometric functions of many variables.

The talk of Andrei Losev presented a quantum mechanics theory based on tropical geometry. He argued that passing from complex geometry to tropical geometry may be interpreted as passing from the quantum field theory to quantum mechanics. In the presented mathematical theory the states of the particles are enhanced with the slopes (corresponding to the slopes of the corresponding tropical rational functions on the tropical line) and the operators of changing the slopes were introduced. The corresponding correlators then can be interpreted as tropical Gromov-Witten invariants.

Andrei Zelevinsky discussed tropical aspects of cluster algebras. He presented three different points of view on cluster algebras, introduced in his joint papers with Sergey Fomin. One starts (in a simplest case) with a skew-symmetric integer-valued matrix and associated group of variables. The operation of mutation changes the matrix as well as the variables. The mutation is a birational operation on variables. Cluster algebra is an associative algebra generated by all mutated variables. The relation to tropical geometry comes out from the observation that the whole theory can be developed over an arbitary semifield. Zelevinsky also discussed the problem of constructing "canonical bases" in cluster algebras and relation of this problem to recent papers by Fock and Goncharov on Langlands duality for cluster varieties.

The talk of Yan Soibelman was devoted to his joint project with Maxim Kontsevich in which they study Donaldson-Thomas invariants for 3-dimensional Calabi-Yau varieties (possibly non-commutative). The whole subject can be roughly described as a counting problem of stable objects in a Calabi-Yau category endowed with a stability structure (e.g. counting of special Lagrangian submanifolds in a $3d$ Calabi-Yau manifold). As the stability structure changes (e.g. as we move in a complexified Kähler cone toward infinity) the (properly defined) number of stable objects can change as one crosses some "wall" of real codimension one. This change of numbers is described by new "wall-crossing formulas". The moduli space of stability structures resembles tropical hyperkahler variety (a.k.a. skeleton of the corresponding maximally degenerate Calabi-Yau variety introduced in the earlier paper by Konstevich and Soibelman). Wall-crossing formulas for non-commutative $3d$ Calabi-Yau varieties generated by a spherical collection (e.g. vanishing Lagrangian spherical cycles in the geometric case) are described by quivers with potentials, and the wall-crossing formulas give rise to cluster transformations.

Vladimir Berkovich discussed his approach to tropical geometry treated as geometry of analytic spaces over the field $\mathbf{F}_1$ of one element. The latter can be thought of as a monoid with two elements $0, 1$ with operation of multiplication. Then $\mathbf{F}_1$-algebras are commutative monoids. One can mimick Berkovich approach to the theory of analytic spaces, which is based on the notion of Berkovich spectrum of a commutative Banach ring. The spectrum is a compact Hausdorff space of bounded multiplicative seminorms. In the non-archimedean case one can glue spectra of affinoid algebras into more complcated spaces (analytic spaces). In the case of the theory over $\mathbf{F}_1$ there is are analogs of affinoid algebras, and their spectra are PL-spaces equipped with sheaves of affine functions, i.e. tropical spaces. Gluing procedure and hence the global theory of such $\mathbf{F}_1$-analytic spaces had not been developed by the time of the workshop.

Talk by Yong-Geun Oh was devoted to Seidel long exact sequence for Floer homology. Floer homology for a pair of Lagrangian submanifolds are not always well-defined. Aim of his talk was to construct certain exact triangle in the derived Fukaya category of a general Calabi-Yau manifold (previously Seidel considered the case of exact Lagrangian submanifolds). More precisely, let L be an exact Lagrangian sphere (parameterized, i.e. the diffeomorphism with S^n is chosen) in a compact sympectic manifold with contact boundary. With such data one can associate the Dehn twist τ_L. Then for any two exact Lagrangian submanifolds L_0, L_1 Seidel constructed an exact triangle of Floer homology groups with the vertices $HF(\tau_L(L_0), L_1)$, $HF(L_0, L_1)$, $HF(L, L_1) \otimes HF(L_0, L)$. Y.-G. Oh described which technical difficulties one should overcome (and how) in order to generalize this result to arbitrary Calabi-Yau manifolds.

Scientific Progress and Outcome of the Meeting

The main outcome of the meeting was in new collaborations of people from different areas of science and different background ranging from Combinatorics to String Theory. The same tropical phenomena appear in different areas in very recent research, so it is very important to set up a uniform terminology and language and it is not less important for researchers in one area to learn the developments that appear in other areas. Grigory Mikhalkin gave an introductory talk based on [12] and [11] with the definition of tropical varieties and other basic notions (such as tropical modifications). The talks were scheduled thematically each day and were followed by informal discussions that allowed experts in one particular area to understand tropical developments in other areas. (The relevant areas include Combinatorics, Algebraic Geometry, Symplectic Geometry, Complex Analysis and Physics, many of these areas can be considered as the parent fields of Tropical Geometry.) It was crucial for the informal discussions that they were guided by such prominent experts in these fields as Eliashberg, Hori, Khovanski, Viro and Zelevinski. In addition, recent developments in the areas bordering Tropical Geometry (Amoebae, Algae, Patchworking, etc) were also discussed formally and informally. Several cross-area collaborations started during the conference.

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Chapter 10

Mathematical developments around Hilbert's 16th problem (07w5021)

Mar 11 - Mar 16, 2007

Organizer(s): Christiane Rousseau (Université de Montréal)

Overview of the Field

In his famous lecture of the 1900 International Congress of Mathematicians, David Hilbert stated a list of 23 problems with deep significance for the advance of mathematical science. There has been intensive research on these problems throughout the 20th century. Hilbert's 16th problem called "Problem of the topology of algebraic curves and surfaces" is one of the few problems which is still completely open. This problem has two parts. The first part asks for the relative positions of closed ovals of an algebraic curve given by the set of points which are solutions of a polynomial equation $P(x, y) = 0$. The maximum number was given by Harnack. As for the relative positions, even if this is a purely algebraic problem, there has been little progress on the general case, while there is progress for small values of the degree of the polynomial P (degree less or equal to 7). The workshop "Mathematical developments around Hilbert's 16th problem", held in BIRS on March 12-16 2007, focused on the second part of the problem which is a problem in ordinary differential equations, but with the components of the vector field given by polynomials, or equivalently by an algebraic Pfaff form. The second part of Hilbert's 16th problem asks for the maximal number $H(n)$ and relative positions of limit cycles of planar polynomial (real) vector fields of a given degree n. This problem, opened for more than a century, has been at the center of many developments in differential equations. The main difficulty of Hilbert's problem is that, although a polynomial vector field is an algebraic object, its trajectories are not algebraic. In the neighbourhood of singular points they may not even be analytic. The fascination of Hilbert's 16th problem comes from the fact that it sits at the confluence of analysis, algebra, geometry and even logic.

As mentioned above, Hilbert's 16th problem, second part, is completely open. It was mentioned in Hilbert's lecture that the problem "may be attacked by the same method of continuous variation of coefficients... ". Even if the problem was stated as early as 1900 it was only in 1987 that Ecalle and Ilyashenko proved independently that a polynomial vector field has a finite number of limit cycles. Both proofs are a real "tour de force" and each requires a 300 pages volume. The idea is to compactify the phase space to the Poincaré disk. Then, as limit cycles are isolated periodic solutions, if there were an infinity of them, they would need to accumulate on a graphic (also called polycycle). By blowing up the singularities it is possible to limit oneself to graphics with hyperbolic or semi-hyperbolic (one nonzero eigenvalue) singularities. For such a graphic one studies the return map in the neighborhood of the graphic and shows that it is not oscillating.

While the result of Ecalle and Ilyashenko shows that each individaul polynomial vector field has a finite number of limit cycles it is impossible to derive from it any uniform estimate on the number of limit cycles. Approaches to an accurate estimate $H(n)$ have come from two sides. On the one hand there is the construction of bounds from below. The best lower bound is of the order $H(n) \geq Cn^2 \ln n$. Such a bound was found by Christopher and Lloyd [3] through the construction of polynomial vector fields of degree n with such a number of limit cycles. This is done with a mixture of bifurcation methods and use of symmetries. So far, the only approaches from above are of the form $H(n) < \infty$ also called "Existential part of Hilbert's 16th problem" or also "Finiteness part of Hilbert's 16th problem". An important contribution is this direction is the program started in 1991 by Dumortier-Roussarie-Rousseau [7] and reducing the proof that $H(2) < \infty$ to the proof that 121 graphics have finite cyclicity. The program followed an idea of Roussarie [38] that, compactifying both the phase space and the space of coefficients, the global finiteness would follow from local finiteness. Indeed limit cycles can only accumulate in the product of phase space times parameter space on limit periodic sets. It hence suffices to show that each limit periodic set has finite cyclicity (i.e. can give rise to a finite number of limit cycles) inside the family of vector fields for which finiteness is expected. This idea of Roussarie is very powerful and has given rise to intense research on developing methods to prove the finite cyclicity of limit periodic sets.

There exists many variants of Hilbert's 16th problem. Several of them consists in addressing Hilbert's question for a simpler class of polynomial equations. In all cases the sub-questions mentioned above are also considered namely bounds from below for the number of limit cycles and also the finiteness problem. Among the subfamilies considered are Abel equations, classical Liénard equations and generalized Liénard equations.

A very important variant of Hilbert's problem is the "tangential" or "infinitesimal part" of Hilbert's 16th problem. This problem is related to the birth of limit cycles by perturbation of an integrable system with an annulus of periodic solutions. Under the perturbations usually only a finite number of periodic solutions remain. When the integrable system is Hamiltonian, then the number of limit cycles appearing in a small perturbation of the system is obtained by counting the number of zeroes of Abelian integrals, at least as long as one remains far from polycycles. The subject is very active and has been covered and discussed widely during the workshop.

It is more than a century that Hilbert's problem was stated and we do not yet know if there exists a uniform upper bound for the number of limit cycles of degree n. Specialists tend to believe that such a bound exists and all efforts are in this direction. It is tempting here to mention Khovanskii's theory of fewnomials. Indeed the essence of Hilbert's problem is that, although the trajectories of a polynomial vector field are not algebraic, the algebraic nature of the system should leave its footprints and imply finiteness properties. The theory of fewnomials applied to Pfaff forms is remarkable in that aspect. It proves non-oscillating properties of separating solutions of Pfaff forms, which in itself is a spectacular demonstration that the phase curves "know" they have some kind of algebraic nature. Moreover the method of Khovanskii is extremely powerful for proving existence of upper bounds on the number of solutions of equations or systems of equations.

The developments around Hilbert's 16th problem and those on complex dynamics, in particular Fatou and Julia sets, had long developed in parallel, although they share some common problematics and methods of solutions. On purpose, researchers representing both groups had been invited together. In particular Adrien Douady had been invited to the workshop. He passed away between the time that the workshop was planned and the effective time of the workshop. His heritage is immense and his far reaching ideas have had an influence far outside his own field of complex dynamics. In particular it is his idea and the thesis of Lavaurs which made possible to identify the space of modules for unfoldings of parabolic points.

Theme and plenary lectures

The workshop brought together researchers making significant contributions to domains of differential equations related to Hilbert's 16th problem. The focus was put on the following subjects:

(i) singularities of differential equations and complex foliations, and related normal forms,

(ii) bifurcations of differential equations and finite cyclicity problems,

(iii) algebro-geometric techniques in differential equations.

In order to present an overview of the most significant directions the workshop started with invited lectures given by Robert Roussarie, Pavao Mardešič, Vadim Kaloshin, Jean-Christophe Yoccoz and Abdelraouf Mourtada with the purpose of describing the principal breakthroughs or main directions in the subject.

The lecture of **Robert Roussarie** enlightened the importance of the study of singular perturbed systems in the study of Hilbert's 16th problem. A brilliant demonstration of this is given in the recent paper [6] of Dumortier–Panazzolo–Roussarie where they give a counter-example to the celebrated conjecture of Lins Neto-Pugh-de Melo on the number of limit cycles of classical Liénard equations stating that a classical Liénard system of degree $2n$ or $2n+1$ has at most n limit cycles. As this conjecture was cited by Smale in his list of problems at the beginning of the twentieth century, it is also called Smale's conjecture. The study of singular perturbed systems is likely to bring new conjectures on the maximum number of limit cycles for polynomial vector fields. Indeed, it is possible by perturbation of such systems to create exponentially small regions in parameter space in which we observe more limit cycles than those which can be created by the standard bifurcation techniques.

The expository lecture of **Pavao Mardešič** focused on techniques and difficulties of what is called the infinitesimal and tangential Hilbert 16-th problem, namely the number of zeroes of Abelian integrals and the number of limit cycles that can be created by perturbing a Hamiltonian vector field. Pavao Mardešič distinguished between the "tangential Hilbert 16-th problem" which is strictly limited to the study of the number of zeroes of Abelian integrals and the "infinitesimal Hilbert 16-th problem" which is concerned with the number of limit cycles that can be created by perturbation of a polynomial Hamiltonian system. The lecture started with the celebrated finiteness result of Varchenko-Khovanskii, stating for any integer N the existence of a uniform bound for the number of zeroes of Abelian integrals of forms of a degree $\leq N$ over ovals of a Hamiltonian of degree $\leq n$. The idea of the proof is to use the Picard-Fuchs equations satisfied by Abelian integrals and to finish the proof by a special study in the neighborhood of the polycycles. Among the recent significant generalizations we find the results of Mourtada and Novikov, both highlights of this workshop. An important property of Abelian integrals is the Chebychev property which allows to give a bound on the number of zeros of the Abelian integrals based only on the dimension of the vector space of these. A geometric explanation of the Chebychev properties of elliptic envelopes was given, coming from the fact that the Picard-Fuchs is 2-dimensional because the homology group is 2-dimensional. The generalization for Hamiltonians of higher was discussed with known examples and counter-examples and open problems.

To make the link with algebraic dynamical systems **Jean-Christophe Yoccoz** lectured on the recent results of Buff and Cheritat on the geometry and size of Siegel disks of quadratic polynomials which allow them in particular to find parameters for which the corresponding Julia set has positive Lebesgue measure. He concentrated on the case where the multiplier λ of the quadratic polynomial $P(z) = \lambda z + z^2$ is of the form $\lambda = \exp(2\pi i\alpha)$ where α is a Liouvillian irrational number and he showed the limiting process which leads to the existence of a Julia set with positive Lebesgue measure.

The lecture of **Aldelraouf Mourtada** summarized his immense work spread over several years to prove the finite cyclicity of hyperbolic polycycles in compact families of analytic vector fields on the sphere S^2, including the introduction of several algebras of quasi-analytic functions and the use of the theory of fewnomials of Khovanskii. The work was started many years ago with the generic case of an attracting or repelling polycycle. The proof sketched at the workshop included the case where the polycycle is an accumulation of cycles (it is the boundary of an annulus of periodic solutions). The ideas of this proof are then used to extend the result of Khovanski-Varchenko about Abelian integrals, to the neighbourhood of hyperbolic polycycles. And this gives rise to the following general result: let H be a Morse polynomial of degree $d+1$ which is generic at infinity (but maybe with multiple critical values). Then there exist a number $N(d)$ (depending only on d), such that every perturbation of dH (of degree d and with non vanishing Abelian integrals) has at most $N(d)$ limit cycles on the real plane.

The lecture of **Vadim Kaloshin** reported on recent breakthroughs in the restricted or planar 3-body problem. Among the problems discussed where the Hausdorff dimension of oscillatory motions, and the Arnold diffusion. In private conversations, Vadim Kaloshin also discussed recent ideas on embedding planar polynomial vector fields in Hamiltonian vector fields of dimension 4 with the hope of using Floer homology to obtain new lower bounds for $H(n)$.

Presentation highlights

The role of singular perturbations in Hilbert's 16th problem

After more than a century of work on Hilbert's 16th problem and long lasting conjectures on the maximum number of limit cycles for some special classes of polynomial systems it became very clear only recently that the study of slow-fast systems is going to be one of the keys in the estimation of uniform upper bounds for the number of limit cycles. Robert Roussarie explained the recent example of Daniel Pannazolo of a classical Liénard system providing a negative answer to Smale's system. What is particularly remarkable in that case is that no "classical method" allows to track this polynomial system with 5 limit cycles where only 4 were expected by the conjecture. It gives a new light on the long standing conjecture that $H(2) = 4$. And it makes the link with the remark of Joan Carlos Artés, Jaume Llibre and Dana Schlomiuk [1] that a corner of the bifurcation diagram of quadratic systems is a very degenerate slow-fast system.

The role of the study of singularities in Hilbert's 16th problem

A group of lectures dealt with the study of singularities of analytic vector fields. This study is one of the most basic and fundamental part of the subject as the singularities are the organizing centers of the foliations. Among the study of singularities lies the problem of the center, which is fundamental in the context of Hilbert's 16th problem. The lecture of Emmanuel Paul sat in that context, where he discussed the Galoisian reducibility for a germ of quasi-homogeneous foliation.

Singularities of Fuchsian systems. The lecture of Caroline Lambert illustrated the link between the unfoldings of the confluent hypergeometric equation and that of the Riccati equation unfolding a saddle-node. In this lecture she showed how to cover all parameter values in the confluence from the hypergeometric equation to the confluent hypergeometric equation. In particular she was the first to identify the parametric resurgence phenomenon in this context and explain it. She could completely calculate the unfolding of the Martinet-Ramis modulus for a Riccati equation unfolding a saddle-node of codimension 1. Rodica Costin discussed the linearization of nonlinear perturbations of Fuchsian systems.

Moduli of analytic classification of families unfolding resonant singularities. This was discussed by the group of lectures of Loïc Teyssier (modulus space for germs of families unfolding saddle-nodes of codimension k), Javier Ribon (analytic classification of unfoldings of resonant diffeomorphisms) and Christiane Rousseau (analytic classification of unfoldings of resonant diffeomorphisms and moduli spaces in the codimension 1 case). The moduli of analytic classification of resonant singularities have been a tool in Ecalle's proof that limit cycles of an analytic vector field cannot accumulate on a polycycle. Later. in the paper [5]. the Martinet-Ramis modulus of a saddle-node was used as a tool to prove the finite cyclicity of some graphics of the DRR program [7]. While these graphics were generic, it became clear that to tackle the same kind of questions for graphics which can produce an annulus of periodic solutions, then it was necessary to control the behaviour in the parameter to be able to control the number of limit cycles which can appear in a perturbation. Christiane Rousseau presented her joint work with Colin Christopher where they determine the space of moduli of germs of generic 1-parameter families unfolding a diffeomorphism with a parabolic fixed point. It is the first time that a space of moduli can be determined for a **family** of dynamical systems. In the same spirit the lecture of Loïc Teyssier explored the higher codimension case for unfolding of saddle-node vector fields. The description was based on the decomposition of a neighborhood of the singularities in sectors over which the space of leaves is given by $\mathbb{C}$. Now a complete modulus of analytic equivalence or conjugacy has been given for a family unfolding the codimension k saddle-node and all the machinery is ready for attacking the description of the space of moduli. Javier Ribon discussed a complete system of analytic invariants for a germ of one-parameter family of diffeomorphisms unfolding a diffeomorphism with a parabolic point. His results complete and extend those of [13]. In particular they are valid for any codimension and the family need not be generic. Better criteria for deciding the analytic conjugacy of two such families are given.

Topology of leaves of analytic foliations on Stein manifolds. This was the topic of the lecture of Tanya Firsova. Her theorem stated that, for a generic singular foliation of a Stein manifold, all leaves except possibly a countable number are topological disks and the rest are topological cylinders. This result is likely to open a new domain in analytic foliations.

Algebraic dynamical systems

The trend on algebraic dynamical systems and iteration of rational maps, started with the lecture of Jean-Christophe Yoccoz and was followed by the lecture of Alexey Glutsyuk presenting that the horospheric lamination of the orbit space of a rational function is topologically transitive, provided that the rational function under consideration does not belong to an explicit list of exceptions.

o-minimal structures and Hilbert's 16th problem

o-minimal structures have been studied for several years in conjunction with Hilbert's 16th problem. This comes from the fact that the properties of quasi-analytic solutions of analytic ordinary differential are well captured by the language of algebra and logic, in the same way as they can be studied by the theory of fewnomials of Khovanskii. For instance it is well known that the non-spiraling leaves of real analytic foliations of codimension 1 all belong to the same o-minimal structure. Some specialists of o-minimal sructures had to cancel their coming to BIRS. Nevertheless the subject was represented at the workshop. As a follow-up to the lecture of Abdelraouf Mourtada, the lecture of Reinhard Schäfke, discussed how non-oscillating trajectories of real analytic vector fields sit inside o-minimal structures and explained that, under certain assumptions, such a trajectory generates an o-minimal and model complete structure together with the analytic functions. The proof uses the asymptotic theory of irregular singular ordinary differential equations in order to establish a quasi-analyticity result from which the main theorem follows. An application was given of an infinite family of o-minimal structures such that any two of them do admit a common extension, and also an example of non-oscillating trajectory of a real analytic vector field in dimension 5 that is not definable in any o-minimal extension of the reals.

Extension of the Varchenko-Khovanskii theorem to the Darboux integrable case

Dmitry Novikov presented a beautiful extension of the Varchenko-Khovanskii theorem to the Darboux integrable case, yielding a bound on the number of zeroes of Abelian integrals in the latter case. The tools introduced for the proof of this result introduced new perspectives in tangential Hilbert's 16th problem as the classical tools for studying Abelian integrals do not work when we switch from a Hamiltonian system to a Darboux integrable one. In particular the Pontrjagin-Melnikov integrals in that case have no natural extension to the complex domain.

Results for particular classes of polynomial vector fields in the spirit of Hilbert's 16th problem.

A special emphasis was put on special subclasses of polynomial vector fields for which Hilbert's 16th problem is much studied.

Quadratic vector fields. They have been studied very systematically by Joan Carlos Artés, Jaume Llibre and Dana Schlomiuk for several years. They now have a complete bifurcation diagrams of quadratic systems with a weak focus of order greater or equal to 2 [1] and they start attacking the case of a weak focus of order 1. Their study of the quadratic systems with weak order of order 2 has enlightened some corners of the bifurcation diagram which they conjecture will produce the larger number of limit cycles in the family of quadratic vector fields. For their work they mix algebro-geometric techniques with numerical simulations and Joan Carlos Artés lectured on the numerous phase portraits expected for a system with a weak focus of order 1.

Classical Liénard systems. The recent work of Freddy Dumortier and Magdalena Caubergh shows the existence of a uniform explicit bound for the number of limit cycles of a classical Liénard system of degree n as long as one stays in a compact subsetof the parameter space, far from the singular perturbed system. The finiteness part of Hilbert's 16th problem for this subfamily is thus reduced to the proof of the finite cyclicity of the graphics in the singular perturbed systems.

The question of the number of critical periods of a polynomial vector field with a center is often studied in parallel and in the same spirit as the questions on the number of limit cycles of a polynomial vector field. Freddy Dumortier presented recent results with Peter De Maesschalck on the period function of the classical Liénard systems. Here again the problem of the existence of a uniform bound is reduced to the study of slow-fast systems.

Unicity of a limit cycle of a vector field and unicity of the critical period in an annulus of periodic solutions surrounding a center. A number of results in this direction were presented. Jordi Villadelprat discussed the period fonction of quadratic centers. Liénard systems were discussed Jaume Llibre. An application of results in Liénard equations to a predator-prey system was discussed in the lecture of Huaiping Zhu. Abel equations were discussed by Armengol Gasull and Rafel Prohens Sastre.

Recent Developments and Open Problems

The most important developments in the last years around Hilbert's 16th problem are the following:

The role of singular perturbations for tracking additional limit cycles.

The techniques of geometric singular perturbation theory become more and more sophisticated and permit now to treat problems where the slow manifold has a large number of local extrema and there are several breaking parameters.

In this context an open problem stated by Roussarie is the following: what can be said of the number and type of fixed points of a composition of applications of the form

$$R_i(x) = \alpha_i + x^{r_i}$$

with $x > 0$ and $\alpha_1 \in \mathbb{R}$ and $r_i > 0$? Such compositions have been studied locally by Mourtada in the case where $\alpha_i \sim 0$ and for x close to 0. The new global problem appears naturally in singular perturbations problem. Although very simple to state, very little is known on this problem. It seems easier to expect lower bounds, but any result on bounds from below or bounds from above would be welcome.

Several finiteness problems for subfamilies of polynomial vector fields in the spirit of Hilbert's 16th problem have been reduced to conjectures on slow-fast systems.

While the methods are becoming more sophisticated it remains the case that precise results on the number of limit cycles are much harder to get when there is more than one limit cycle. This comes from the fact that the analysis must be pushed to a much finer level in order to be able to conclude

The recent progress in the tangential (infinitesimal) Hilbert's 16th problem.

The generalization by D. Novikov of the finiteness result of Varchenko-Khovanskii for zeroes of Abelian integrals to the case of a perturbation of an integrable foliation of Darboux type is a real breakthrough. Varchenko-Khovanskii's theorem states for any integer N the existence of a uniform bound for the number of zeroes of Abelian integrals of forms of a degree $\leq N$ over ovals of a Hamiltonian of degree $\leq n$. The generalized theorem by Novikov states the existence of such a bound when one replaces a Hamiltonian foliation by an integrable foliation of Darboux type. Such a foliation has a first integral of the form $\prod P_i^{\alpha_i}$, where the P_i are bivariate polynomials such that the algebraic curves $P_i = 0$ are invariant for the foliation. The existence of the uniform bound depends only on the degree of the rational form defining the foliation and the degree of the pertubation. The difficulty of the proof comes from the fact that the Abelian integrals do not satisfy any more a Picard-Fuchs equation. Also the level curves of the first integral are no more nice Riemann surfaces on which it is possible to extend the Abelian integrals. The method used for the proof is a clever application of Khovanskii theory. As in the case of the Khovanskii-Varchenko theorem it uses a special study of the asymptotic expansion of the Abelian integrals in the neighborhood of the polycycles. One difficulty is that these extensions can have small denominators. However a solution can be found using the fact that the singular points in the corners of the polycycles are integrable.

This result opens the new field of the study of Abelian integrals appearing in the perturbations of Darboux integrable systems in several directions. The bound obtained by Novikov is not explicit. Also it is only obtained under generic conditions on the first integral. Two natural generalizations are in the direction of obtaining explicit bounds under more precise conditions and to generalize to the limit cases of generalized Darboux integrable systems: these occur when two or more algebraic invariant curves coallesce.

The first space of modulus for a germ of analytic family of vector fields unfolding a resonant singularity is identified.

While it is already known for some years that the Ecalle-Voronin modulus of a germ of diffeomorphism with a parabolic fixed point or the Martinet-Ramis modulus of a saddle-node can be unfolded to yield a modulus for the unfolded system, the dependence of the unfolded modulus on the parameter was completely open. This came from the fact that no construction of this modulus could make a full turn in the parameter. All constructions yield to multiple descriptions of the modulus for some values of the parameter. The key for understanding the dependence on the parameter was to express that these two descriptions yielded the same dynamics. This “compatibility condition” ensures then that the unfolded modulus was $1/2$-summable in the parameter.

This result opens many questions. Several of them are concerned with the generalization to higher codimension. Others concern the applications. For instance what can we say of the indifferent fixed points which are born by perturbation of a parabolic fixed point. It is certainly interesting to also consider applications to problems of finite cyclicity of graphics not satisfying a genericity condition.

The fruitful interactions of the studies of polynomial vector fields and algebraic dynamical systems. The common thread between these two domains is the fact that the singularities organize the dynamics. Moreover the study of singularities of 2-dimensional vector fields can sometimes be reduced to the study of singularities of 1-dimensional dynamical systems. Although the workshop produced no specific output on this particular item, all participants appreciated the fruitful discussions between the two fields.

The reduction of the finiteness part of Hilbert’s 16th problem for classical Liénard equations to the study of singular perturbations in this family. Classical Liénard equations are ones for which the tools of singular perturbations work quite efficiently. The reduction performed by Freddy Dumortier and Magdalena Caubergh opens the hope that a proof of uniform finte cyclicity be given soon, at least for degrees not too high.

The better understanding of the leaves of the foliation of a saddle-node. The lecture of Loïc Teyssier was particularly impressive with his programming of the leaves in the neighborhood of a saddle-node or an unfolding of a saddle-node. In particular he illustrated in a brilliant way how the leaves near a hyperbolic point could have both a node or saddle behaviour depending in which direction we approach the singular point. Examples of his drawings appear in Figures 10.1 and 10.2.

Outcome of the Meeting

The meeting was a real success. This was the opinion of al participants. Most of them really appreciated the broad sense given to the theme and the wide spectrum of expertises. The mixing of specialists from

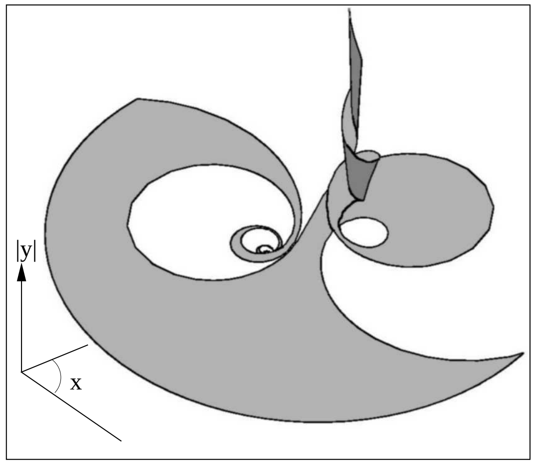

Figure 10.1: Modulus of a leaf a vector field unfolding a saddle node. The drawing justifies the terms "node type" (on the left of the figure) and "saddle type" (on the right) qualifying the singular points.

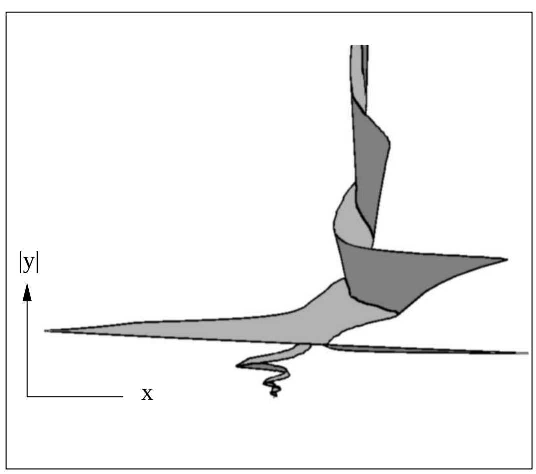

Figure 10.2: Another view of the same leaf.

different areas permitted animated discussions and mixing of ideas during the workshop. Several talks have been absolutely exceptional in quality, in particular the plenary talks which played the role of presenting a mature view of the state of the subject. Moreover several participants were hearing for the first time the details of some of the new significant results of the subject. Some students and young researchers attended the workshop and could discuss and exchange with the senior researchers. Two students: Tanya Firsova and Caroline Lambert gave lectures. Moreover several subgroups used the opportunity to start new work.

While there has been important developments around Hilbert's 16th problem in the last years, it is clear that no complete solution is expected in the near future and there is a consensus that new ideas are still needed in order to make a breakthrough towards a complete solution.

Appendix: list of participants and titles of lectures

- Waldo Arriagada-Silva (Montreal)
- Joan C. Artés (UAB, Barcelona): *Quadratic vectors fields of codimension 1*
- Patrick Bonckaert (Hasselt University): *Invariant manifolds close to linear non-hyperbolic singularities*
- Magdalena Caubergh (Hasselt University): *Large Amplitude Limit Cycles for Liénard systems*
- Rodica Costin (Ohio State): *Nonlinear perturbations of Fuchsian systems: linearization criteria and classification*
- Freddy Dumortier (Hasselt University): *The period function of classical Liénard equations*
- Remy Etoua (Montrea)
- Tanya Firsova (Toronto): *Topology of leaves of analytic foliations on Stein manifolds*
- Armengol Gasull (UAB Barcelona): *Some results on periodic orbits for Abel-type equations*
- Alexey Glutsyuk (ENS, Lyon): *On density of horospheres in dynamical laminations*
- Vadim Kaloshin (Penn. State): *Oscillatory motions and instabilities for the planar 3-body problem*
- Caroline Lambert (Montreal): *Confluence of the hypergeometric equation and Riccati equation*
- Jaume Llibre (UAB, Barcelona): *On the limit cycles of the Liénard differential systems*
- Pavao Mardesic (Dijon): *Infinitesimal and tangential 16-th Hilbert problem*
- Abdelraouf Mourtada (Dijon): *Hilbert's 16th problem for hyperbolic polycycles and extension of Khovanskii-Varchenko theorem to algebraic polycycles*
- Dmitry Novikov (Weizmann): *Extension of the Varchenko-Khovanskii theorem to the integrable case*
- Emmanuel Paul (Toulouse): *Galoisian reducibility for a germ of quasi-homogeneous foliation*
- Rafel Prohens Sastre(Illes Balears, Spain): *On the number of limit cycles of some systems on the cylinder*
- Javier Ribón (IMPA): *Analytic classification of unfoldings of resonant diffeomorphisms*
- Robert Roussarie (Dijon): *Slow-fast systems and Sixteenth Hilbert's Problem*
- Christiane Rousseau (Montréal): *The space of modules of unfoldings of germs of generic diffeomorphisms with a parabolic point*
- Reinhard Schfke (Strasbourg): *Quasi-analytic solutions of analytic ordinary differential equations and o-minimal Structures*

- Loc Teyssier (Strasbourg): *Confluence of singular points in a family of holomorphic vector fields*
- Jordi Villadelprat (Universitat Rovira i Virgili,Spain): *A new result on the period function of quadratic reversible centers*
- Jean-Christophe Yoccoz (ENS, Paris): *Siegel disks and Julia sets of quadratic polynomials, according to X.Buff and A.Chritat*
- Huaiping Zhu (York): *Bifurcation of limit cycles from a Nilpotent Center in a Near-Hamiltonian System*

List of Participants

Arriagada, Waldo (Université de Montréal)
Artes, Joan C. (Universitat Autonòma de Barcelona)
Bonckaert, Patrick (Hasselt University)
Caubergh, Magdalena (Hasselt University)
Costin, Rodica (Ohio State University)
Dumortier, Freddy (Hasselt University)
Etoua, Remy (Université de Montréal)
Firsova, Tatiana (University of Toronto)
Gasull, Armengol (Universitat Autonòma de Barcelona)
Glutsyuk, Alexey (École normale supérieure de Lyon)
Kaloshin, Vadim (Pennsylvania State University)
Lambert, Caroline (Université de Montréal)
Llibre, Jaume (Universitat Autonòma de Barcelona)
Mardesic, Pavao (Université de Bourgogne)
Mourtada, Abderaouf (Université de Bourgogne)
Novikov, Dmitry (Weizmann Institute)
Paul, Emmanuel (Université Paul-Sabatier)
Prohens, Rafel (Universitat de les Illes Balears)
Ribṕn Herguedas, Javier (IMPA)
Roussarie, Robert (Université de Bourgogne)
Rousseau, Christiane (Université de Montréal)
Schäfke, Reinhard (Université Louis-Pasteur)
Teyssier, Loïc (Université Louis-Pasteur)
Villadelprat, Jordi (Universitat Rovira i Virgili)
Yoccoz, Jean-Christophe (Collège de France)
Zhu, Huaiping (York University)

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Chapter 11

Contemporary Schubert Calculus and Schubert Geometry (07w5112)

Mar 18 - Mar 23

Organizer(s): James Carrell (University of British Columbia), Frank Sottile (Texas A&M University)

A Brief Overview of Schubert Calculus and Related Theories

Schubert calculus refers to the calculus of enumerative geometry, which is the art of counting geometric figures determined by given incidence conditions. For example, how many lines in projective 3-space meet four given lines? This was developed in the 19th century and presented in the classic treatise ”Kälkul der abzählanden Geometrie” by Herman Cäser Hannibal Schubert in 1879. Schubert, Pieri, and Giambelli subsequently developed algorithms to solve enumerative geometric problems concerning linear subspaces of vector spaces, which we now understand to be computations in the cohomology ring of a Grassmannian. Their vision and technical skill exceeded the rigorous foundations of this subject, and Hilbert, in his 15th problem, asked for a rigorous foundation of the subject. This was largely completed by the middle of the 20th century, based on the cohomology of rings of Grassmannians.

The basic tools of Schubert calculus are the Grassmannians $G_{k,n}$ consisting of the k-dimensional linear subspaces of $\mathbb{P}^n$ and Schubert varieties, which are certain subvarieties of $G_{k,n}$ made up of the k-dimensional subspaces which satisfy certain incidence relations (called Schubert conditions) with respect to a fixed flag: A flag in $\mathbb{P}^n$ is an increasing sequence $W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset \mathbb{P}^n$ where W_i is a linear subspace of dimension i. Given a sequence of integers $0 \leq a_0 < a_1 < \cdots < a_k \leq n$, the Schubert variety $\Omega(a_0, a_1, \dots, a_k)$ is the set of all $V \in G_{k,n}$ for which $\dim(V \cap W_{a_j}) \geq j$. Another description of $\Omega(a_0, a_1, \dots, a_k)$ is obtained by noting that the group $GL_{n+1}(\mathbb{C})$ of nonsingular $n \times n$ matrices acts transitively on $G_{k,n}$, and let B denote the subgroup of all upper triangular matrices in $GL_{n+1}(\mathbb{C})$. Then $\Omega(a_0, a_1, \dots, a_k)$ is the Zariski closure of the orbit BV_0, where V_0 is the (unique) k-plane in $\Omega(a_0, a_1, \dots, a_k)$ spanned by standard coordinate points $e_{a_0}, \dots, e_{a_k}$. The key fact, known as the ”Basissatz”, is that the Schubert varieties form a basis of the integral homology groups of $G_{k,n}$, and dually determine a basis of its cohomology algebra. The basic formulas of Schubert calculus are given by the famous identities of Giambelli and Pieri and the determinantal formula.

By the 1940’s it had been proved that multiplication in cohomology ring of $G_{k,n}$ with the natural basis formed by the classes dual to the Schubert varieties is the same as multiplication of Schur functions in the algebra of symmetric functions. That is, if $\sigma(a_0, a_1, \dots, a_k)$ denotes the class dual to $\Omega(a_0, a_1, \dots, a_k)$ in the cohomology algebra $H^*(G_{k,n}, \mathbb{Z})$, then when one expresses the product $\sigma(a_0, a_1, \dots, a_k)\sigma(b_0, b_1, \dots, b_k)$ as a sum of classes $\sigma(c_0, c_1, \dots, c_k)$, the coefficient of $\sigma(c_0, c_1, \dots, c_k)$ is a particular Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$. To be precise, if μ is the partition $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_k$ where $\mu_i = n - i - a_i + i$ and ν and

λ are the partitions defined analogously for $(b_0, b_1, \ldots, b_k)$ and $(c_0, c_1, \ldots, c_k)$, then

$$\{\mu\}\{\nu\} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\{\lambda\},$$

where $\{\mu\}$, $\{\nu\}$ and $\{\lambda\}$ are the Schur functions corresponding to the three partitions. This is extremely fundamental since the Littlewood-Richardson rule gives a combinatorial formula for the $c^{\lambda}_{\mu\nu}$ in terms of Young tableaux. Furthermore, it establishes a close relationship between Schubert calculus and the representation ring of the general linear group with its basis of irreducible Weyl modules.

In a famous unpublished 1951 paper, Chevalley proposed a far reaching generalization of the notion of a Schubert variety which illuminated the close connection between Schubert geometry and the theory of algebraic groups and initiated the systematic investigation of the natural generalizations of Schubert varieties for an arbitrary algebraic homogeneous space: that is, a projective variety of the form G/P, where P is a parabolic subgroup of a semi-simple algebraic group over an algebraically closed field (see [6]). His major contribution was to define Schubert varieties in a G/P as the closures of B-orbits, where B is a Borel subgroup of G: that is, a maximal solvable connected subgroup of G. This paper motivated many, many papers about the structure of Schubert varieties, the nature of their singularities and the combinatorial properties of the Weyl group of G and its connection with the geometry of the flag variety G/B. The Littlewood-Richardson rule for the cohomology ring of G/B in particular remains an important open question and was one of the major problems that the workshop addressed. In fact, the Littlewood-Richardson problem now refers to the analogous question for the general cohomology theories relevant to algebraic geometry, such as equivariant cohomology, K-theory and quantum cohomology, as well as equivariant versions of these theories.

Recent Developments and Open Some Problems

In the last 20 years, Schubert calculus has come to refer to the study of the geometric, combinatoric and algebraic aspects of the Schubert basis in the various cohomological setting mentioned above along with their relation to the rest of mathematics. There has been an explosion of progress in the subject, especially since the major impetus provided by the Oberwolfach workshop in 1997 at which people from combinatorics and geometry met each other for the first time at the very time Gromov-Witten theory, quantum cohomology and moduli spaces were being studied at a year-long meeting in Sweden. At about the same time, Knutson and Tao [15] proved the saturation conjecture, which, with work of Klyachko, established the Horn conjecture about the eigenvalues of hermitian matrices. While their methods were purely combinatorial, the relation to the Horn problem was through the geometry of Schubert calculus. We put recent results into three categories.

The Quantum and Equivariant Schubert Calculus,

Here, the highlights include proofs by Aaron Bertram of the Pieri, Giambelli and determinantal formulas for Grassmannians in the setting of quantum cohomology [1]; the work of Allen Knutson and Ezra Miller [14] giving a natural cohomological setting for Schubert polynomials; positivity results of Graham [11] for equivariant cohomology of G/B (see also [12] and [8]); and Brion's work on positivity in the K-theory of the flag variety [2].

Generalizations of the Littlewood-Richardson Rule and Combinatorics

The equivariant Littlewood-Richardson rule for general cohomologies remains an important open problem where, so far, only partial results, many very recent, have been attained. Knutson, Tao and Woodward [16], using versions of the Littlewood-Richardson rule stated in terms of honeycombs and puzzles, re-prove all the main theorems about eigenvalues in the context of honeycomb borders. This gives a beautiful connection between the Littlewood-Richardson rule and the eigenvalue problem just mentioned. Also, Anders Buch [3] has proved the Littlewood-Richardson rule for the K-theory of Grassmannian; Furthermore, Ravi Vakil, working over an arbitrary field, obtained a purely geometric proof of the Littlewood-Richardson rule [25]. Shortly before the workshop, Izzet Coşkun announced a solution to the Littlewood-Richardson problem for the small quantum cohomology of a Grassmannian (see [7]). A key notion on the combinatorial side of the Littlewood-Richardson problem is jeu de taqin, introduced by Schützenberger [22]. Recently, using the language of *jeu de taqin*, Chaput and Perrin [5] have computed the Littlewood-Richardson constants for the Λ-miniscule elements of the Weyl group introduced by Peterson (see [4]). This generalizes the recent result of Thomas and Yong proving the Littlewood-Richardson rule for cominuscule homogeneous spaces [24].

Real Schubert Calculus

Another research thrust has concerned real solutions to Schubert problems. A by-product of Vakil's geometric Littlewood-Richardson rule was a proof that every Schubert problem on the Grassmannian can have only real solutions [26]. Sottile and his collaborators [23, 21] have studied a subtle conjecture of Boris and Michael Shapiro and its generalizations. Eremenko and Gabrielov [9] proved it for Grassmannians of codimension-2 planes by showing that a rational function with only real critical points must be real. Just before the workshop, Mukhin, Tarasov, and Varchenko gave a general proof for all Grassmannians [17], and generalized this to related geometric problems [18].

Presentation Highlights

The following talks were presented at the workshop.

Anders Buch (Rutgers)
Title: *Equivariant Gromov-Witten invariants of Grassmannians*
Abstract: I will speak about joint work with L. Mihalcea, in which we prove that all equivariant (3-point, genus zero) Gromov-Witten invariants on Grassmannians are equal to equivariant triple-intersections on two-step flag varieties. This is a continuation of work with A. Kresch and H. Tamvakis, which established this result for the ordinary Gromov-Witten invariants. The non-equivariant case was obtained by showing that the curves counted by a Gromov-Witten invariant are in bijection with their kernel-span pairs, which consist exactly of the points in a triple-intersection of two-step Schubert varieties. Since the equivariant Gromov-Witten invariants have no enumerative interpretation (and also because they are defined relative to Schubert varieties that are not in general position), the proof in the equivariant case must be based on intersection theory. The main new construction is a blow-up of Kontsevichs moduli space that makes it possible to assign a kernel-span pair of the expected dimensions to every curve. By utilizing a construction of Chaput, Manivel, and Perrin, these results can be extended to all (co)minuscule homogeneous spaces.

Izzet Coşkun (MIT)
Title: *The geometry of flag degenerations*
Abstract: My goal is to explain the flat limits of certain subvarieties of partial flag varieties under one-parameter specializations. I will explain how this study leads to many new Littlewood-Richardson rules. I will try to get the audience to actively participate in the calculations, so we will have graph paper and colored pencils.

Takeshi Ikeda (Okayama University of Science)
Title: *Equivariant Schubert calculus for isotropic Grassmannians*
Abstract: We describe the torus-equivariant cohomology ring of isotropic Grassmannians by using localization maps to any torus-fixed point. We present two formulas for equivariant Schubert classes of these homogeneous spaces. The first formula is given as a weighted sum over combinatorial ob jects which we call excited Young diagrams. The second one is written in terms of factorial Schur Q- or P -functions. As an application, we give a Giambelli-type formula for the equivariant Schubert classes. This is a joint pro ject with Hiroshi Naruse.

Allen Knutson (UC San Diego)
Title: *Matroids, shifting, and Schubert calculus*
Abstract: In A geometric Littlewood-Richardson rule, Vakil starts with an intersection of two Schubert varieties in a Grassmannian, and alternately degenerates, then decomposes into geometric components; the end result is a list of Schubert varieties. The primary geometric miracle is that each degeneration stays generically reduced, with the consequence that the original cohomology class (the product of two Schubert classes) is written as a multiplicity-free sum of Schubert classes. There is an obvious generalization of this procedure

to the flag manifold (where the geometric miracle is as yet unknown) but it doesnt give a combinatorial rule, just a geometric one. In this talk Ill explain how to combinatorialize Ravis procedure, replacing varieties with matroids, and degeneration with shifting (originally invented by Erdos-Ko-Rado for extremal combinatorics). One upshot will be that each of Ravis varieties is defined by linear equations, allowing us to extend his proof to K-theory. Another is an easily stated conjecture for flag manifold Schubert calculus.

Thomas Lam (Harvard)
Title: *Affine Schubert calculus*
Abstract: I will discuss some recent progress in understanding the (co)homology of the affine Grassmannian from the point of view of Schubert calculus. In particular, I will explain how to obtain polynomial representatives for Schubert classes and analogues of Pieri rules. If time permits, I hope also to explain some connections with calculations of the (co)homology ring of the affine Grassmannian due to Bott, to Ginzburg, to Bezrukavnikov, Finkelberg and Mirkovic and to Peterson. Part of this talk is based on joint work with Lapointe, Morse and Shimozono.

Cristian Lenart (SUNY Albany)
Title: *K-theory and quantum K-theory of flag varieties*
Abstract: I previously presented Chevalley-type multiplication formulas in the T -equivariant K-theory of generalized flag varieties G/P; these formulas were derived in joint work with A. Postnikov. In the first part of this talk, I will present a model for $KT = G/P$ in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. This model is based on the Chevalley-type formulas mentioned above, and has potential applications to deriving more general multiplication formulas. In the second part of the talk, I will show the way in which a presentation of the quantum K-theory of the classical flag variety leads to the construction of certain polynomials, called quantum Grothendieck polynomials, that are conjectured to represent Schubert classes. We present evidence for this conjecture; this includes the fact that the quantum Grothendieck polynomials satisfy a multiplication formula which is the natural generalization of the Chevalley-type formula mentioned above and of the corresponding formula in quantum cohomology. This talk is based on joint work with T. Maeno.

Elena Marchisotto (California State University, Northridge)
Title: *Evaluating the research of Mario Pieri (1860-1913) in algebraic geometry*
Abstract: Mario Pieri was an active member of the research groups surrounding Corrado Segre and Giuseppe Peano at the University of Turin around the turn of the nineteenth century. Pieri played a ma jor role in foundations of mathematics. Can the same be said of his role in algebraic geometry? Was he the first to introduce the Schubert calculus to Italy? Did he accomplish all he set out to do with the appropriate rigor? Do his papers contain there valuable ideas for contemporary work? Are his contribu- tions to enumerative geometry limited to the multiplication formula and theorem for correspondences on an n-dimensional pro jective space that are discussed in Fultons book on intersection theory? In my talk I will share my initial research in attempts to answer the above questions. I will provide an overview of Pieri the man and Pieri the algebraic geometer. I will briefly discuss Pieris papers in algebraic geometry, in particular his results in enumerative geometry and his use of the Schubert calculus. I will report on existent letters of Castelnuovo, de Paolis, Enriques, Fouret, Schubert, Severi, and Zeuthen.

Leonardo Mihalcea (Duke)
Title: *Chern-Schwartz-MacPherson classes for Schubert cells in the Grassmannian*
Abstract: A conjecture of Deligne and Grothendieck states that there is a functorial theory of Chern classes on possibly singular varieties, viewed as a natural transformation from the group of constructible functions to the homology (or Chow group) of a compact variety. This conjecture was solved in 1973 by R. MacPherson; the classes he defined (now known as Chern-Schwarz-MacPherson - or CSM) turned out to be the same as those defined by M.H. Schwartz, by different methods. In joint work with Paolo Aluffi, we give explicit formulae for the CSM classes of the Schubert varieties in the Grassmannian. The main tools used in the computation are a new formula for the CSM classes recently discovered by P. Aluffi and a Bott-Samelson resolution of a Schubert variety. Given that Schubert varieties are singular, an unexpected feature is a certain effectivity satisfied by these classes; we have proved it in few cases, and it is conjectured to hold in general.

Steve Mitchell (University of Washington)
Title: *Smooth and palindromic Schubert varieties in affine Grassmannians*
Abstract: The affine Grassmannian associated to a simple complex algebraic group G is an infinite- dimensional projective variety that is very much analogous to an ordinary Grassmannian. In particular, it comes equipped with a decomposition into Schubert cells whose closures are ordinary projective varieties. It turns out that for each fixed G, only finitely many of these Schubert varieties are smooth, and except in type A only finitely many are even palindromic. In fact it is possible to determine the smooth and palindromic Schubert varieties explicitly. For example, in type C a Schubert variety is smooth if and only if it is a closed parabolic orbit, in which case it is a symplectic Grassmannian. It is palindromic if and only if it is either smooth or a subcomplex of the Schubert variety associated to the lowest nontrivial anti-dominant coroot lattice element; this variety is just the Thom space of a line bundle over projective space. The most eccentric type is type A, where in each rank there are two infinite families of singular palindromics. These latter varieties have interesting topological properties, which were studied in the 80s by myself and (independently) Graeme Segal.

Evgeny Mukhin (IUPUI)
Title: *On generalizations of the B. and M. Shapiro conjecture*
Abstract: A proof of the B. and M. Shapiro conjecture is based on the consideration of the periodic quantum Gaudin model. We show what results are obtained in the similar way from quasiperiodic quantum Gaudin and from quasiperiodic XXX models.

Nicolas Perrin (Paris)
Title: *Some combinatorial aspects of cominuscule geometry*
Abstract: In this talk we shall explain combinatorics tools (quivers) generalising Young diagrams that appear in the study of cominuscule homogeneous spaces. We shall see how to read on these quivers some geometric properties of the Schubert varieties like their singularities, the fact that they are Gorenstein, locally factorial or even that they admit or not small resolutions. These quivers are used by A. Yong and H. Thomas for classical Schubert calculus. We shall see how these quivers also help for computing some Gromov-Witten invariants.

James Ruffo (TAMU)
Title: *A straightening law for the Drinfeld Lagrangian Grassmannian*
Abstract: We present a structured set of defining equations for the Drinfeld compactification of the space of rational maps of a given degree in the Lagrangian Grassmannian. These equations give a straightening law on a certain ordered set, which allows the use of combinatorial arguments to establish useful geometric properties of this compactification. For instance, its coordinate ring is Cohen-Macaulay and Koszul.

Mark Shimozono (Virginia Tech)
Title: *Kac-Moody dual graded graphs and affine Schubert calculus*
Abstract: Motivated by the dual Hopf algebra structures on the homology and cohomology of the affine Grassmannians, particularly their Schubert structure constants, we exhibit families of dual graded graphs (in the sense of Fomin) associated to any Kac-Moody algebra, dominant weight and positive central element, yielding enumerative identities involving chains in the strong and weak Bruhat orders.

Hugh Thomas (University of New Brunswick)
Title: *A combinatorial rule for (co)minuscule Schubert calculus*
Abstract: I will discuss a root system uniform, concise combinatorial rule for Schubert calculus of minuscule and cominuscule flag manifolds G/P . (The latter are also known as compact Hermitian symmetric spaces.) We connect this geometry to work of Proctor in poset combinatorics, thereby generalizing Schützenbergers jeu de taquin formulation of the Littlewood-Richardson rule for computing intersection numbers of Grassmannian Schubert varieties. I will explain the rule and, time permitting, discuss ideas used in its proof, including cominuscule recursions, a general technique relating the Schubert constants for different Lie types. I will also discuss cominuscule dual equivalence, a generalization of a concept due to Haiman. We use this to provide an independent proof of Proctors jeu de taquin results in our context.

Julianna Tymoczko (Iowa)
Title: *Divided difference operators for Grassmannians* Abstract: We construct divided difference operators for Grassmannians. More precisely, we give an explicit combinatorial formula for divided difference operators on the equivariant cohomology of G/P, for any parabolic subgroup P and any complex reductive linear algebraic group G. One application generalizes a flag-variety result of Sara Billeys to compute the localizations of equivariant Schubert classes for G/P. Another provides equivariant Pieri rules for Grassmannians. These divided difference operators are constructed using GKM (Goresky-Kottwitz-MacPherson) the- ory, which gives a combinatorial construction of equivariant cohomology for many suitably nice algebraic varieties. We will focus on how the theory works for Grassmannians of k-planes in complex n-dimensional space.

Alexander Varchenko (North Carolina)
Title: *The B. and M. Shapiro conjecture in real algebraic geometry and Bethe ansatz.*
Abstract: I shall discuss the proof of Shapiros conjecture by methods of math physics and representation theory. Shapiros conjecture says the following. If the Wronskian of a set of polynomials has real roots only, then the complex span of this set of polynomials has a basis consisting of polynomials with real coefficients.

Jan Verschelde (University of Illinois-Chicago)
Title: *Numerical homotopy algorithms for enumerative geometry*
Abstract: In a 1998 paper on Numerical Schubert Calculus, Birk Huber, Frank Sottile, and Bernd Sturmfels proposed numerical Pieri homotopy algorithms to solve problems in enumerative geometry. Im- plementations of these algorithms ran on specific examples of intersection problems whose solution set is entirely real. Jointly with Yusong Wang, the Pieri homotopies were applied to the output pole placement problem in linear systems control. Following Vakils geometric proof of the Littlewood-Richardson rule, Sottile, Vakil, and I designed deformation methods to solve general Schubert problems. Current work is directed to implement these new Littlewood-Richardson homotopies.

Alex Yong (University of Illinois)
Title: *Governing singularities of Schubert varieties*
Abstract: We present a combinatorial and computational commutative algebra methodology for studying singularities of Schubert varieties of flag manifolds. We define the combinatorial notion of interval pattern avoidance. For reasonable invariants P of singularities, we geometrically prove that this governs (1) the P -locus of a Schubert variety, and (2) which Schubert varieties are globally not P. The prototypical case is P=singular; classical pattern avoidance applies admirably for this choice [Lakshmibai-Sandhya 90], but is insufficient in general. Our approach is analyzed for some common in- variants, including Kazhdan-Lusztig polynomials, multiplicity, factoriality, and Gorensteinness, extending [Woo-Yong05]; the description of the singular locus (which was independently proved by [Billey-Warrington 03], [Cortez 03], [Kassel-Lascoux-Reutenauer03], [Manivel01]) is also thus reinterpreted. Our methods are amenable to computer experimentation, based on computing with Kazhdan-Lusztig ideals (a class of generalized determinantal ideals) using Macaulay 2. This feature is supplemented by a collection of open problems and conjectures.

Scientific Progress Made

An emerging theme at the workshop was the affine Schubert calculus. Previously, the primary focus in Schubert calculus was on the (generalized) cohomology rings of finite-dimensional flag varieties G/P. Work of Lam, Shimozono, and others pointed to the possibility to understand the homology and cohomology Hopf algebras of the infinite dimensional affine flag varieties. The workshop also provided an opportunity for some to understand Coşkun's new geometric proof of the that Littlewood-Richardson rule in the quantum cohomology of the Grassmannian.

Outcome of the Meeting

This meeting, on the 10th anniversary of the original meeting on Schubert calculus at Oberwolfach was an opportunity for the major participants in Schubert calculus to asses the current state of affairs of the subject. Some of the advances since that meeting had their genesis there. Discussions about the importance of transversality helped inspire Mukhin, Tarasov, and Varchenko's second proof of the Shapiro Conjecture [19]. Purbhoo became interested in transversality and their work, and this led to his geometric construction of the basic algorithms of Young tableaux [20]. Subsequent to the meeting, significant progress has been made in the equivariant K-theory Schubert calculus; again that had some beginnings at the meeting. Also, the affine Schubert calculus has matured since, living up to its early progress.

List of Participants

Belkale, Prakash (University of North Carolina, Chapel hill)
Billey, Sara (University of Washington)
Buch, Anders (Rutgers University)
Carrell, James (University of British Columbia)
Chen, Linda (Ohio State University)
Coskun, Izzet (Massachusetts Institute of Technology)
Elizondo, E. Javier (Instituto de Matematicas, U.N.A.M)
Firsova, Tatiana (University of Toronto)
Goldin, Rebecca (George Mason University)
Harada, Megumi (McMaster University)
Hering, Milena (Institute for Mathematics and its Applications (IMA))
Hillar, Christopher (Texas A&M University)
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Varchenko, Alexander (University of North Carolina)
Verschelde, Jan (University of Illinois at Chicago)
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Chapter 12

Interactions of Geometry and Topology in Low Dimensions (07w5033)

Mar 25 - Mar 30, 2007

Organizer(s): Denis Auroux (Massachusetts Institute of Technology), Hans Boden (McMaster University), Olivier Collin (Université du Québec à Montréal (UQAM)), John Etnyre (Georgia Institute of Technology)

Introduction

This workshop focussed on interactions between symplectic geometry, gauge theory, contact topology, and applications to low-dimensional manifolds. While each of these areas has been very active for many years in an independent fashion, the theory of low-dimensional manifolds has greatly benefited from interactions with the other subjects represented at the event. Our workshop can be seen as a follow-up to the BIRS-MSRI Hot Topics event of November 2003 on Floer homology. Four years after this event, it was very productive to gather gauge theorists, contact topologists, symplectic geometers and topologists all together to share their insights and to foster collaborative investigation and research.

One of the (pleasant) difficulties in organizing such an event is to carefully select a group of 40 world experts with a good balance between research areas, experience versus budding mathematical activity and try to represent as accurately as possible the most important current trends. In the end, this was achieved with great success. In particular, while some of the most established and senior researchers were present (Kirby, Stern, Akbulut, Boyer, Kirk, etc), it was a great inspiration to witness the emergence of young mathematicians through their important scientific work, but also through a lively presence at the conference, asking many questions, making good observations and stimulating discussions, as exemplified by the likes of Hedden, Perutz, Ng, Grigsby among others.

Overview of the Field

In recent years collaborations between contact and symplectic geometers, gauge theorists, and low-dimensional topologists have been highly fruitful, leading to solutions to long-standing conjectures in topology, illuminating the world of contact and symplectic manifolds, and providing new perspectives on fundamental questions in low-dimensional topology.

For some time, gauge theory has provided geometric topologists with powerful techniques yielding spectacular results on the classification problem for 4-dimensional manifolds, including the early 4-manifold invariants of Donaldson, the Seiberg-Witten invariants, and the more recent invariants of Ozsvath-Szabo. In each case, when applied to 4-manifolds with boundary, the invariants take values in the Floer homology groups of the bounding 3-manifold. These Floer homology groups are important 3-manifold invariants in their own right and have been applied to a variety of problems, such as knot theory and the structure of the homology cobordism group in dimension three. Frequently, these invariants are easier to compute in the presence of extra structure. For example, all the above mentioned invariants are non-zero for symplectic 4-manifolds. These non-vanishing results are instrumental in the interactions between contact and symplectic geometry and low-dimensional topology.

In 2001 P. Ozsvath and Z. Szabo introduced the Heegaard Floer homology groups. Their theory was motivated by Seiberg-Witten theory, but is defined in a completely different way: it is a variant of lagrangian Floer homology that uses much of the 3-manifold topology in its definition (in particular a Heegaard decomposition). In many instances, this new theory has been able to reprove fundamental results obtained in gauge theory over the last few decades and has also provided many new applications. One of its advantages is that there are powerful exact sequences enabling one to obtain hard topological information. For example, it was shown by Y. Ni that it detects if a knot in S^3 fibres, see also [12]. This theory is conjecturally equivalent to Seiberg-Witten Floer homology by an analogue of the Atiyah-Floer conjecture, and a very interesting research program of Yi Jen Lee proposes an approach for proving this conjecture. Heegaard Floer homology is also (conjecturally) closely related to the contact homology groups introduced by Y. Eliashberg, H. Hofer and others and it is possible that a tool derived from this set-up, namely the embedded contact homology of Hutchings will provide a relation between Heegaard-Floer and Seiberg-Witten theories. A good explanation of these correspondences would help determine exactly what exactly Heegaard Floer groups measure.

Emmanuel Giroux revolutionized contact geometry by proving an equivalence between contact structures on 3-manifolds up to isotopy and open book decompositions up to stabilization. Open book decompositions are a classical topological concept and have been studied for some time. This result is analogous to Simon Donaldson's proof that symplectic 4-manifolds always admit Lefschetz pencils, and the analogy is strengthened by Bob Gompf's proof that all Lefschetz pencils admit symplectic structures. Both these correspondences relate geometric concepts to topological ones and have been the foundation for many of the applications of symplectic geometry to questions in low-dimensional topology. For example, this correspondence leads to two notable results, namely Giroux and Noah Goodman's positive resolution of Harer's conjecture that all fibered knots in S^3 are related by Hopf plumbings, and Ozsvath and Szabo's proof of Gordon's conjecture that the unknot is the only knot on which p surgery yields the lens space $-L(p, 1)$. The correspondence also has implications in the other directions as well. It is the basis of the non-vanishing of the Ozsvath Szabo invariant of symplectic 4-manifolds mentioned above. It is also the key tool in Eliashberg and John Etnyre's proof that any symplectic filling of a contact manifold can be embedded in a closed symplectic manifold. This result in turn is an integral part of Peter Kronheimer and Tom Mrowka's proof of the Property P conjecture that a nontrivial surgery on a nontrivial knot in S^3 has nontrivial fundamental group or that $\mathbb{R}\mathrm{P}^3$ cannot be obtained by Dehn surgery along a non-trivial knot in S^3.

Recent Developments and Open Problems

There were many active areas represented at the workshop, but if pressed to determine one or two where the breakthroughs have been most important recently, one would have to look in the direction of combinatorial approaches to Heegaard-Floer homology and the impact of symplectic field theory ideas in the contact world through the embedded contact homology of Hutchings, with expected applications to 3-manifold topology.

The problem of combinatorially constructing Heegaard-Floer groups without resorting to counting pseudo-holomorphic curves had taken a very promising turn a few months before the workshop when knot Floer homology was given a purely combinatorial interpretation. This was an underlying theme in several of the talks at the workshop, in particular Ng's talk focused on applications of these ideas to the classification of transverse knots in contact manifolds (see [13]), while Plamenevskaya explained how the Heegaard-Floer contact invariant can be defined in the combinatorial context of Manolescu, Ozsvath and Sarkar, see [58].

Further, as the knot Heegaard Floer homology categorifies the Alexander polynomial in much the same way that Khovanov cohomology categorifies the Jones polynomial, an interesting open problem is to relate the Heegaard Floer and Khovanov cohomologies. A few of the talks at the workshop focused on this aspect, and the most recent result along these lines is the existence of a spectral sequence interpolating between the two theories.

A few months before the conference, Taubes took everyone by surprise in the area of symplectic topology by proving in [15] the 3-dimensional case of the Weinstein conjecture (for any compact oriented 3-manifold M and α a contact 1-form on M, the vector field that generates the kernel of the 2-form $d\alpha$ has at least one closed integral curve). The proof in [15] involved a come-back of the Seiberg-Witten equations, but it also gave very good reasons to focus on further properties of the embedded contact homology (ECH). In turn, Taubes' proof can be seen as a step towards showing that ECH is isomorphic to Seiberg-Witten-Floer homology.

Another example of the interactions between the subjects represented at the workshop is the study of slice numbers of knots. Many talks at the event (Owens, Grigsby, Jabuka, Hedden, Chantraine) involved work on the slice number of classes of knots. The tools for studying this are today quite varied - Heegaard-Floer homology, classical gauge theory, contact topology methods - and there was much discussion at the conference about these problems, including recent work of Lisca (January 2007) on the very classical slice-ribbon conjecture (every slice knot in S^3 is a ribbon knot) which settled the issue in the case of 2-bridge knots.

Presentation Highlights

The meeting featured 22 one-hour talks on various aspects of 3-dimensional and 4-dimensional manifold theory. The central themes were:

- **4-manifolds, smooth and symplectic structures.** *(Talks 1, 2, 5, 12, 19, 20)*
- **3-manifolds, contact structures and invariants.** *(Talks 3, 6, 7, 8, 9, 10, 14, 15, 17, 18, 22)*
- **Heegaard-Floer homology and analogues.** *(Talks 6, 7, 10, 11, 13, 14, 15, 16, 18, 21)*
- **Knots and invariants (especially the knot concordance problem).** *(Talks 4, 6, 11, 15, 16, 21)*

Below is a detailed list of speakers, titles, and brief descriptions of their talks.

1. **Scott Baldridge** (Louisiana State) *Small symplectic building blocks and the geography problem*
   Constructions were given of many simply connected and non-simply connected symplectic and smooth manifolds, including a minimal symplectic manifold homeomorphic to $\mathbb{CP}^2\#3(\overline{\mathbb{CP}}^2)$ containing a symplectic genus 2 surface with simply connected complement.

2. **Inanc Baykur** (Michigan State) *Folded-Kähler structures on 4-manifolds*
   This talk focused on the existence of a generalization of symplectic structures on arbitrary closed smooth oriented 4-manifolds, called "folded-Kähler structures". This result comes with a decomposition theorem, which states that every closed smooth oriented 4-manifold can be decomposed into two compact Stein manifolds (one with reversed orientation), such that the induced contact structures agree on a separating convex hypersurface. There is also a natural topological counterpart for these structures: "folded Lefschetz fibrations".

3. **Steven Boyer** (UQAM) *On families of virtually fibred Montesinos link exteriors*
   William Thurston conjectured over twenty years ago that every compact hyperbolic 3-manifold whose boundary is a possibly empty union of tori is virtually fibred, that is, has a finite cover which fibres over the circle. If true, it provides a significant amount of global information about the topology of such manifolds. To date, there has been remarkably little evidence to support the conjecture. For instance, there is only one published non-trivial example of a closed virtually fibred hyperbolic rational homology 3-sphere. (Non-trivial in this context means that the manifold neither fibres nor semi-fibres.)

   This talk outlined a proof of this conjecture for the exteriors of many Montesinos links and constructed an infinite family of closed virtually fibred hyperbolic rational homology 3-spheres. As a result, one obtains a finite index bi-orderable subgroup in the fundamental groups of the exteriors of many Montesinos links.

4. **Baptiste Chantraine** (UQAM) *Cobordisms of Legendrian knots*
One generalization of cobordism theory of knots in the Legendrian category is asking that such cobordisms are realised by Lagrangian surfaces in the symplectization. From a study of relative Gromov-Lee theorems and the behaviour of the classical invariants, one sees that this relation is rigid (unlike in the topological case where only the Thom-Pontryagin construction is needed). The first step in studying this relation (and the relation of Lagrangian concordance) is to show that Legendrian-isotopic knots are Lagrangian concordant. This talk gave the basic definitions and theorems which are the starting point of the theory and provided non-trivial examples of Lagrangian cobordisms implying that this relation is non-symmetric. As an application, a contact-topology proof of the local Thom conjecture was obtained.

5. **Stefan Friedl** (UQAM) *Symplectic 4-manifolds with a free circle action*
Let W be a symplectic 4-manifold with free circle action. It was shown that if the fundamental group of the orbit space satisfied certain separability properties, then the orbit space fibers over S^1 . Using the Lubotzky alternative, the same result can be proved if the canonical class is trivial.

6. **Julia Grigsby** (Columbia) *Knot concordance and Heegaard Floer homology invariants in branched covers*
The smooth concordance order of a knot K is defined as the smallest positive integer n for which the connected sum of n copies of K bounds a smoothly-embedded disk in the four ball. This talk described two new invariants which yield an obstruction to a knot having finite smooth concordance order. These invariants are defined by examining analogues of "classical" Heegaard Floer homology invariants in the double-branched cover of K. Using a simple combinatorial description of these invariants in the case where K is a two-bridge knot, we are able to conclude that all two-bridge knots of 12 or fewer crossings for which the concordance order was previously unknown have infinite concordance order.

7. **Matthew Hedden** (MIT) *Lens space surgeries, contact structures, the braid group, and algebraic curves*
This talk gave a proof of the following statement: If Dehn surgery on a knot K yields a lens space, then K arises as the transverse intersection of an algebraic curve in $\mathbb{C}^2$ with the three-sphere. Furthermore, the genus of the piece of the curve inside the four-ball is equal to the Seifert genus of K. Knots arising in this way are more general than the well-understood links of singularities, as their singular sets may be more complicated. The result follows from a theorem stating that an invariant defined using Ozsvath-Szabo theory detects when fibered knots arise from algebraic curves with a genus constraint as above. This theorem, in turn, follows from connections between Ozsvath-Szabo theory and Giroux's work on three-dimensional contact geometry, and work of Rudolph relating the knot theory of algebraic curves to the braid group.

8. **Benjamin Himpel** (Bonn) *A splitting formula for spectral flow and the SU(3) Casson invariant for spliced sums*
The $SU(3)$ Casson Invariant does not behave well under spliced sums, however, for splicings of complements of torus knots, there is a conjectured formula relating the $SU(3)$ Casson invariant to the $SU(2)$ Casson invariant of the two knots. A proof of this conjecture was presented, and the main tool for this is a splitting formula for the $su(n)$ spectral flow of the twisted odd signature for 3-manifolds cut along a torus coupled to a path of $SU(n)$ connections.

9. **Ko Honda** (Southern California) *Invariants of exact Lagrangian cobordisms*
Constructions of exact Lagrangian cobordisms between two Legendrian knots in the symplectization of the standard contact $\mathbb{R}^3$ were given, and invariants arising from Legendrian contact homology and Khovanov homology were discussed.

10. **Michael Hutchings** (UC Berkeley) *Embedded contact homology*
The goal of this talk was to interpret Seiberg-Witten and Ozsvath-Szabo Floer homology via contact geometry in 3-dimensions and J-holomorphic curves in 4-dimensions. The candidate theory is embedded contact homology (ECH), which is a kind of Floer theory defined for a contact 3-manifold Y , whose differential counts certain embedded pseudoholomorphic curves in the symplectization $\mathbb{R} \times Y$. This talk defined the theory and discussed some associated open problems and conjectures.

11. **Stanislav Jabuka** (Nevada, Reno) *Knot concordance and Heegaard Floer homology*
After describing some of the open questions for knot concordance, some of them dating back to the seminal work of J. Milnor and R. Fox in the mid 1960's, some new results obtained using Heegaard Floer homology were presented. An obstruction for a knot to be of order $n > 0$ in the concordance group was defined and examined for small crossing knots (as of this writing there are still 8-crossing knots with unknown concordance order), this work obviously being related to the problem of existence of n-torsion knots for > 2 in the concordance group. Also presented was a Heegaard Floer proof of a theorem first obtained by Fintushel and Stern according to which the only 3-stranded pretzel knot $K(p, q, r)$ (with p, q, r odd) with trivial Alexander polynomial and which is of finite concordance order, is the unknot. Generalizations of this result to pretzel knots with nontrivial Alexander polynomial were given.

12. **Hee Jung Kim** (McMaster) *Topological triviality of smoothly knotted surfaces in 4-manifolds*
Some generalizations and variations of the Fintushel-Stern rim surgery are known to produce smoothly knotted surfaces. If the fundamental groups of their complements are cyclic, then these surfaces are topologically unknotted. Using a twist-spinning construction from high-dimensional knot theory, examples of knotted surfaces whose complements have cyclic fundamental groups were constructed.

13. **Thomas Mark** (Virginia) *On perturbed Heegaard Floer invariants*
This talk constructed a version of Heegaard Floer homology with coefficients in certain Novikov rings depending on a choice of perturbation, namely a 2-dimensional real cohomology class. For nontrivial perturbations, the reducible part of the Floer homology vanishes, in close analogy with Seiberg-Witten theory. This leads to analogues of the Ozsvath-Szabo invariants for closed 4-manifolds with $b^+ > 0$ and well-behaved relative invariants for 4-manifolds with boundary.

14. **Gordana Matic** (Georgia) *Open books and contact class in Heegaard Floer Homology*
An alternate, simple description of the Ozsvath-Szabo contact class in Heegaard Floer homology for a closed contact 3-manifold was given, and some consequences of this approach were discussed. This variant of the contact class was extended to define a new contact class for a contact manifold with convex boundary which lives in Juhasz's sutured Floer homology.

15. **Lenhard Ng** (Duke) *Transverse knots and Heegaard Floer homology*
The recently discovered combinatorial form for knot Floer homology has an unexpected application in contact geometry. In fact, one can use knot Floer homology to produce an effective invariant of transverse knots, and this talk gave several new examples of transversely non-simple knot types.

16. **Brendan Owens** (Louisiana State) *Slicing numbers of knots*
The slicing number $u_s(K)$ of a knot K in S^3 is the least number of crossing changes to convert K to a slice knot. This gives an upper bound for the slice genus $g_s(K)$. Livingston defined an invariant $U_s(K)$ which takes into account signs of crossings, with $g_s \leq U_s \leq u_s$. Heegaard Floer theory and also Donaldson's theorem can be used to give information on these numbers, and this talk described an infinite family of knots K_n with slice genus n and $U_s > n$.

17. **Tim Perutz** (Cambridge) *A symplectic Gysin sequence and the Floer homology of connected sums*
This talk described a project to study the structure of symplectic models for gauge-theoretic TQFTs on singularly-fibred 3-and 4-manifolds. The Gysin sequence for the cohomology of a sphere-bundle has a symplectic Floer-theoretic counterpart, which in turn is precisely analogous to a sequence describing Floer homology for connected sums of 3-manifolds.

18. **Olga Plamenevskaya** (SUNY Stony Brook) *A combinatorial description of the Heegaard Floer contact invariant*
Manolescu, Ozsvath, and Sarkar recently proved that certain Heegaard Floer homologies admit a purely combinatorial description. In particular, Sarkar and Wang developed an algorithm to modify a given Heegaard diagram for a 3-manifold so that the holomorphic disks can be combinatorially understood. Using the geometric description of the contact invariant due to Honda, Kazez, and Matic, a version of this algorithm was described in the context of open books to show that the Heegaard Floer contact invariant is combinatorial.

19. **Nikolai Saveliev** (Miami) *Dirac operators on manifolds with periodic ends*
This talk focused on relationships between classical invariants of low-dimensional topology and certain invariants arising in 4-dimensional gauge theory, and in particular on Dirac operators on non-compact spin manifolds with periodic ends of dimension at least four. A necessary and sufficient condition for such an operator to be Fredholm for a generic endperiodic metric was given, and these end-periodic Dirac operators were used to prove that an invariant introduced by Cappell and Shaneson in the 1970's provides an obstruction to the existence of metrics of positive scalar curvature on some non-orientable 4-manifolds. As an application, it was shown that some exotic 4-manifolds do not admit a metric of positive scalar curvature (in some cases, even if their orientation double covers do).

20. **Ron Stern** (UC Irvine) *Reverse engineering and smooth structures on simply-connected smooth manifolds*
This talk introduced a procedure called "reverse engineering" which can be used to construct infinite families of smooth 4-manifolds in a given homeomorphism type. This is a very general technique that recovers many of the known techniques for producing smooth structures on a given simply-connected 4-manifold.

    Reverse engineering is a three step process for constructing infinite families of distinct smooth structures on simply connected 4-manifolds. One starts with a model manifold which has nontrivial Seiberg-Witten invariant and the same euler number and signature as the simply connected manifold that one is trying to construct, but with $b_1 > 0$. The second step is to find b_1 essential tori that carry generators of H_1 and to surger each of these tori in order to kill H_1 and, in favorable circumstances, to kill π_1. The third step is to compute Seiberg-Witten invariants. After each surgery one needs to be careful to preserve the fact that the Seiberg-Witten invariant is nonzero. One can construct exotic structures on many simply-connected smooth manifolds starting with an irregular complex surface.

21. **Hao Wu** (Massachusetts) *The Khovanov-Rozansky cohomology and Bennequin inequalities*
Bennequin type inequalities were established using various versions of the Khovanov- Rozansky cohomology. A new proof of a Bennequin type inequality was established and new Bennequin type inequalities for knots using Gornik's version of the Khovanov-Rozansky cohomology were given. These generalize results by Shumakovitch, Plamenevskaya and Kawamura using the Rasmussen invariant.

22. **Chris Wendl** (MIT) *Intersection theory and compactness for holomorphic curves in low dimensions*
This talk described a recent result strengthening the standard compactness theorem for a geometrically natural class of embedded holomorphic curves in contact 3-manifolds: it turns out the intersection theory of punctured holomorphic curves can be used to rule out multiple covers in the limit, so that transversality is never a problem. This has applications to the theory of finite energy foliations (generalizations of planar open book decompositions), and also suggests an approach for defining distinctly low dimensional versions of Contact Homology and SFT. Also described were some related results in symplectic 4-manifolds and nontrivial symplectic cobordisms. These are part of a larger program to justify the statement that "nice holomorphic curves degenerate nicely".

Scientific Progress Made

The workshop brought together leading experts from several different areas, and this sparked much scientific interaction. Despite the large number of proposed talks, the organizers were committed to the idea that such a workshop based in interactions between subjects should have more than enough scheduled time for scientific discussions. This was accomplished thanks to a careful scheduling of long breaks at the lunch period and not overloading the talk timetable so that informal discussions were many during the evenings. In some instances this also provided valuable time for teams of researchers who rarely get to meet to work together.

The meeting also established several collaborations that have already born fruit. Two examples readily come to mind. One evening while discussing in the BIRS lounge, the post-doctoral researcher Paolo Ghiggini and Ph.D. student Jeremy van Horn-Morris elaborated with Ko Honda a way to prove that Giroux torsion kills the contact invariant of Ozsvath-Szabo (see [4] for a write up of their results) and since then, Ghiggini and Honda have been working on extensions of this work to the case of twisted coefficients.

The workshop was also the perfect stage for intense discussions around the classification problem for smooth structures and the geography problem for symplectic 4-manifolds. In the weeks leading to the conference, many independent approaches to these problems were put forward in the form of preprints, with competing claims made by various authors. A large group of the researchers involved in these developements attended the workshop: Scott Baldridge, Inanc Baykur, Paul Kirk, Doug Park, and Ron Stern. It was at this BIRS meeting that many issues about rigorous proofs were raised and discussions between the various teams sorted out the details and even brought many of the key players together in collaboration, as evidenced by the recent papers [1, 2].

We conclude with the following bibliography which is far from being extensive, its goal being mostly to provide some background articles, all of them still in preprint form, to some of the latest topics discussed at the workshop and articles that have emerged since then. Consulting these articles and, in turn, their own references will give a much broader perspective on the various areas represented at the event.

List of Participants

Baldridge, Scott (Louisiana State University)
Baykur, Inanc (Michigan State University)
Boden, Hans (McMaster University)
Boyer, Steve (Université du Québec à Montréal)
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Collin, Olivier (Université du Québec à Montréal (UQAM))
Friedl, Stefan (Université du Québec à Montréal)
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Gordon, Cameron (University of Texas at Austin)
Grigsby, Julia Elisenda (Columbia University)
Hedden, Matthew (Massachusetts Institute of Technology)
Herald, Chris (University of Nevada - Reno)
Himpel, Benjamin (University of Bonn)
Honda, Ko (University Southern California)
Hutchings, Michael (University of California Berkeley)
Jabuka, Stanislav (University of Nevada-Reno)
Kim, Hee Jung (McMaster University)
Kirby, Robion (University of California - Berkeley)
Kirk, Paul (Indiana University)
Lee, Yi-Jen (Purdue University)
Mark, Thomas (University of Virginia)
Matic, Gordana (University of Georgia)
Ng, Lenny (Duke University)
Owens, Brendan (Louisiana State University)
Park, B. Doug (University of Waterloo)
Park, Jongil (Seoul National University)
Perutz, Tim (Cambridge University)
Plamenevskaya, Olga (State University of New York at Stony Brook)
Saveliev, Nikolai (University of Miami)
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Stern, Ronald (University of California, Irvine)
Sullivan, Michael (University of Massachusetts)
Van Horn-Morris, Jeremy (University of Texas, Austin)
Wendl, Chris (Massachusetts Institute of Technology)
Wu, Hao (University of Massachusetts)

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Chapter 13

Stochastic Dynamical Systems and Climate Modeling (07w5007)

Apr 15 - Apr 20, 2007

Organizer(s): Jinqiao Duan (Illinois Institute of Technology), Boualem Khouider (University of Victoria), Richard Kleeman (Courant Institute, New York University), Adam Monahan (University of Victoria)

Overview

The workshop on Stochastic Dynamical Systems and Climate Modelling gathered about 11 mathematicians and 15 geoscientists, from around the world, to talk about stochastic dynamical systems and their application to atmosphere and ocean sciences. At least 9 of the geoscientists work for a government lab and thus use stochastic and/or probabilistic methods in real world applications for climate predictions and/or day-to-day weather forecasts. On top of that 14 students and post-docs have attended the workshop. There were 7 women participants.

The speakers have covered a quite rich and diverse scientific program, ranging from pure mathematical issues in the new area of stochastic dynamical systems, such as,

- Lyapunov exponents for random dynamical systems,
- dynamical boundary conditions,
- synchronization in stochastic dynamical systems,
- singular perturbation theory and Navier-Stokes shell models
- stochastic mode reduction in large deterministic systems,

to systematic applied math strategies for modelling and simulating geophysical flow phenomena based on stochastic and statistical techniques:

- mode reduction and hidden Markov techniques for atmospheric low frequency variability and for the predictability of the associated annular modes,
- first passage time for stochastic climate models,
- nonlinear optimal perturbation for prediction and ensemble forecast,
- stochastic models for unresolved processes such as wave breaking, cumulus convection, and the underlying moist dynamics

to application of those techniques in the reality of climate predictions and day to day weather forecast and to explore the variability in the coupled atmosphere-ocean system, including:

- sea-surface temperature and sea surface winds,
- El Nineo-Southern Oscillation,
- low stratospheric dynamics,
- quasi-geostrophic turbulence,
- climate predictability,
- data assimilation,
- model error in weather and climate models,
- regimes and metastability in atmospheric dynamics, etc.

Some specific main topics

Mode reduction and hidden Markov chains

Given a dynamical system with a large number of dynamical variables. Imagine that we only are interested in the large scale/low-frequency variability associated with this system. This is particularly typical for climate predictions. Common sense suggests that in a suitable variable–coordinate system only a few 'slow' variables are capable are enough to capture at least a qualitative behaviour of those large scale/low-frequency features. However, often in practise the 'fast' variables have affect the slow dynamics. How many mode should be considered and what should one with the fast variables are some of the questions the systematic mode reduction techniques try to answer. This was on of the topics addressed by Andrew J. Majda from Courant Institute (NYU). His lecture was about systematic mathematical strategies for low-dimensional mode reduction for large dimensional dynamical system using stochastic methods and their application for low frequency variability. The main idea is to exploit well know teleconnection patterns that dominate the atmospheric low frequency variance in the Northern hemisphere and use them as basis functions to develop cheap climate predictability models with highly reduced degrees of freedom. To explain the basics of the stochastic mode reduction technique, he started his lecture with a simple solvable pedagogical model with three modes to explain the underlying MTV (for Majda-Timofeyev-Vanden Ejden, see Timofeyev's talk) theory. Then, he showed how the feasibility of the mode reduction when applied to an actual atmospheric model with a thousand degrees of freedom effectively reduced to a stochastic model will only ten modes, which captures the essential statistical dynamics. The rest of the lecture was devoted to investigating the existence of metastable states in atmospheric low-frequency variability, despite the nearly Gaussian behaviour in the associated probability distribution, through a technique know as the hidden Markov chains (HMV, see Franke's talk). Apparently, those metastable regimes are not part of the low-frequency dynamics but related to turbulent effect from high frequency mode. However, the mode reduction methods were capable in capturing those metastable low frequency regimes when both a suitable nonlinearity and a multiplicative noise were added into the stochastic differential equation.

Ilya Timofeyev, from U of Houston, explained that the mode reduction techniques, introduced earlier by Majda, consists on reducing a large dimensional dynamical system into a few stochastic differential equations for the slow variables, under the assumptions of ergoticity and a time scale separation, between the slow and fast variables. This is demonstrated by using a truncated Hopf-Burgers (THB) equation system as a test case. The THB obtained basically by using discrete Fourier transforms for the Hopf-Burger equation and non-paying attention to the aliasing errors introduced by the unresolved non-linear interaction, which introduce a huge amount of noise into the discretized system. He then showed how the MTV method is applied to derive a reduced number of differential equations where the noise carried by the high frequency modes is represented by a stochastic noise. He also demonstrated the importance of the scale separation by artificially increasing the timescale gap between the slow and fast variables.

Christian Franzke from the National Center for Atmospheric Research, used the hidden Markov model (HMM) technique, discussed in Majda talk, to identify meta-stable states in time series (data) taken from different atmospheric models. According to Christian the HMM approach is useful for describing situations where we are given a time series of a zonal flow, for e.g., but without knowledge of other flow fields which are crucial for dynamics of the given variable. Those 'hidden'–unknown variables are represented by a Markov chain, whose probability density function is calculated from the given time series. It turns out that even if the "observed"–given time series has a nearly Gaussian distribution the hidden Markov chain can exhibit meta-stable states. Tests are then carried for some atmospheric models, such as a barotropic flow over topography and multilayer quasi-geostrophic flow, with well know atmospheric flow patterns, and the HMM technique was able to effectively describe those regimes.

Stochastic models for cumulus convection

The physical processes associated with clouds occur at length and time scales ranging from a few millimeters and a few fractions of seconds for the droplet growth and formation to a few kilometers and a few hours for the updrafts and downdrafts in convective cells including various intermediate radiative and turbulent effects. This particularly makes it impossible in practice to represent all those processes, from first principle, on a single grid. Rather, the grid is truncated at a certain level and all the unresolved feature are represented by often ad hoc or empirical set of equations called a parameterisation. In typical general circulation model (GCM) used for climate and long range weather predictions, the grid resolution varies from 10 km to 200 km. At this level even the convective cell and the associated latent heat release feeding back into the large scale–resolved variables need to be parameterised. This is often referred to as the cumulus or convective parameterisation problem. The latent heat release from deep convective clouds is the main heat source that drives tropical atmospheric circulation and storms.

Two talks at this meeting were concerned with convective parameterisations using stochastic models. The first talk was by George Craig from DLR, Germany. In his group, they propose a statistical mechanic model based on an equilibrium–canonical distribution for the convective upward mass flux of an an ensemble of non-interacting clouds, given the cloud base mass flux which can be estimated from the large scale thermo-dynamics. He then presented some (direct) numerical simulation using cloud resolving modeling–where the convection is explicitly represented, to demonstrate the theoretical predictions. He particularly put emphasis on the fact the simulated results approach the theory when the external forcing (here radiation) is increased so that the convective clouds are not strongly linked with each, they appear and disappear in almost ad hoc fashion. This is known as the unorganized convection regime, as opposed to the organized convection were clouds clusters and superclusters as seen to form and propagate, especially in the tropical region. Finally, G. C. stressed on the efforts his group are putting together to apply this model in a realistic GCM simulation.

The second talk on this subject was by Boualem Khouider form the University of Victoria, and was on his joint work with A. Majda on birth-death stochastic models for convective inhibition. Convective inhibition refers to a stable layer located above the mixed boundary layer, where convective parcels typically originate. It constitutes an energy barrier for the rising parcel to reach it level of free convection where it becomes positively buoyant through condensational heating. They propose a simple order parameter defined on microscopic sites 1 to 10 km apart and takes values 1 or 0 according to whether convection is inhibited or there is potential for deep convection. The sites interact with each other and with the ambient (external) large-scale variables in manner similar to the Ising model used for magnetization and phase transition, i.e, via a Hamiltonian and an invariant Gibs measure. The resulting microscopic stochastic model evolves in time according to some intuitive spin-flip rules so that convection will preserve its organization character. B.K then derived a coarse grained-stochastic birth-death process by averaging the microscopic model on the "GCM" grid box. The resulting model is then tested on a toy GCM model with a bad convective parametrization and showed how the stochastic model can change the model results and capable to drive the system back and forth to a climate regime know to persist in the tropics. Moreover, B.K. showed some new results exploring some interesting parameter regimes with metastable states elucidating the behavior seen in his simulations.

Data assimilation and predictability

Cecile Penland (Climate Diagnostics Center, CIRES, University of Colorado and Physical Sciences Division/ESRL/NOAA) presented her research on adaptive stochastic modeling using data assimilation. Most operational centers are at least investigating stochastic parameterizations of unresolved processes in the ensemble forecasts, and ECMWF has already implemented a version for increasing their ensemble spread. The problem is that stochastic modeling is a real pain. To do it right, we need to look into issues of both parameter estimation and numerical integration. She and Jim Hansen and came up with a neat way to approach both problems at once and figured out why it worked.

Adam Monahan from the University of Victoria presented some new results regarding the probability density function of sea-surface momentum fluxes. These surface momentum exchanges, which exert a drag on surface winds and play an essential role in driving the ocean circulation, are largely determined by the surface wind field itself. Starting from previous work in which the pdf of surface vector winds was characterised, Monahan presented empirical and mechanistic models of the pdf of surface momentum fluxes. He demonstrated that the simulation of the first four moments (mean, standard deviation, skewness, and kurtosis, the last two of which may be parameterised in terms of the first) of the surface vector wind field is sufficient to characterise these moments of the momentum fluxes. Furthermore, an idealised stochastic boundary layer model was shown to provide a qualitatively accurate characterisation of the relationships between momentum flux moments seen in observations.

Mu Mu (Institute of Atmospheric Physics, Chinese Academy of Sciences) reported his recent work on conditional nonlinear optimal perturbation and its applications in predictability study, ensemble forecast, and adaptive observation.

Peter C. Chu (Department of Oceanography, Naval Postgraduate School, USA) reported his work on the first passage time for climate index and model prediction. Climate variability is simply represented by teleconnection patterns such as the Arctic Oscillation (AO), Antarctic Oscillation (AAO), North Atlantic Oscillation (NAO), Pacific/North American Pattern (PNA), and Southern Oscillation (SO) with associated indices. Two approaches can be used to predict the indices: forward and backward methods. The forward method is commonly used to predict the index fluctuation at time t with a given temporal increment . Using this method, it was found that the index (such as for NAO) has the Brownian fluctuations. On the base of the first passage time (FPT) concept, the backward method is introduced in this study to predict the typical time span ( ) needed to generate a fluctuation in the index of a given increment . After the five monthly indices (AO, AAO, NAO, PNA, SO) running through the past history, the FPT density functions (inverse Gaussian distribution) are obtained. FPT presents a new way to detect the temporal variability of the climate indices.

FPT can also be used as metrics to evaluate climate model predictability. FPT is defined as the time period when the prediction error first exceeds a pre-determined criterion (i.e., the tolerance level). It depends not only on the instantaneous error growth, but also on the noise level, the initial error, and tolerance level. The model predictability skill is then represented by a single scalar, FPT. The longer the FPT, the higher the model predictability skill is. A theoretical framework on the base of the backward Fokker-Planck equation is developed to determine FPT.

Youmin Tang (University of Northern BC) and Richard Kleeman (Courant Institute, NYU) presented their work on comparison of information-based measures of predictability in ensemble ENSO prediction. Ensemble predictions of the El Nino Southern Oscillation (ENSO) were conducted for the period from 1981-1998 using two hybrid coupled models. Several recently proposed information-based measures of predictability, including relative entropy (R), predictive information (PI), predictive power (PP) and mutual information (MI), were explored in terms of their ability of estimating a $priori$ the predictive skill of the ENSO ensemble predictions. The address was put on examining the relationship between these measures of predictability that do not use observations and the model prediction skills of correlation and root mean square error (RMSE) that make use of observations. The relationship identified here offers a practical means of estimating the potential predictability and the confidence level of an individual prediction.

It was found that the MI is a good indicator of overall skill. When it is large, the prediction system has high prediction skill whereas small MI often corresponds to a low prediction skill. In a perfect model scenario, this suggests the MI to be a good indicator of the actual skill of the models. The R and PI have a nearly identical average (over all predictions) as should be the case in theory. Comparing the different information-based measures reveals that R is a better predictor of prediction skill than PI and PP, especially when correlation-based metrics are used to evaluate model skill. A "triangular relationship" emerges between R and the model skill, namely that when R is large, the prediction is likely to be reliable whereas when R is small, the prediction skill is much variable. A small R is often accompanied by a relatively weak ENSO variability. The possible reasons why R is superior to PI and PP as a measure of ENSO predictability will also be discussed.

Stochastic modeling and parameterizations

Geoff Vallis (Princeton University) discussed about deterministic and stochastic variability in the coupled atmosphere-ocean-climate system. Geoff also presented a poster on the dynamics of the NAO and annular modes with insight from stochastic models.

Prashant Sardeshmukh (Climate Diagnostics Center, CIRES, University of Colorado and Physical Sciences Division/ESRL/NOAA) discussed how to reconcile non-Gaussian climate statistics with linear dynamics.

Juan Restrepo (University of Arizona) presented his research on stochastic parametrization of wave breaking.

Balasubramanya Nadiga (Los Alamos National Laboratory) discussed stochastic parameterizations of unresolved scales in geophysical large scale flows, in the context of large eddy simulations.

The immense number of degrees of freedom in large scale turbulent flows as encountered in the world oceans and atmosphere makes it impossible to simulate these flows in all their detail in the foreseeable future. On the other hand, it is essential to represent these flows reasonably accurately in Ocean and Atmospheric General Circulation Models (OGCMs and AGCMs) so as to improve the confidence in these model components of the earth system in ongoing effort to study climate and its variability. Furthermore, it is very often the case that in highly resolved computations, a rather disproportionately large fraction of the computational effort is expended on the small scales) whereas a large fraction of the energy resides in the large scales. It is for these reasons that the ideas of Large Eddy Simulation (LES)—wherein the large scale unsteady motions driven by specifics of the flow are explicitly computed, but the small (and presumably more universal) scales are modelled—are natural in this context.

Given the great interest in large scale geophysical flows with its small vertical to horizontal aspect ratio, we restrict ourselves to two-dimensional or quasi two-dimensional flows. Previous models of the small scales in how they affect the large scales in the momentum equations or equivalently the vorticity equation in incompressible settings have mostly been confined to an enhanced eddy viscosity or nonlinear eddy viscosity like that of Smagorinsky or biharmonic viscosity. Given the non-unique nature of the small scales with respect to the large scales, the aforementioned use of deterministic and dissipative closures seem rather highly restrictive. On the other hand, it would seem desirable to actually represent a population of eddies that satisfy overall constraints of the flow rather than make flow specific parametric assumptions. This has led to recent investigations of the possibility of using stochastic processes to model the effects of unresolved scales in geophysical flows. More recently, subgrid scale (SGS) stresses have been analysed in simple but resolved flows as a possible way to suggest stochastic parameterizations. These efforts have been preceded, of course, by various attempts to model anomalies in geophysical flow systems as linear Langevin equations and the analysis of stochastic models in isotropic and homogeneous three dimensional turbulence.

Nadiga (with Jinqiao Duan) analyzed the stochastic approach to parameterization in the barotropic vorticity equation and show that (i) if the stochastic parameterization approximates the SGS stresses, then the stochastic large eddy solution approximates the "true" solution at appropriate scale sizes; and that (ii) when the filter scale size approaches zero, then the solution of the stochastic LES approaches the true solution.

Leslie Smith (University of Wisconsin) reported research on reduced models for wave and vortical interactions in stochastically forced dispersive systems.

Timothy DelSole (George Mason University) presented his work on stochastic models of quasigeostrophic turbulence.

Judith Berner (European Centre for Medium-Range Weather Forecasts, UK) talked about stochastic parametrizations for representing model error in weather and climate models.

Jinqiao Duan (Illinois Institute of Technology, Chicago, USA) discussed about a stochastic approach for parameterizing unresolved scales in a simple system with a time-integral memory term. When applying to more complicated systems relevant to climate dynamics, it is noted that more physical mechanisms should be incorporated.

Philip Sura (Climate Diagnostics Center, CIRES, University of Colorado and Physical Sciences Division/ESRL/NOAA) talked about non-Gaussian SST variability.

Paul Williams (University of Reading, UK) presented work on noise-induced phenomena in low-order stratospheric dynamics.

Recent advances in random dynamical systems

Kening Lu (Brigham Young University, USA) presented a new exciting result on multiplicative ergodic theorem for random dynamical systems in a Banach space. This sets the foundation for further study of random dynamical systems in infinite dimensions.

Bjorn Schmalfuss (University of Paderborn, Germany) talked about stochastic partial differential equations models with dynamical boundary conditions. The results include random attractors and asymptotic dynamics.

Tomas Caraballo (Universidad de Sevilla, Spain) discussed synchronization of a stochastic reaction-diffusion system on a thin two-layer domain.

Hakima Bessaih (University of Wyoming, USA) discussed about a stochastic shell model, i.e., the stochastic Gledzer-Ohkitani-Yamada model. It is a simplified Fourier system. the topics include existence and uniqueness of invariant measure, non-viscous limit and asymptotic exponents.

Barbara Gentz (University of Bielefeld, Germany) discussed a geometric singular perturbation theory with application to simple stochastic climate models. This is a constructive approach to the quantitative description of the effect of noise on multiscale dynamical systems. This method, developed in collaboration with Nils Berglund (CPT-CNRS Marseille, France), consists in the construction of small sets in which the sample paths of the corresponding coupled system of stochastic differential equations are typically concentrated, and provides precise bounds on the exponentially small probability to observe atypical behaviour.

A variant of Stommel's box model for the North-Atlantic thermohaline circulation served as our key example. We first showed how to estimate the effect of random fluctuations on the fast variable which models the difference in temperature between boxes. This allows to study the reduced system for the slow variable, modelling the difference in salinity between the boxes. Depending on the freshwater flux, the salinity difference may approach a bifurcation point. In such a situation, due to noise, a transition to a different stable regime may occur even before the deterministic bifurcation point is reached. The presented approach yields qualitative estimates on such transition times and probabilities.

List of Participants

Berner, Judith (European Centre for Medium-Range Weather Forecasts)
Bessaih, Hakima (University of Wyoming)
Caraballo, Tomas (Universidad de Sevilla)
Chen, Baohua (Illinois Institute of Technology)
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Craig, George (DLR Germany)
Culina, Joel (University of Victoria)
Davoudi, Jahanshah (University of Toronto)
DelSole, Timothy (Center for Ocean-Land-Atmosphere Studies)
Duan, Jinqiao (Illinois Institute of Technology)
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Ge, Fanghua (Chinese Academy of Sciences)
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Godlovitch, Dan (University of Victoria)
Khouider, Boualem (University of Victoria)
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Majda, Andrew (New York University - Courant Institute of Mathematical Sciences)
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Chapter 14

The Many Strands of the Braid Groups (07w5104)

Apr 22 - Apr 27, 2007

Organizer(s): Joan Birman (Barnard College, Columbia University), Patrick Dehornoy (University of Caen), Roger Fenn (University of Sussex), Vaughan Jones (University of California, Berkeley), Dale Rolfsen (University of British Columbia)

The meeting gathered mathematicians from nine countries: Brazil, Canada, France, Great Britain, Japan, Korea, New Zealand, Spain, United States. It was devoted to recent progress and new interactions involving Artin's braid groups (see below for a synopsis).

The very intense scientific program comprised thirty-two talks of twenty-five and forty-five minutes, plus a problem session, and a recollection of the work and life of X.S.Lin (1957–2007), to whose memory the meeting was dedicated. In order to take advantage of the connections between talks, the schedule was organized to provide homogeneous sessions, each devoted to one particular aspect.

According to the feedback from most participants, the meeting was a great success. In particular, the problem session opened promising perspectives for future developments (see below).

The more detailed report below has four parts.

1. A mathematical synopsis of the scientific theme of the meeting.
2. The schedule of the talks and their abstracts.
3. The list of questions raised in the problem session.
4. The complete list of participants to the meeting.

Braid groups: definitions and results

The braid groups B_n were introduced by E. Artin in 1926 [1] (see also [2]). They have been of importance in many fields – algebra, analysis, cryptography, dynamics, topology, representation theory, mathematical physics – and many of these aspects were represented in the BIRS workshop. This workshop involved not only leading experts in the field, but also, importantly, a number of young researchers, postdoctoral fellows and several graduate students. This made for an exciting and informative mix of ideas on the subject. Female mathematicians were well represented, and were among the leading contributors.

Many equivalent definitions

The importance of the braid groups is based, in part, on the many ways in which they can be defined. Below are six different definitions of the braid groups.

Definition 1: Braids as particle dances. Consider n particles located at distinct points in a plane. To be definite, suppose they begin at the integer points $\{1, \ldots, n\}$ in the complex plane $\mathbb{C}$. Now let them move around in trajectories

$$\beta(t) = (\beta_1(t), \ldots, \beta_n(t)), \quad \beta_i(t) \in \mathbb{C}, \quad 0 \le t \le 1.$$

A *braid* is then such a time history with the proviso that the particles are noncolliding:

$$\beta_i(t) \neq \beta_j(t) \quad \text{if} \quad i \neq j$$

and end at the spots they began, but possibly permuted:

$$\beta_i(0) = i, \quad \beta_i(1) \in \{1, \ldots, n\}, \quad i = 1, \ldots, n.$$

If one braid can be deformed continuously into another (through the class of braids), the two are considered equivalent – we will say equal.

Braids α and β can be multiplied: one dance following the other, each at double speed. The product is associative but not in general commutative. The identity dance is to stand still, and each dance has an inverse; doing the dance in reverse time. These (deformation classes of) dances form the group B_n.

A braid β defines a permutation $i \to \beta_i(1)$ which is a well-defined element of the permutation group Σ_n. This is a homomorphism with kernel, by definition, the subgroup P_n of *pure* braids. P_n is sometimes called the *colored* braid group, as the particles can be regarded as having identities, or colors. P_n is of course normal in B_n, of index $n!$, and there is an exact sequence

$$1 \to P_n \to B_n \to \Sigma_n \to 1.$$

Definition 2: Braids as strings in 3-D. This is the usual and visually appealing picture. A braid can be viewed as the graph, or timeline, of a braid as in the first definition, drawn in real x, y, t-space, monotone in the t direction. The complex part is described as usual by $x + y\sqrt{-1}$. The product is then a concatenation of braided strings.

This viewpoint provides the connection with knots. A braid β defines a knot or link $\hat{\beta}$, its closure, by connecting the endpoints in a standard way so that no new crossings are introduced. J. W. Alexander showed that all knots arise as the closure of some braid and by a theorem of Markov (see [5] for a discussion and proof) two braids close to equivalent knots if and only if they are related by a finite sequence of moves and their inverses: conjugation in the braid group and a stabilization, which increases the number of strings.

Definition 3: B_n as a fundamental group of a configuration space. In complex n-space $\mathbb{C}^n$ consider the big diagonal

$$\Delta = \{(z_1, \ldots, z_n); \quad z_i = z_j, \quad \text{some} \quad i < j\} \subset \mathbb{C}^n.$$

Using the basepoint $(1, 2, \ldots, n)$, we see that

$$P_n = \pi_1(\mathbb{C}^n \setminus \Delta).$$

In other words, pure braid groups are fundamental groups of complements of a special sort of complex *hyperplane arrangement*, itself a deep and complicated subject.

To get the full braid group we need to take the fundamental group of the *configuration space*, of orbits of the obvious action of Σ_n upon $\mathbb{C}^n \setminus \Delta$. Thus

$$B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/\Sigma_n).$$

Notice that since the singularities have been removed, the projection

$$\mathbb{C}^n \setminus \Delta \longrightarrow (\mathbb{C}^n \setminus \Delta)/\Sigma_n$$

is actually a covering map. As is well-known, covering maps induce injective homomorphisms at the π_1 level, so this is another way to think of the inclusion $P_n \subset B_n$.

Finally, we note that the space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ can be identified with the space of all complex polynomials of degree n which are monic and have n distinct roots

$$p(z) = (z - r_1) \cdots (z - r_n).$$

This is one way in which the braid groups play a role in classical algebraic geometry, as fundamental group of the space of such polynomials.

Definition 4: The algebraic braid group. B_n can be regarded algebraically as the group presented with generators $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the braid with one crossing, with the string at level i crossing over the one at level $i+1$ and the other strings going straight across.

These generators are subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad |i-j| = 1.$$

We can take a whole countable set of generators $\sigma_1, \sigma_2, \ldots$ subject to the above relations, to define the infinite braid group B_∞. If we consider the (non-normal) subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$, these algebraically define B_n. Notice that this convention gives "natural" inclusions $B_n \subset B_{n+1}$ and $P_n \subset P_{n+1}$.

Definition 5: B_n as a mapping class group. Going back to the first definition, imagine the particles are in a sort of planar jello and pull their surroundings with them as they dance about. Topologically speaking, the motion of the particles extends to a continuous family of homeomorphisms of the plane (or of a disk, fixed on the boundary). This describes an equivalence between B_n and the mapping class of D_n, the disk D with n punctures (marked points). That is, B_n can be considered as the group of homeomorphisms of D_n fixing ∂D and permuting the punctures, modulo isotopy fixing $\partial D \cup \{1, \ldots, n\}$.

Definition 6: B_n as a group of automorphisms. A mapping class $[h]$, where $h : D_n \to D_n$, gives rise to an automorphism $h_* : F_n \to F_n$ of free groups, because F_n is the fundamental group of the punctured disk. Using the interpretation of braids as mapping classes, this defines a homomorphism

$$B_n \to Aut(F_n),$$

which Artin showed to be faithful, i. e. injective.

The generator σ_i acts as

$$x_i \to x_i x_{i+1} x_i^{-1}; \quad x_{i+1} \to x_i; \quad x_j \to x_j, \quad j \neq i, i+1.$$

Thus B_n may be considered a group of automorphisms of $Aut(F_n)$ satisfying a condition made precise by Artin.

Representations of the braid groups

One of the most active aspects of braid theory is the study of linear representations. A major breakthrough has been the proof in 2000 by S. Bigelow [4] and D. Krammer [15] of the long-standing conjecture that Artin's braid groups B_n are linear groups. That is, there exists a faithful representation of B_n in a finite-dimensional linear group. The Lawrence–Krammer representation that provides a linear representation of B_n has dimension $n(n-1)/2$. After the result was established, considerable efforts have been made to better understand the algebraic underlying socle on which the representations arise. The general question is to identify the non-trivial finite-dimensional quotients of the group algebra $\mathbb{C}B_n$, on the shape of the Iwahori–Hecke algebra investigated in the past decades. The general philosophy is: the bigger the quotient algebra, the better the results. Until recently, the biggest known algebra was the Birman–Murakami–Wenzl algebra [7].

I. Marin discussed the image of representations of the braid groups and certain generalizations in $GL(N)$ and showed that their images are Zariski-dense. This has important algebraic consequences, discussed in his abstract listed below.

By way of representations which are not linear, F. Castel reviewed certain faithful representations of B_n in mapping class groups of surfaces. He showed that in some sense, these constitute all the possible embeddings, utilizing Nielsen-Thurston theory of surface automorphisms and rigidity of the embeddings involved.

Applications to knot theory

The most obvious applications of braid theory are to the study of knots. About two decades ago, work of V. Jones [10] established a new powerful knot invariant via representations of B_n. This work led to exciting and unsuspected connections with operator theory, statistical mechanics and other aspects of mathematical physics. It was also generalized to the so-called HOMFLYPT polynomial, the Kauffman polynomial and a plethora of other knot invariants.

An outstanding open question is whether the Jones polynomial detects the unknot. In other words, if the Jones polynomial $V_K(t)$ of a knot K is trivial, does it imply that K is unknotted? The corresponding question for links of two or more components was settled very recently by Eliahou, Kauffman and Thistlethwaite [10], who displayed infinite families of links with the same Jones polynomial as the unlink, but which are nontrivially linked.

It is also well-known that there are many examples of distinct knots with the same Jones (and HOMFLYPT) polynomial, using various techniques: Conway mutation, a construction of Kanenobu (producing an infinite family with common Jones polynomial), etc. H. Morton showed that, in the presence of extra symmetry, mutant knots have satellites which (unlike knots in general) also cannot be distinguished by their HOMFLYPT polynomials. Representation theory provided the tool for Morton's proof.

J. Birman spoke of the fascinating connection between a certain family of knots which arise in dynamical systems, called Lorenz knots, and number theory. These knots were originally studied as closed trajectories of a 3-dimensional dynamical system defined by the meteorologist E. Lorenz in 1963, contained in a celebrated "strange attractor." Work by Etienne Ghys, showing that they arise in a certain "modular flow" has inspired renewed interest in this family of knots. A wonderful exposition of this is in [13].

Knot homology theories

It was shown recently by Khovanov that the Jones polynomial can be considered as a sort of Euler characteristic of a homology theory related to a given knot. Several of the talks focussed on Khovanov theory, including a presentation by L. Watson of knots which cannot be distinguished by their Khovanov homology and a proof of a functoriality property by S. Morrison. Przytycki described a relationship between Khovanov homology and the more classical Hochschild homology theory. Another recent, and very fruitful, development in low-dimensional topology is Heegaard-Floer homology. Originally defined using methods of complex analysis, a new combinatorial version of this homology theory was presented by D. Thurston at the meeting.

Three-dimensional manifolds and TQFT's

Topological quantum field theory was codified by Atiyah [3] and Witten [16] in 1988. Witten showed that the Jones polynomial, originally defined using representations of the braid groups, could also be expressed as a certain configuration space integral. One of the most important tools in the study of 3-manifolds is the Casson invariant $\lambda(M)$, defined by A. Casson for any integral homology 3-sphere M. The original definition by Casson in 1984 involved counting $SU(2)$ representations of the fundamental group of M. G. Kuperberg and D. Thurston showed, in 1999, how to express $\lambda(M)$ as a configuration space integral. Other new invariants of manifolds have been devised using TQFT methods, and the connection between TQFT and braid theory remains an active area of research. Loop spaces of configuration spaces were shown at the meeting by T. Kohno to be instrumental in developing new invariants of knots and links.

Finite-type invariants, following Vassiliev, have been extremely important in the study of knots and 3-manifolds. C. Lescop presented surgery formulas for finite-type invariants associated with rational homology 3-spheres, that is, orientable 3-manifolds with trivial rational homology in dimension one.

Braids, combinatorics and algorithms

A very active area which was well-represented at the conference concerns ideas surrounding Garside's 1969 solution to the word and conjugacy problems in the braid groups [12]. An equivalent way of describing the framework is to introduce the notion of a Garside groupoid (small catgeory where all arrows are invertible). Technically, an extended Garside structure is specified by axiomatizing the intervals $[a, a\Delta]$ of a Garside monoid, where Δ is a Garside element. The main interest of this extended framework is to make it possible to define completely new Garside structures on braid groups — and, possibly, on more general mapping class groups, but this remains a conjecture. The construction starts with considering the braid group B_n as acting on a disk with n punctures, as in Definition 5 above.

Now, the new ingredient is to add q marked points on the boundary circle. By considering certain cell decompositions of such ”bi-punctured” disks (punctures in the interior and on the boundary) up to isotopy, one obtains a lattice and, under a convenient version of Dehn's half-twist in which the boundary punctures are shifted, one obtains an action of the braid group B_n on that lattice. In the case $q = 2$ (only the North and the South poles of the disk are marked), the action is simply transitive, and one obtains the standard Garside structure of B_n. For $q \geq 3$, the action is not transitive, and one obtains a completely new structure. In particular, for $q = 3$ (3 punctures on the boundary disk), the lattice can be described explicitly, and, surprisingly enough, the famous MacLane pentagon shows up, and, more generally, the intervals $[a, a\Delta]$ are closely related with the Stasheff associahedra. This opens a new, fascinating connection between Artin's braid group and Richard Thompson's groups, and certainly much more is still to come.

The word and conjugacy problems in the braid groups have importance for their role in public key cryptography. It is well known that the complexity of the word problem in the braid group B_n is $(|W|^2 n)$, where $|W|$ is word length and n is braid index, whereas all solutions to the conjugacy problem known at this time are exponential. Codes have been designed which are based on the assumption that the conjugacy problem is fundamentally exponential, so a polynomial solution to the conjugacy problem would be of major importance. J. Gonzalez-Meneses outlined an ambitious program, with J. Birman and V. Gebhardt, to develop a polynomial-time algorithm to solve the conjugacy problem in braid groups, as well as the closely-related conjugacy search problem.

Generalizations of the braid groups

Because of the many definitions of the braid groups, there are various natural ways to generalize them, some of which have far-reaching applications. Several such generalizations were considered in the BIRS workshop, namely Artin groups (an algebraic generalization), mapping class groups (also known as modular groups), configuration spaces and their algebraic properties.

Artin groups and reflection groups

Deligne [13] and Brieskorn-Saito [6], introduced a family now referred to as Artin groups, which generalizes the braid groups and is also closely related to the so-called Coxeter groups which arise in the study of Lie groups and symmetries of Euclidean space. For a fixed positive integer n, consider an n by n matrix $M = \{m_{ij}\}$, where m_{ij} is a positive integer or ∞, with the assumption that $m_{ij} = m_{ji} \geq 2$ and $m_{ii} = 1$. The corresponding Artin group has a presentation with generators $x_1, \ldots, x_n$ and, for each pair i, j there is a relation:

$$x_i x_j x_i \cdots = x_j x_i x_j \cdots$$

where the product on each side has length m_{ij} ($m_{ij} = \infty$ indicates no relation is present). If one adjoins relations $x_i^2 = 1$, the result is the so-called Coxeter group corresponding to the given matrix.

In this context, the $n+1$ by $n+1$ matrix with entries equal to 3 just above and below the diagonal, and 2 in entries farther from the diagonal, corresponds exactly to the braid group B_n; in this case the Coxeter group is the symmetric group Σ_n. The Artin groups for which the corresponding Coxeter group is finite are an important subclass, referred to as “spherical.” As with the braid groups, Artin groups of spherical type correspond to fundamental groups of configuration spaces associated to hyperplane arrangements.

The finite Coxeter groups can be considered as groups of reflections of $\mathbb{R}^n$, acting on configuration spaces, as described in Definition 3 for the case of the braid groups. Periodic elements in the sphrical Artin groups were described at the meeting by D. Bessis. B. Wiest outlined algorithmic solutions of the conjugacy problem in an important class of Artin groups, called right-angled as their corresponding reflection groups involve right angles.

Mapping class groups

The mapping class group $Mod(S)$ of an orientable surface S is well-known to be generated by Dehn twists about simple closed curves in S. An important subgroup of this is the Torelli subgroup, consisting of (classes of) homeomorphisms which induce the identity on the homology of S. In particular, the subgroup K of $Mod(S)$ generated by twists along separating curves of S, called the Johnson kernel, lies in the Torelli subgroup. An important advance in our understanding of this family of groups was described by D. Margalit, who gave an explicit calculation of the cohomological dimension of the Torelli group. Another aspect which promises to be quite fruitful was discussed by D. Kraamer, who showed that the Torelli groups can be analyzed using a structure similar to that used by Garside to solve the conjugacy problem in braid groups.

J. Marché's talk dealt with so-called quantum representations of $Mod(S)$, showing that asymptotically they are faithful, converging in a certain sense to the space of regular functions on a certain character variety.

Surface braid groups, string links and orderings

If S is a Riemann surface, one can consider braids in the product of S with an interval, just as classical braids are defined over the disk. This defines surface braid groups $B_n(S)$, codified by Fox and Neuwirth [11] in 1962. Many aspects of these surface braid groups are still not well understood. It is known that the only surfaces whose braid groups contain elements of finite order are the sphere and projective plane. At the conference J. Guaschi and D. Goncalves described their recent compilation of exactly which finite groups which can occur as subgroups of the braid group of the sphere.

String links are another generalization of braid groups, in which the strands are no longer required to be monotone in the "time" direction. Just as braids, they may be multiplied by concatenation, but they no longer form a group, as inverses do not always exist. N. Habegger and X.-S. Lin (in whose memory the conference was dedecated) showed that under J. Milnor's notion of link homotopy, in which one allows strands to pass through themselves but not each other, string links do form a group. They also derived an algorithm for comparing string links. The student K. Yurasovskaya discussed her recent work, showing that the groups of homotopy string links can be endowed with a strict total ordering which is invariant under left-multiplication. This was inspired by the celebrated result of P. Dehornoy that the classical braid groups are left-orderable. Another student, A. Clay, presented his work showing the seemingly paradoxical result that, although the Dehornoy ordering is discrete (in the sense of orderings), certain important subgroups – for example the commutator subgroup of B_n are order-dense, under the same ordering. An important open question is whether $B_n(S)$ is left-orderable for surfaces of positive genus.

The above discussion is just a sample of the progress in braid theory discussed at the BIRS conference. Further details may be found in the abstracts which are given below.

The talks

General organization

As mentioned above, we managed to organize, as much as possible, homogeneous sessions. Roughly speaking, the themes were as follows:

- Monday morning: applications of braids to knot theory (3 talks),
- Monday afternoon: algebraic properties of braids (4 talks),
- Tuesday morning: connections with Heegard and Floer homology (4 talks),
- Tuesday afternoon: geometric aspects of braids (5 talks),
- Wednesday morning: Garside's theory of braids (4 talks),
- Thursday morning: braids and mapping class groups (4 talks),
- Thursday afternoon: more on Garside's theory (4 talks),
- Friday morning: geometric and ordered aspects of braids (4 talks).

Schedule

Here is the complete schedule and the abstracts of the talks.

Monday

9:15–9:45 Recollection of X.Lin and his work, by Joan Birman
10:00–10:25 H.Morton, Mutants with symmetry
11:00–11:45 H.Murakami, On a generalization of the volume conjecture
12:00–12:25 G.Zhang, Concordance crosscap number of a knot

14:45–15:30 P.Dehornoy, Alternating normal forms of braids
16:00–16:25 A.Clay, Normal subgroups of the braid groups and the Dehornoy ordering
16:45–17:30 J.Guaschi & D.Goncalves, Finite subgroups of the sphere braid groups
17:45–18:10 S.Humphries, Subgroups of braid groups generated by powers of Dehn twists

Tuesday

9:00–9:45 J.Przytycki, Two–braid intersection of Hochschild and Khovanov homologies
10:00–10:25 L.Watson, Knots with identical Khovanov homology
10:50–11:35 D.Thurston, Combinatorial Heegard–Floer homology for knots via grid diagrams
11:45–12:30 S.Morrison, Functoriality for Khovanov homology in S^3

14:45–15:30 C.Lescop, Surgery formulae for finite type invariants of rational homology 3–spheres
16:00–16:25 W.Menasco, A calculus for Legendrian and transversal knots
16:35–17:00 R.Fenn, Welded braids, links, their configuration spaces and other properties
17:10–17:35 H.Matsuda, A calculus on links via closed braids
17:45–18:10 J.Birman, Lorenz knots, templates and closed braids

Wednesday

9:00–9:45 D.Krammer, A Garside type structure on the Torelli group
10:00–10:25 D.Margalit, Dimension of the Torelli group
10:50–11:35 D.Bessis, Periodic elements in spherical type Artin groups
11:45–12:30 J.Gonzalez–Meneses, A project to find a polynomial solution to the conjugacy problem in braid groups

Thursday

9:00–9:45 J.Marché, On asymptotics of quantum representations of mapping class groups via skein theory
10:00–10:25 K.Kawamuro, Braid index and algebraic crossing number
11:00–11:45 F.Castel, Rigidity of the representations of the braid group in the mapping class group
12:00–12:25 S.Kamada, On braid presentation of knotted surfaces and the enveloping monoidal quandle

14:45–15:30 S.Lee, Translation numbers in Garside groups
16:00–16:25 E.Lee, Super summit property of abelian subgroups of Garside groups
16:45–17:30 I.Marin, Generalized braid groups as Zariski–dense subgroups of GL_N
17:40–18:05 B.Wiest, The conjugacy problem in right–angled Artin groups and their subgroups

20:00–21:30 Problem session

Friday

9:00–9:25 E.Kin, The ratio of the topological entropy to the volume for pseudo–Anosov braids
9:40–10:05 T.Kohno, Loop spaces of configuration spaces and link invariants
10:30–10:55 E.Yurasovskaya, String links and orderability
11:00–11:25 D.Rolfsen, Ordered groups and pseudo-Anosov maps

Abstracts

Here are (in alphabetic order by speaker surname) the abstracts of these talks.

Speaker: **David Bessis** (Ecole Normale Supérieure, Paris, France)
Title: *Periodic elements in spherical type Artin groups*
Abstract: In the braid group on n strings, the classification of periodic elements (elements with a central power) follows from a classical theorem of Kerekjarto. We generalize this to the other spherical type Artin groups and obtain a complete description of periodic elements, their conjugacy classes and their centralizers. A key ingredient is a categorical reformulation of a theorem by Bestvina.

Speaker: **Joan Birman** (Columbia University, New York, USA)
Title: *Lorenz knots, templates and closed braids*
Abstract: Lorenz knots were first defined in a 1983 paper that Bob Williams and I wrote. They arise as the periodic orbits in the flow associated to solutions to a particular ODE in 3-space which has since become a paradigm for chaos. They are of renewed interest right now because of work by Etienne Ghys, who proved that the identical family of knots (and their defining 'template') are as the closed orbits in the classical modular flow on the complement of the trefoil knot.

Speaker: **Fabrice Castel** (Université de Dijon, France)
Title: *Rigidity of the representations of the braid group in the mapping class group*
Abstract: In 1995, Perron and Vannier proved that the morphism from the braid group into the mapping class group of an orientable surface, that sends the generators of the braid group on Dehn twists, is injective. We show that under some restrictions on the genus, the embeddings between these two groups arise all from the embedding defined by Perron and Vannier. Using the rigidity of such embeddings, one can for instance compute the group of automorphisms of the braid group as well as the group of automorphisms of the mapping class group. The proof of the theorem is based on Nielsen-Thurston theory and of a simultaneous action of the mapping class group on itself, on the complex of curves and on the complex of subsurfaces.

Speaker: **Adam Clay** (University of British Columbia, Vancouver, Canada)
Title: *Normal subgroups of the braid groups and the Dehornoy ordering*
Abstract: The braid groups admit a left-ordering, discovered by Dehornoy, which is discrete as an ordering. I will show that normal subgroups interact with the Dehornoy ordering in such a way that "nearly all" normal subgroups of the braid groups are densely ordered with respect to this ordering. In particular, some popular normal subgroups–such as the commutator subgroup and kernels of the Burau representations–can be easily analyzed. This is joint work with Dale Rolfsen.

Speaker: **Alissa Crans** (Loyola Marymount University, Los Angeles, USA)—talk cancelled due to illness
Title: *Analogues of self-distributivity*
Abstract: This is joint work with Scott Carter, Mohamed Elhamdadi, and Masahico Saito. Self-distributive binary operations have appeared extensively in knot theory in recent years, specifically in algebraic structures called 'quandles.' A quandle is a set equipped with two binary operations satisfying axioms that capture the essential properties of the operations of conjugation in a group. The self-distributive axioms of a quandle correspond to the third Reidemeister move in knot theory. Thus, quandles give a solution to the Yang-Baxter equation, which is an algebraic distillation of the third Reidemeister move. We formulate analogues of self-distributivity in the categories of coalgebras and Hopf algebras and use these to construct additional solutions to the Yang-Baxter equation.

Speaker: **Patrick Dehornoy** (Université de Caen, France)
Title: *Alternating normal forms of braids*
Abstract: We describe new types of normal forms for braid monoids, Artin-Tits monoids, and, more generally, all monoids in which divisibility has some convenient lattice properties ("locally Garside monoids"). We show that, in the case of braids, one of these normal forms turns out to coincide with the normal form introduced by Burckel and deduce that the latter can be computed easily. This approach leads to a new, simple description for the canonical well-order of B_n^+ in terms of that of B_{n-1}^+ which, in turn, leads to unprovability

statements for certain games involving braids.

Speaker: **Roger Fenn** (University of Sussex, Brighton, GB)
Title: *Welded braids, links, their configuration spaces and other properties*
Abstract: Configuration spaces of the classical braids are well known. A configuration space for welded braids is given with a suggestion for possible invariants.

Speaker: **Daciberg Lima Goncalves** (Universidade de Sao Paulo, Brazil) & **John Guaschi** (Université de Toulouse, France)
Title: *Finite subgroups of the sphere braid groups*
Abstract: It is well known that the sphere braid groups $B_n(S^2)$ have torsion elements. Such elements were characterised by Murasugi. In this talk, we classify the finite subgroups of $B_n(S^2)$. Our work is partly motivated by the study of the generalisation of the Fadell-Neuwirth short exact sequence for pure braid groups to the 'mixed' subgroups of the full braid groups. By giving explicit constructions, we prove that for all $n \geq 3$, $B_n(S^2)$ contains subgroups isomorphic to the dicyclic groups of order $4n$ and $4(n-2)$. It follows that $B_n(S^2)$ contains two non-conjugate copies of the quaternion group of order 8 for all $n \geq 4$ even, one of which lies in the commutator subgroup of $B_n(S^2)$, the other not. Finally we classify the finite subgroups of $B_n(S^2)$: the maximal finite subgroups of $B_n(S^2)$ are either cyclic, dicyclic or binary polyhedral groups (their realisation depending on n). Two corollaries of this classification are: a) the binary tetrahedral group is a subgroup of $B_n(S^2)$ for all $n \geq 4$ even; b) if n is odd then the finite subgroups of $B_n(S^2)$ are cyclic or dicyclic.

Speaker: **Juan Gonzalez-Meneses** (University of Seville, Spain)
Title: *A project to find a polynomial solution to the conjugacy problem in braid groups*
Abstract: This is a joint work with Joan S. Birman and Volker Gebhardt. We present a project to find a polynomial solution to the conjugacy decision problem and the conjugacy search problem in braid groups, whose outline is the following. First we need to determine the geometric type of the braids involved, that is, to classify a given braid as periodic, reducible or pseudo-Anosov. In the periodic case, we give a polynomial solution by using some Garside structures of the braid groups and of Artin-Tits groups of type B. In the reducible case, one needs to find the reducing curves, and also to solve the question of finding the generators of the centralizer of a braid. In the pseudo-Anosov case, we show how one can simplify the situation by taking powers of the original braids, and reducing the problem to the conjugacy search problem for "rigid" braids. We will present our achievements, together with the open problems that remain.

Speaker: **Stephen Humphries** (Brigham Young University, Utah, USA)
Title: *Subgroups of braid groups generated by powers of Dehn twists*
Abstract: Let $F =< x_1, \dots, x_n >$ be the free group on n generators and let $P_n =< A_{12}, \dots, A_{n-1,n} >$ be the pure braid group with its standard (Dehn twist) generators. We identify F_n with the subgroup $< A_{1,n+1}, \dots, A_{n,n+1} >$ of P_{n+1}. We are interested in the related questions: (1) when is a subgroup of F_n which is generated by a set of powers of conjugates of $x_1, \dots, x_n$, of finite index in F_n; and (2) when is a subgroup of P_n which is generated by a set of powers of conjugates of $A_{12}, \dots, A_{n-1,n}$, of finite index in P_n? For example, we give necessary and sufficient conditions for a subgroup of P_n of the form $< A_{12}^{e_{12}}, \dots, A_{n-1,n}^{e_{n-1,n}} >$ to have finite index in P_n. The answer to question (1) involves Schur's theory of S-rings.

Speaker: **Seiichi Kamada** (Hiroshima University, Japan)
Title: *On braid presentation of knotted surfaces and the enveloping monoidal quandle*
Abstract: We introduce a method to describe a knotted surface in 4-space by a sequence of braids, Alexander's and Markov's theorem in dimension 4. It is natural to regard such a sequence as an element of the enveloping monoidal quandle in the sense of Kamada and Matsumoto.

Speaker: **Keiko Kawamuro** (Rice University, Houston, USA)
Title: *Braid index and algebraic crossing number*
Abstract: I will discuss a conjecture that the maximal Bennequin number of a knot is realized at its minimal

braid representatives.

Speaker: **Eiko Kin** (Tokyo Institute of Technology, Japan)
Title: *The ratio of the topological entropy to the volume for pseudo–Anosov braids*
Abstract: We consider two invariants of pseudo-Anosov mapping classes. One is the dilatation of pseudo–Anosov homeomorphisms and the other comes from the volume of mapping tori. Both invariants measure a kind of complexity of pseudo–Anosov mapping classes. The mapping class group on the n–punctured disk is identified with the n–braid group up to full twist braids, and it makes to sense to speak of the dilatation and the volume for pseudo–Anosov braids. We are interested in a relation of these two invariants, the dilatation and the volume. In this talk we focus on the ratio of the logarithm of the dilatation namely the (topological) entropy to the volume. We show that there is a constant $c > 0$ such that the ratio of the entropy to the volume for the pseudo–Anosov 3–braids is greater than c. We also extend this result for a family of pseudo–Anosov braids with many strands. This is a joint work with Mitsuhiko Takasawa (Tokyo Institute of Technoloty).

Speaker: **Toshitake Kohno** (University of Tokyo, Japan)
Title: *Loop spaces of configuration spaces and link invariants*
Abstract: It is know by F. Cohen and S. Gitler that the homology of the loop spaces of configuration spaces of ordered points in the Euclidean space is a graded algebra defined by infinitesimal pure braid relations. Based on this result we give a description of a link homotopy invariant as an integral of de Rham cohomology class of the loop space of a configuration space.

Speaker: **Daan Krammer** (University of Warwick, GB)
Title: *A Garside type structure on the Torelli group*
Abstract: In 1969, Garside solved the word and conjugacy problems for braid groups. We now say that he proved braid groups to be Garside groups. In 1998 another Garside structure on the braid group was discovered by Birman-Ko-Lee (BKL).

A well-known class of groups generalising braid groups are the surface mapping class groups. The Torelli group of a surface is the subgroup of the mapping class group of those elements which act trivially on the first homology $H_1(S,Z)$ of the surface.

I will present a Garside type structure on the Torelli group. It depends on the choice of a lexicographic total ordering on $H_1(S,Z)$. It is a close relative of the BKL Garside structure on the braid group.

It is not precisely a Garside structure for a number of reasons:

(1) Rather than as a group, it should be regarded as a groupoid whose object set looks a lot like a topological space;

(2) The distinguished path between two points in general has an infinite number of intermediate stops in a mild way.

Still, the most important properties of Garside groups, such as the grid property, still hold.

Speaker: **Eon-Kyung Lee** (Sejong University, Seoul, Korea)
Title: *Super summit property of abelian subgroups of Garside groups*
Abstract: Garside groups provide a lattice-theoretic generalization of braid groups and finite type Artin groups. In the talk, we show that for every abelian subgroup H of a Garside group, some conjugate $x^{-1}Hx$ consists of super summit elements. Using this property, we show that the centralizer of H is a finite index subgroup of the normalizer of H. Combining with the results on translation numbers in Garside groups, we obtain an easy proof of the algebraic flat torus theorem for Garside groups.

Speaker: **Sangjin Lee** (Konkuk University, Korea)
Title: *Garside groups and translation numbers*
Abstract: The translation number of an element in a combinatorial group is defined as the asymptotic word length of the element. The discreteness properties of translation numbers have been studied for geometric groups such as biautomatic groups and hyperbolic groups. The Garside group is a lattice-theoretic generalization of braid groups and Artin groups of finite type. In this talk, we discuss recent results on the discreteness properties of translation numbers in Garside groups, and their applications to the conjugacy problem.

Speaker: **Christine Lescop** (Université of Grenoble, France)
Title: *Surgery formulae for finite type invariants of rational homology* 3*-spheres*
Abstract: I wish to present four graphic surgery formulae for the degree n part Z_n of the Kontsevich-Kuperberg-Thurston universal finite type invariant of rational homology spheres. Each of these four formulae determines an alternate sum of the form $\sum_{I\subset N}(-1)^{\sharp I}Z_n(M_I)$ where N is a set of disjoint operations to be performed on a rational homology sphere M, and M_I denotes the manifold resulting from the operations in I. The first formula treats the case when N is a set of $2n$ Lagrangian-preserving surgeries (a *Lagrangian-preserving surgery* replaces a rational homology handlebody by another such without changing the linking numbers of curves in its exterior). In the second formula, N is a set of n rational surgeries on the components of a boundary link. The third formula deals with the case of $3n$ surgeries on the components of an algebraically split link. The fourth formula is for $2n$ surgeries on the components of an algebraically split link in which all Milnor triple linking numbers vanish. In the case of homology spheres, these formulae can be seen as a refinement of the Garoufalidis-Goussarov-Polyak comparison of different filtrations of the rational vector space freely generated by oriented homology spheres (up to orientation-preserving homeomorphisms).

Speaker: **Julien Marché** (Université Paris 7, France)
Title: *On asymptotics of quantum representations of mapping class groups via skein theory*
Abstract: We explain a simple proof of the asymptotic faithfulness of quantum representations of the mapping class group of a surface S. The idea is to show that in some sense, the quantum representations converge to the representation $H(S)$, where $H(S)$ is the space of regular functions on the character variety of S in $SL(2, C)$.

Speaker: **Dan Margalit** (University of Utah, USA)
Title: *Dimension of the Torelli group*
Abstract: In joint work with Mladen Bestvina and Kai-Uwe Bux, we prove that the cohomological dimension of the Torelli group for a closed surface of genus g at least 2 is equal to $3g - 5$.

Speaker: **Ivan Marin** (Université Paris 7, France)
Title: *Generalized braid groups as Zariski-dense subgroups of* GL_N
Abstract: Embeddings of every (irreducible) spherical-type Artin group in some GL_N have been described in recent years. We show that these embedings have Zariski-dense image, and use this to prove group-theoretical results on Artin groups. In particular we show that these groups are residually torsion-free nilpotent, and compute their Frattini and Fitting subgroups. We also generalize a classical result of D. Long which says that normal subgroups of braid groups which are not included in the center intersect non-trivially. The density result is based on a simple interpretation of these embeddings as monodromy representations, that we shall describe if time permits.

Speaker: **Hiroshi Matsuda** (Columbia University, New York, USA)
Title: *A calculus on links via closed braids*
Abstract: We improve "Markov Theorem Without Stabilization" of Birman and Menasco.

Speaker: **William Menasco** (University at Buffalo, USA)
Title: *A calculus for Legendrian and transversal knots*
Abstract: Using an extended example of the Etnyre-Honda (2,3) cabling of the (2,3) torus knot we discuss a calculus of isotopies associated with Legendrian and transversal knots in the standard contact structure of S^3 (Joint work with Douglas Lafountain, University at Buffalo).

Speaker: **Scott Morrison** (University of California, Berkeley, USA)
Title: *Functoriality for Khovanov homology in* S^3
Abstract: (Joint work with Kevin Walker.) I'll tell you what I mean by the Khovanov homology of a knot in S^3 (as opposed to the usual B^3). We can show that Khovanov homology is still functorial in this case, but it takes a bit more work beyond checking the 15 movie moves needed for functoriality in B^3.

Speaker: **Hugh Morton** (University of Liverpool, GB)

Title: *Mutants with symmetry*
Abstract: Mutants with certain extra symmetry, for example the pretzel knots $K(a_1, ..., a_k)$ with k and all a_i odd, can be shown to share many more of their Homfly satellite invariants than is the case for a general mutant. The proofs make use of representation theory of quantum $sl(N)$ modules.

Speaker: **Hitoshi Murakami** (Tokyo Institute of Technology, Japan)
Title: *On a generalization of the volume conjecture*
Abstract: The volume conjecture says that the large N limit of the N-colored Jones polynomial of a knot evaluated at the N-th root of unity would determine the volume of the knot complement. In this talk we will consider what happens if we change the evaluation.

Speaker: **Jozef Przytycki** (George Washington University, Washington DC, USA)
Title: *Two-braid intersection of Hochschild and Khovanov homologies*
Abstract: We show that Khovanov homology and Hochschild homology theories share common structure. In fact they overlap: Khovanov homology of a $(2, n)$-torus link can be interpreted as a Hochschild homology of the algebra underlining the Khovanov homology. In the classical case of Khovanov homology we prove the concrete connection. In the general case of Khovanov-Rozansky, $sl(n)$, homology and their deformations we conjecture the connection. The best framework to explore our ideas is to use a comultiplication-free version of Khovanov homology for graphs developed by L. Helme-Guizon and Y. Rong and extended here to M-reduced case, and in the case of a polygon to noncommutative algebras. In this framework we prove that for any unital algebra A the Hochschild homology of A is isomorphic to graph cohomology over A of a polygon.

Speaker: **Dale Rolfsen** (University of British Columbia, Vancouver, Canada)
Title: *Ordered groups and pseudo-Anosov maps*
Abstract: This is a report on work in progress regarding finding orderings of groups invariant under a given automorphism. One goal is to show that for a pseudo-Anosov homeomorphism of a surface, there is an ordering of the surface group invariant under the action of the induced mapping. This would imply the bi-orderability of fundamental groups of hyperbolic 3-manifolds which fibre over the circle.

Speaker: **Dylan Thurston** (Barnard College, Columbia University, New York, USA)
Title: *Combinatorial Heegaard-Floer homology for knots via grid diagrams*
Abstract: We give a combinatorial definition of Heegaard-Floer homology. In particular, this yields a very simple algorithm for computing the knot genus. Our method is based on grid diagrams, a representation for knots that, with restrictions on the allowed moves, also yields transverse or Legendrian knots or closed braids up to isotopy.

Speaker: **Liam Watson** (Université du Québec Montréal, Canada)
Title: *Knots with identical Khovanov homology*
Abstract: While it is well know that mutation is not detected by the Jones polynomial, it is presently unknown if mutation of knots preserves Khovanov homology. In this talk we will present a technique for producing pairs of distinct knots that cannot be distinguished by Khovanov homology. As an application, this construction may be applied to produce families of examples of mutant pairs that have identical Khovanov homology.

Speaker: **Bert Wiest** (Université de Rennes, France)
Title: *The conjugacy problem in right-angled Artin groups and their subgroups*
Abstract: We prove that the conjugacy problem in right-angled Artin groups and a large class of their subgroups can be solved in linear time. This concerns in particular all graph braid groups. Some of this talk is joint work with J.Crisp, some with J.Crisp and E.Godelle.

Speaker: **Ekaterina Yurasovskaya** (University of British Columbia, Vancouver, Canada)
Title: *String links and orderability*
Abstract: The group of homotopy classes of string links $H(k)$ has first been described by Nathan Habegger and Xiao-Song Lin in 1990 and provided the main tool to classify links up to link-homotopy. Since then $H(k)$ became an object of interest in itself. I shall discuss $H(k)$ as an example of orderable groups appearing

in topology.

Speaker: **Gengyu Zhang** (Tokyo Institute of Technology, Japan)
Title: *Concordance crosscap number of a knot*
Abstract: We define the concordance crosscap number of a knot as the minimum crosscap number among all the knots concordant to the knot. The four-dimensional crosscap number is the minimum first Betti number of non-orientable surfaces smoothly embedded in 4-dimensional ball, bounding the knot. Clearly the 4-dimensional crosscap number is smaller than or equal to the concordance crosscap number. We construct two infinite sequences of knots for which the 4-dimensional one is strictly smaller than the concordance one. In particular, the knot 7_4 is one of the examples.

The problem session

The problem session on Thursday night was a great moment. We think that the following list, which grew out of the discussion and was subsequently elaborated, contains very interesting and deep problems.

Question 1 Famous open question Is it true that if two elements of PB_n do not commute, then they generate a free group?

Question 2 Joan S. Birman What are the (interesting) finite quotients of B_n? Find a constructive proof of the known fact that the braid groups B_n are residually finite. That is, a proof that produces (interesting) finite quotients of B_n.

Question 3 Józef H. Przytycki What if we adjoin the relation $\sigma_i^p = 1$ and $\Delta^4 = 1$ to B_n? (For which n, p is the quotient finite?)

Question 4 Dan Margalit Let B_n^k be the subgroup of B_n fixing the first k punctures. Note that $B_n^0 = B_n$, $B_n^1 = A(B_n)$, $B_n^{n-1} = B_n^n = PB_n$. What is $Aut(B_n^k)$? (The answer is known in the aforementioned cases [Dyer-Grossman, Ivanov, Charney-Crisp, Bell-Margalit]. Bell-Margalit proved that $Aut(B_n^k)$ surjects onto $Aut(B_n^k/Z(B_n^k))$, i.e. the automorphism group can exchange any of the fixed punctures with the boundary of the disk.

Question 5 Dan Margalit Define the kth term of term of the Johnson filtration to be the kernel of the map $B_n \to Aut(F_n/F_n^k)$, where F_n^k is the kth term of the lower central series of F_n. What is this filtration? Note that the first two terms are B_n and PB_n. (Proposed generating set: push punctures about elements of F_n^k.)

Question 6 Ivan Marin What is the topological closure of the Lawrence-Krammer representation of B_n, depending on the two complex parameters?

Question 7 Joan S. Birman An open problem is whether there is a solution to the conjugacy search problem in the braid groups which is polynomial both in the braid index and word length of a braid. If one wishes to use the Nielsen-Thurston classification, then a subquestion concerns its complexity. We suggest as a starting point to compute the complexity of the existing algorithms to decide the Nielsen-Thurston type of a braid. If it turns out that none of the existing algorithms have the desired polynomial properties, we suggest that this problem be studied.

Question 8 Patrick Dehornoy Let M_n be the incidence matrix for the standard greedy normal form of braids, i.e., the matrix with rows and columns indexed by permutation braids such that the (x, y)-entry is 1 if (x, y) is left weighted, and 0 otherwise. Is the spectrum of M_n included in the spectrum of M_{n+1}? (true for $n \leq 13$; ref: J. Combinatorial Th. Series A; 114 (2007) 389-409.)

Question 9 Juan González-Meneses What is a pseudo-Anosov element in an Artin-Tits group? (There are natural definitions of reducible and periodic elements in these groups, but pseudo-Anosov ones are just defined as "none of the above". Some Artin-Tits groups embed into the braid group, so this can give a partial answer. The others embed in E_8. So the question could be: What is a pseudo-Anosov element in E_8? But a general answer would be much better.)

Question 10 Stephen Humphries Does the Artin-Tits group E_8 embed in $Aut(F_n)$ for some n? (A positive answer would solve the previous question, since an element of E_8 would be pseudo-Anosov just as maps to an infinite order, irreducible automorphism of F_n)

Question 11 Dale Rolfsen Are spherical type Artin-Tits groups left-orderable? (true provided it is true for E_8)

Question 12 Ivan Marin Do the exceptional type pure spherical Artin-Tits groups admit non-abelian free normal subgroups?

Question 13 Ivan Marin Can they be decomposed as an iterated semi-direct product of free groups?

Question 14 Fabrice Castel What is the kernel of the standard embeddings of Artin-Tits groups E_6, E_7 and E_8 in a mapping class group? What are the outer automorphisms of these groups?

Question 15 Dan Margalit Let B_n^k be the subgroup of B_n fixing the first k punctures. Note that $B_n^0 = B_n$, $B_n^1 = A(B_n)$, $B_n^{n-1} = B_n^n = PB_n$. What is $Aut(B_n^k)$? (The answer is known in the aforementioned cases [Dyer-Grossman, Ivanov, Charney-Crisp, Bell-Margalit]. Bell-Margalit proved that $Aut(B_n^k)$ surjects onto $Aut(B_n^k/Z(B_n^k))$.

Question 16 Seiichi Kamada Is there an algorithm to decide if two given n-tuples of elements of a group G are in the same orbit under the Hurwitz action of B_n on G^n? (specially for $G = B_n$ and $G = MCG(\Sigma)$)

Question 17 Daciberg Lima Goncalves For which integers $m < n$ is $B_m(S^2)$ a subgroup of $B_n(S^2)$? For $m = 3$ it is known to be true if and only if $n \equiv 0, 2 \ (mod\ 3)$.

Question 18 Daan Krammer Luis Paris proved that Artin monoids embed into groups, but his proof is rather indirect. Distill a combinatorial proof from his methods.

Question 19 Daan Krammer Is every Garside group linear? (Guess: no). Find combinatorial necessary conditions for a Garside group to be linear (more precisely, for it to have a faithful representation over a totally ordered field which realises the Garside structure).

Question 20 Daan Krammer One of the equivalent definitions of Garside groups (namely, the grid property) is a combinatorial analog of convex sets in real vector spaces, or equivalently, in real hyperbolic space. Weaken Garside groups by modelling them on convex sets in complex hyperbolic space.

Question 20 Ivan Marin Is the Frattini subgroup of a Garside group always trivial/central?

Question 21 Dale Rolfsen Assume Σ is a surface of positive genus. Is $B_n(\Sigma)$ left orderable? [$P_n(\Sigma)$ is bi-orderable]

Question 22 Joan Birman Can one find a bound on the volume of the complement of a Lorentz knot?

Question 23 Hugh Morton Let us regard a Lorentz knot as a framed knot (by the template); describe a Lorentz pattern as a framed pattern in the standard annulus by including the Lorentz pattern in the annulus. If a Lorentz knot is a satellite, is it constructed as a satellite of a Lorentz knot using a Lorentz pattern? (the satellite of any Lorentz knot using a Lorentz pattern is always a Lorentz knot).

Question 24 Roger Fenn Are there interesting polynomials that are invariants of welded links (other than the Alexander...)?

Question 25 Dale Rolfsen Is there a practical algorithm to decide for a braid β whether $\widehat{\beta}$ is fibred?

Question 26 Michel Boileau What are the positive braid presentations of a torus knot? (think of the Lorentz presentations)

Question 27 Michel Boileau Let Σ be a closed surface, φ a pseudo-Anosov homeomorphism of Σ. Then $\Sigma \rtimes S^1$ has a representation in $PSL(2, \mathbb{C})$ which induces a representation ρ of $\pi_1(\Sigma)$ in $PSL(2, \mathbb{C})$. Consider $H_1(\Sigma)$ with coefficients twisted by ρ^*. Look at the action of φ on $H_1(\Sigma)$. It is described by a matrix $M(\varphi)$ which is nontrivial. What does $M(\varphi)$ say about the dynamics of φ?

Question 28 Patrick Dehornoy Can one (fruitfully) use self-distributive systems that are not racks in knot theory? (In other words: algebraic systems that encode invariance under Reidemeister move III, but not necessarily move II; comment: highly non-trivial examples of such systems are known.)

List of Participants

Bachman, David (Pitzer College)
Bessis, David (DMA - Ecole normale supérieure)
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Castel, Fabrice (University of Bourgogne)
Clay, Adam (University of British Columbia)
Dehornoy Patrick (University of Caen)
Digne, Francois (Université de Picardie Jules-Verne)

Fenn, Roger (University of Sussex)
Gebhardt, Volker (University of Western Sydney)
Gonzalez-Meneses Juan (University of Seville)
Guaschi, John (Université Paul Sabatier)
Humphries, Stephen (Brigham Young University)
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Kamada, Seiichi (Hiroshima University)
Kania-Bartoszynska, Joanna (National Science Foundation)
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Krammer, Daan (University of Warwick)
Lee, Sang-Jin (Konkuk University)
Lee, Eon-Kyung (Sejong University)
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Lima Goncalves, Daciberg (Universidade de Sao Paulo)
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Menasco, William (University at Buffalo)
Michel, Jean (University Paris 7)
Morrison, Scott (University of California, Berkeley)
Morton, Hugh (University of Liverpool)
Murakami, Hitoshi (Tokyo Institute of Technology)
Przytycki, Jozef (George Washington University)
Rolfsen, Dale (University of British Columbia)
) **Thurston, Dylan** (Barnard College, Columbia University)
Watson, Liam (L'Université du Québec à Montréal)
Wiest, Bert (Universite de Rennes)
Yurasovskaya, Ekaterina (University of British Columbia)
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Chapter 15

New Applications and Generalizations of Floer Theory (07w5010)

May 13 - May 18, 2007

Organizer(s): Octav Cornea (Universite de Montreal), Viktor Ginzburg (University of California Santa Cruz), Ely Kerman (University of Illinois at Urbana-Champaign), Francois Lalonde (Universite de Montreal)

Overview of the Field

In the mid-eighties, Vladimir Arnold made several seminal conjectures which predicted that the number of fixed points of Hamiltonian diffeomorphisms, and more generally intersection points of Lagrangian submanifolds, should be greater than the bounds that one obtains from standard differential topology, [Ar]. These conjectures signaled the beginning of the study of symplectic topology and motivated much of the remarkable progress which has taken place in this active field during the intervening years.

One of the most important outcomes of this activity was Andreas Floer's development of his homology theory, [Fl1, Fl2, Fl3, Fl4, Fl5]. In essence, Floer theory is a set of techniques which makes it possible to extend certain aspects of the Morse theory of finite-dimensional manifolds to infinite dimensional examples. Within symplectic topology, Floer used Gromov's notion of pseudo-holomorphic curves to develop the Morse theory of the variational principles which underlie the Hamiltonian fixed-point and Lagrangian intersection problems. Instead of considering the total gradient flow of the corresponding functionals, Floer theory uses only those gradient trajectories with finite energy. It is Floer's great insight that these trajectories can be viewed as perturbed holomorphic curves, and that, under suitable assumptions, Gromov's compactness theorem provides the space of finite energy trajectories with the algebraic structure necessary to construct a chain complex whose homology is (an often rich) invariant. With this remarkable new tool, Floer proved some of the most significant of Arnold's conjectures under certain topological hypotheses.

The ground-breaking methods developed by Floer have had a profound influence on many areas of mathematics and mathematical physics. They have been applied to the Yang-Mills functional by Floer and Donaldson, and most recently to Seiberg- Witten theory in the work of Kronheimer and Mrowka. Another primary example of this influence is Heegaard Floer theory of Oszvath and Szabo which is a prominent new tool in low dimensional topology. Symplectic versions of Floer theory have also had a profound influence on other areas. For instance, the new and rapidly developing field of string topology is deeply related with the Floer homology of cotangent bundles and the symplectic field theory of unit cotangent bundles via the work of Viterbo, Salamon-Weber, Abbondandolo-Schwarz and Ceileibak-Latchev.

State of the field and open questions

Floer homology is still the most important tool in symplectic topology. Recently, there has been a great deal of activity in the field to generalize Floer theory and to apply it in new ways and in new settings. Many of these efforts focus on the analytic underpinnings of Floer theory with the intention of overcoming several natural restrictions of the original theory. Other efforts concern applications of the rich algebraic structures of Floer theories, which arise from its compactness statements. There is also a great effort to utilize the ideas of Floer theory to address new questions in Hamiltonian dynamics and symplectic rigidity. This workshop brought together researchers at the forefront of many of these developments.

Presentation Highlights

The following sixteen talks were delivered at the meeting.

A non-displaceable Lagrangian torus in T^*S^2
Peter Albers

Abstract: Leonid Polterovich exhibited a beautiful Lagrangian torus in T^*S^2 and asked if this torus is Hamiltonianly displaceable. In joint work with Urs Frauenfelder we prove that the Lagrangian Floer homology does not vanish, indeed equals the singular homology of the torus. In particular, this gives a negative answer to Polterovich's question. In the talk, we will describe the construction of the Lagrangian torus and present the computation of the Lagrangian Floer homology which is based on an symmetry argument.

An exact sequence for symplectic and contact homology
Frederic Bourgeois

Abstract: Given a symplectic manifold (W, ω) with contact type boundary (M, ξ), one can define the symplectic homology of (W, ω) and the linearized contact homology of (M, ξ) with respect to its filling. We introduce a Gysin-type exact sequence relating these invariants and describe one of the maps therein in terms of rational holomorphic curves in the symplectization of (M, ξ). This is joint work with Alexandru Oancea.

Floer-Novikov homology and Lagrangian embeddings in the cotangent bundle
Mihai Damian

Abstract: We use a non-Hamiltonian version of Floer theory to establish some obstructions on the existence of exact Lagrangian embeddings in the cotangent bundle of a manifold which fibers over the circle.

Intersection rigidity in symplectic topology: some rigid sets are more rigid than the others
Michael Entov

Abstract: A central and well-known rigidity phenomenon in symplectic topology says that certain sets in symplectic manifolds cannot be displaced by a Hamiltonian isotopy (even though they can be displaced by a smooth one). The talk will concern an hierarchy of intersection rigidity properties of sets beyond such a non-displaceability by a Hamiltonian isotopy: as it turns out, some sets cannot be displaced by symplectomorphisms (including non-Hamiltonian ones!) from more sets than the others. I will also present new examples of rigidity of intersections involving, in particular, specific fibers of moment maps of Hamiltonian torus actions, monotone Lagrangian submanifolds (following the previous work of Peter Albers) as well as certain, possibly singular, sets defined in terms of Poisson-commutative subalgebras of smooth functions.

The results are based on the machinery of partial symplectic quasi-states. These are certain real-valued non-linear functionals on the space of all continuous functions on a closed symplectic manifold which are constructed by means of the Hamiltonian Floer theory and which conveniently encode a part of information contained in it.

This is a joint work with Leonid Polterovich.

Trivial Curves in Symplectic Field Theory
Oliver Fabert

Abstract: Unlike trivial cylinders themselves, branched covers of trivial cylinders might come with the right index to contribute to the algebraic invariants of SFT. However, one has to add abstract perturbations to the Cauchy-Riemann operator, using e.g. the polyfold theory by Hofer, Wysocki and Zehnder, before counting these curves. Using obstruction bundles we prove that the resulting relative virtual moduli cycles are zero. In particular, we show how to deal with the codimension one boundary of moduli spaces of punctured curves in order to define Euler numbers for Fredholm problems. It follows that the differential in SFT is indeed strictly (!) decreasing with respect to the natural action filtration.

Rabinowitz's action functional for very negative line bundles
Urs Frauenfelder

Abstract: This is joint work with Kai Cieliebak. The motivation for this work comes from an alternative attempt to prove the Arnold conjecture, avoiding abstract perturbation theory. We consider a very negative line bundle over an integral symplectic manifold. Such a line bundle is itself a symplectic manifold allowing a Hamiltonian circle action coming from rotations in the fibres. Its Marsden-Weinstein quotient is conformally symplectomorphic to the base symplectic manifold. Rabinowitz's action functional is a Lagrange multiplier functional whose critical points lie in the Marsden-Weinstein quotient and project down to critical points of the action functional of classical mechanics. If the line bundle is negative enough there are generically no holomorphic spheres in the line bundle so that Rabinowitz's action functional has better compactness properties than the action functional of classical mechanics.

Normally polynomial perturbation : revisited
Kenji Fukaya

Abstract: In this talk I will explain the first half of Chapter 8 of our book on Lagrangian Floer theory written jointly with Oh, Ohta, and Ono. There the construction of filtered A-infinity algebras over the integers is given for semi-positive Lagrangian submanifolds. The main part of the construction is to use single-valued abstract perturbations of the Kuranishi map of the Kuranishi structure constructed on the moduli space of pseudo-holomorphic discs. Various techniques are used for this purpose including representation theory of groups, real algebraic geometry, and the Whitney stratification of vector bundles over a stack.

The Generalized Weinstein-Moser Theorem and Periodic Orbits of Twisted Geodesic Flows
Basak Gurel

Abstract: The Weinstein–Moser theorem asserts the existence of a certain number of distinct periodic orbits of an autonomous Hamiltonian flow on a symplectic Euclidean space on every energy level near a non-degenerate extremum. A similar question is of interest and has been extensively studied in the case where the Euclidean space is replaced by any symplectic manifold and the non-degenerate extremum at a point is replaced by a Morse-Bott non-degenerate symplectic extremum.

Along these lines, in a recent joint work with Viktor Ginzburg, we prove the existence of periodic orbits of an autonomous Hamiltonian flow on all energy levels near a Morse–Bott non-degenerate symplectic extremum of the Hamiltonian, provided that the ambient manifold meets certain topological conditions.

As an immediate application of the generalized Weinstein–Moser theorem, we establish the existence of periodic orbits of a twisted geodesic flow on all low energy levels, provided that the "magnetic field" form is symplectic and spherically rational.

In this talk I will discuss these results and outline the proof of the main theorem.

Gluing pseudoholomorphic curves along branched covered cylinders
Michael Hutchings

Abstract: We discuss how to glue together two pseudoholomorphic curves in the symplectization of a contact 3-manifold together with an index zero branched cover of an $\mathbb{R}$-invariant cylinder between them. The number of such gluings is given by a count of zeroes of a certain section of an obstruction bundle over a noncompact moduli space of branched covers. We obtain a combinatorial formula for this count. We deduce that $d^2 = 0$ in embedded contact homology. (joint work with Cliff Taubes)

From Monopoles to curves on manifolds with cylindrical ends
Yi-Jen Lee

Abstract: I will describe an extension of Taubes's "$SW \to Gr$" theorem in the context of manifolds with cylindrical ends. The key new ingredient of the proof is an estimate of the Seiberg- Witten "topological energy", which grows linearly with r–the parameter of perturbation–for large r. In fact, its asymptotic slope is precisely $1/4\pi$ of a Gromov notion of energy by Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder. This technical result is useful for a program towards the proof of the equivalence of Seiberg-Witten and Heegaard Floer homologies.

Spectral invariants in Lagrangian Floer homology
Remi Leclercq

Abstract : We generalize the spectral invariants introduced by Y.-G. Oh and M. Schwarz to the case of Lagrangian intersections Floer theory. They are the homological counterpart of Lagrangian spectral invariants of higher order that we also introduce. We provide a way to distinguish one from the other via a purely topological object and estimate their differences in terms of a geometric quantity. We show that this property induces interesting corollaries regarding the homological invariants. Finally, we show that they carry strictly more information than their homological counterpart, even in the Morse case, by making explicit computations in a particular case.

A structure equation for moduli spaces of holomorphic discs
Klaus Mohnke

Abstract: The moduli spaces of holomorphic discs with boundary on a Lagrangian submanifold define an element in the String Topology algebra of the submanifold. This element satisfies an equation similar to the Maurer-Cartan equations. This was observed and exploited by K. Fukaya. He pointed out that this element can be viewed as a flat connection, and applications arise if one applies gauge theoretic arguments. I will try to shed some light on these ideas.

Monotone Lagrangian tori in $\mathbb{C}P^n$
Felix Schlenk

Abstract: We construct many (about 2^n) different monotone Lagrangian tori in $\mathbb{C}P^n$. This is work joint with Yuri Chekanov.

The ring isomorphism between the pair-of-pants and the Chas-Sullivan product
Matthias Schwarz

Abstract: This joint work with Alberto Abbondandolo establishes an explicit ring isomorphism between Floer homology on cotangent bundles with the pair-of-pants product and the Chas-Sullivan product on the free loop space. We consider also the more general case of Floer homology for paths with conormal boundary conditions.

Floer field theory and Lagrangian correspondences
Katrin Wehrheim

Abstract: In joint work with Chris Woodward we prove an isomorphism of Floer homologies for embedded composition of Lagrangian correspondences. This isomorphism is obstructed by a novel type of bubble, which can however be excluded in monotone settings. This isomorphism is the crucial step in building a symplectic 2-category whose morphism spaces are Donaldson/Fukaya-type categories of (generalized) Lagrangian correspondences and Floer homology classes. The algebraic structures are defined by counts of "pseudoholomorphic quilts". We also obtain a "categorification 2-functor". One application is a general machinery for constructing new topological invariants by associating smooth Lagrangian correspondences to "simple morphisms" (e.g. 3-cobordisms or tangles with one critical point) and checking that the Cerf moves (which connect equivalent decompositions into simple morphisms) correspond to embedded composition of Lagrangian correspondences.

Exact triangles for fibered Dehn twists

Chris Woodward

Abstract: Seidel proved an exact triangle for Dehn twists in Floer homology which is a key ingredient in his computational algorithm for computing the Fukaya category of Lefschetz fibrations. I will discuss a generalization of this triangle to the case of fibered Dehn twists, suggested by Seidel-Smith, and its applications to exact triangles for various Floer theories such as $SU(n)$ knot homology. The proof uses pseudoholomorphic quilts. This is joint work with Katrin Wehrheim.

Outcome of the Meeting

The talks given at the workshop represented a broad range of research programs which concern or utilize Floer homology.

Various aspects of the analytic underpinnings of Floer theory were addressed in the talks of Fabert, Fukaya, Hutchings, and Lee. Kenji Fukaya presented the latest developments in his joint work with Oh, Ohta and Ono to use Floer theory to associate a filtered A^∞-algebra to a semi-positive Lagrangian submanifold. This construction plays a major role in the field of Mirror Symmetry. In his talk, Fukaya described a new approach to the transversality problem for the underlying spaces of holomorphic curves, which uses a variety of techniques. Michael Hutchings presented a new procedure for enumerating the number of ways of gluing holomorphic curves in symplectizations along branched $\mathbb{R}$-invariant holomorphic cylinders of index zero. This is an important step in his joint project with Taubes to construct Embedded Contact Homology and to prove that it agrees with an appropriate version of Seiberg-Witten Floer homology. In particular, this gluing result implies that the square of the differential is zero. On a related topic, Oliver Fabert presented a proof that branched $\mathbb{R}$-invariant holomorphic cylinders of index zero do not contribute the differential Symplectic Field theory. One important implication of this result is that the differential preserves the action filtration Yi-Jen Lee presented new energy estimates which will play a crucial role in her program to relate the Seiberg-Witten and Ozsvath-Szabo Floer theories.

In another direction, Urs Frauenfelder described a joint project with Kai Cieliebak aiming at making use of Rabinowitz's action functional to overcome analytic difficulties in the proof of Arnold's conjecture.

The talks of Bourgeois, Leclercq, Mohnke and Schwarz described new applications and constructions involving the algebraic structure of Floer theory. Frederic Bourgeois described his joint work with Oancea to construct an exact sequence relating the symplectic homology of a symplectic manifold with the linearized contact homology of its boundary. Remi Leclercq outlined a construction of new spectral invariants for Lagrangian manifolds generalizing spectral invariants in Lagrangian Floer homology. The talk of Klaus Mohnke focused on the structure of the string theory algebra of a Lagrangian submanifold, generalizing its Lagrangian Floer homology. Matthias Schwarz described an isomorphism between the ring of Hamiltonian Floer homology of a cotangent bundle and the homology of the loop space of the base equipped with the Chas–Sullivan product.

The talk of Felix Schlenk concerned his joint project with Chekanov to construct and classify monotone Lagrangian tori, which are central objects in the study of Lagrangian Floer homology. Peter Albers reported on a new computation of Lagrangian Floer homology, obtained with Urs Frauenfelder, which is relevant to recent work of Eliashberg, Kim, and Polterovich.

There were also several talks concerning new applications of Floer homology to Hamiltonian dynamical systems and symplectic intersection phenomena. Basak Gurel presented a solution, obtained in collaboration with Viktor Ginzburg, of an old problem of Arnold concerning the existence of periodic orbits for low energy charged particles moving in a nondegenerate magnetic field. Mihai Damian described new obstructions to exact Lagrangian embeddings into cotangent bundles obtained using Floer homology for symplectic maps which are not isotopic to the identity. Michael Entov discussed a hierarchy of symplectic intersection results, obtained, together with Leonid Polterovich, using their newly developed theory of partial symplectic quasi-states.

List of Participants

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Albers, Peter (New York University)
Bourgeois, Frédéric (Université Libre de Bruxelles)
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Damian, Mihai (IRMA, Universite Louis Pasteur, Strasbourg)
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Fabert, Oliver (University of Munich)
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Fukaya, Kenji (Kyoto)
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Hind, Richard (University of Notre Dame)
Hofer, Helmut (New York University)
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Leclercq, Remi (University of Montreal)
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Bibliography

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Chapter 16

The Mathematics of Knotting and Linking in Polymer Physics and Molecular Biology (07w5095)

May 20 - May 25, 2007

Organizer(s): Kenneth Millett (University of California Santa Barbara), Eric Rawdon (University of Saint Thomas), Christine Soteros (University of Saskatchewan), Andrzej Stasiak (University of Lausanne), Stuart Whittington (University of Toronto)

Introduction

The focus of this workshop was the mathematics associated with an array of cutting edge problems in polymer physics and molecular biology showing promise for immediate progress at the interfaces between mathematics and the physical and life sciences.

The first targeted area concerns the presence of knotting of DNA in living cells at a steady-state level lower than the thermodynamic equilibrium expected for a system in which inter-segmental passages within long DNA molecules occurs at random. Can one develop a systematic approach to understanding the wide range of potential topoisomerase mechanisms and their application in diverse settings? Is there a selective topoisomerase mechanism by which knotting is kept below a topological equilibrium or are there specific constraining mechanisms promoting this relaxation of knots? The study of the characteristics of the equilibrium now include geometric, spatial, and topological aspects that may be implicated in these mechanisms as well as the characteristics of polymers, for example under theta conditions. Computational, experimental and theoretical aspects of this area were featured in many of the presentations and discussions.

The second targeted area concerns the mathematical, statistical, and computational tools under development for the study of knotting and linking of open and closed macromolecules. One example is the collection of strategies developed to quantify and characterize the entanglement, e.g. knotting and linking, of open macromolecules which show promise for practical application of polymers. Another is the development of different methods for the selection of random equilateral polygons, with respect to the natural measure on the space of equilateral polygons. With efforts to quantify a wide range of new spatial features of these random equilateral polygons, greater care is necessary in order to demonstrate that the selection process is sufficient to provide statistically accurate estimations of critical quantities. Still another concerns the methods used to identify the topological type of the knotted polygons. Many of these methods are based on calculations of the Alexander polynomial or the more recent Jones and HOMFLY polynomials. While these have worked well to date, research questions are now moving into the range of 1500 edges (or Kuhn statistical segments) and, therefore, many thousands of crossings in a generic projection. Still another, distinct, computational thrust concerns efforts to achieve optimal spatial configurations when measured by the ropelength. With effort by several teams, this work faces challenging theoretical and computational obstacles.

The third focus is the application of the theory and methods above to the study of macromolecules in confined geometries, for example polymers between two parallel planes as in models of steric stabilization of dispersions or in DNA molecules contained in a capsid. Macromolecules so confined exhibit significantly different average and individual structures in comparison with those in free environments. Effective confining arises in the case of macromolecules that have specific hydrophobic and hydophilic regions or when regions have restricted flexibility or torsion. While, in general, one might expect that much is now known concerning the knotting of macromolecules in such environments, in fact little is known rigorously and many fundamental questions appear to be beyond immediate reach, both theoretically or via numerical studies.

Knotting in DNA and polymers

One of the key themes of this workshop was the focus upon the implications of experimental results in the context of theoretical models to understand them and their physiological implications. Setting the theme, **Lynn Zechiedrich**'s opening session described the role of knotting on gene function by leading to a significant increase in mutation. DNA must be long enough to encode for the complexity of an organism, yet thin and flexible enough to fit within the cell. The combination of these properties greatly favors DNA collisions, which can tangle the DNA. Despite the well-accepted propensity of cellular DNA to collide and react with itself, it is not clear what the physiological consequences are. When cells are broken open, the classified knots have all been found to be the mathematically interesting twist knots. These remarkable knots can have very high knotting node numbers (complexity), but can be untied in only one strand passage event. Zechiedrich's group used the Hin site-specific recombination system to tie twist knots in plasmids in E. coli cells to assess the effect of knots on the function of a gene. Knots block DNA replication and transcription. In addition, knots promote DNA rearrangements at a rate four orders of magnitude higher than an unknotted plasmid. These results show that knots are potentially toxic, and may help drive genetic evolution. The enzymes that untie knots are the type II topoisomerases. How they carry out their function to unknot and not knot DNA is largely unknown. Although domains of type II topoisomerases have been crystallized and the atomic structures solved, no complete, intact, active enzyme structure is known and no co-crystals with DNA have been obtained. Zechiedrich's group used electron cryomicroscopy (CryoEM) to generate the first three-dimensional structure of any intact, active type II topoisomerase. The data suggest a simple one-gate mechanism for enzyme function.

Jennifer Mann described how human topoisomerase II α resolves DNA twist knots in a single step. Cellular DNA knotting is driven by DNA compaction, topoisomerization, replication, supercoiling-promoted strand collision, and DNA self-interactions resulting from transposition, site-specific recombination, and transcription. Type II topoisomerases are ubiquitous, essential enzymes that inter-convert DNA topoisomers to resolve knots. These enzymes pass one DNA helix through another by creating an enzyme-bridged transient break. How type II topoisomerases accomplish their unknotting feat is a central question. Will a type II topoisomerase resolve a DNA twist knot in one cycle of action? Each crossing reversal performed by a type II topoisomerase requires energy. Within the cell, DNA knots might be pulled tight by forces such as those which accompany transcription, replication, and segregation, thus increasing the likelihood of DNA damage. The results show DNA knots can be lethal and promote mutations. Therefore, it would be advantageous for type II topoisomerases to resolve DNA knots in the most efficient manner. Mann's data show that purified five- and seven-noded twist knots are converted to the unknot by human topoisomerase II α with no appearance of either trefoils or five-noded twist knots which are intermediates if the enzyme acted on one of the inter-wound nodes.

Dorothy Buck presented a topological model that predicts which knots and links are the products of site-specific recombination. Buck described the topology of how DNA knots and links are formed as a result of a single recombination event, or multiple rounds of (processive) recombination events, starting with substrate(s) consisting of an unknot, an unlink, or a $(2, n)$-torus knot or link. The model relies on only three assumptions and Buck provided biological evidence for each of these assumptions. This talk presented the biological background, evidence, and applications of the model that was further explored in the talk of Erica Flapan. The biological determination is accomplished by describing the topology of how DNA knots and links are formed as a result of a single recombination event, or multiple rounds of (processive) recombination events, starting with substrate(s) consisting of an unknot, an unlink, or a $(2, n)$-torus knot or link.

Giovanni Dietler reported on the properties of knotted DNA in respect to the critical exponents and the localization of the knot crossings. He showed that probably two universality classes exist in this case and that localization of the knot crossings could explain the activity of the topoisomerases. Gel electrophoresis of DNA knots was discussed and simulations as well as experiments were presented in which the knot complexity and its topology play an essential role. Some hydrodynamics experiments with knots were presented at the end.

Mathematical, statistical, and computational methods

Discussing models employed in modeling DNA molecules, **Alexander Vologodskii** put the attention on the discrete worm-like chain, a carefully tested model that leads to a reliable analysis of enzymatic topological transformations. First, he described what exactly can be computed by the method, and how the computational results can be used to test a particular model of the enzyme action used in the simulation. He showed how two kinds of experimental data can be compared with the simulation results and discussed the major assumptions and theoretical bases of the approach. Then the key elements of the simulation were briefly considered. This general description of the approach was illustrated by specific examples.

Hue Sun Chan described the statistical mechanics of how recognition of local DNA juxtaposition geometry may underlie the unknotting and decatenating actions of type II topoisomerases. Topoisomerases may unknot and decatenate by recognizing specific DNA juxtapositions. The statistical mechanical viability of this hypothesis was investigated by considering lattice models of single-loop conformations and two-loop configurations of ring polymers. Using exact enumerations and Monte Carlo sampling, the statistical relationship between the local geometry of a juxtaposition of two chain segments on one hand, and whether a single loop was knotted or whether two loops were linked globally on the other was determined; and it was ascertained how the knot/unknot topology and global linking were altered by a topoisomerase-like segment passage at the juxtaposition. Presented results showed that segment passages at a “free” juxtaposition tend to increase knot probability but segment passages at a “hooked” juxtaposition cause more transitions from knot to unknot than vice versa, resulting in a steady-state knot probability far lower than that at topological equilibrium. Similarly, the selective segment passage at hooked juxtapositions can lower catenane populations significantly. A general exhaustive analysis of 6,000 different juxtaposition geometries showed that the ability of a segment passage to unknot and decatenate correlates strongly with a juxtaposition’s “hookedness.” Most remarkably, and consistent with earlier experiments on type II topoisomerases from different organisms, the unknotting potential of a juxtaposition geometry in the presented model correlates almost perfectly with its corresponding decatenation potential. These quantitative findings suggest that it is possible for type II topoisomerases to disentangle by acting selectively on juxtapositions with hook-like geometries.

Andrzej Stasiak presented another perspective on a model of selective simplification of DNA topology by DNA topoisomerases. The presented model tested the hypothesis that type II DNA topoisomerases maintain the steady state level of DNA knotting below the thermodynamic equilibrium by acting as topological filters that recognize preferentially certain geometrical arrangements of juxtaposed segments, “hooked relationships”. It was shown that such specificity can result in two interrelated topological consequences: maintaining the steady-state knot probability level below the topological equilibrium and selecting a specific way of relaxation of more complex knots. It was observed, in addition, that local structures in random configurations of a given knot statistically behave as analogous local structures in ideal geometric configurations of the corresponding knot types.

Mariel Vazquez contributed to the theme of modeling DNA topology simplification. Random cyclization of linear DNA can result in knotted DNA circles. Experiments on DNA confined inside P4 viral capsids have found knotting probabilities as high as 0.95. A full description of the complicated knots remains unavailable. Type II topoisomerases unknot DNA very efficiently by performing strand-passage on DNA strands. Motivated by these biological observations, Vazquez and colleagues studied random state transitions in knot space for all prime knots with 8 or fewer crossings and fixed length. The main goal was to quantify unknotting under different geometrical constraints. The long term goal is to understand the mechanism of action of type

II topoisomerases, and to characterize the knots extracted from the P4 capsids. They used the Monte Carlo based BFACF algorithm to generate ensembles of self-avoiding polygons (SAP) in Z^3 with identical knot type and fixed length. The BFACF algorithm produces a reducible Markov chain whose ergodicity classes are the knot types. They performed random strand-passage on these knots, computed state transitions between knot types, and steady-state distributions after repeated strand-passages. Introducing different topological biases resulted in various probability distributions. The large amount of knots used in their model made it possible to gather additional information regarding knots and their projections. They computed minimal lattice knots, and in some cases improve existing lower bounds. They also provided other physical measures such as the writhe and average crossing number. Finally, using an algorithm that removes Reidemeister I and II moves simultaneously, they computed the average number of crossings before and after Reidemeister removal.

Christine Soteros discussed the asymptotics of knotting after a local strand passage. On the macroscopic scale, circular DNA can be viewed simply as a ring polymer. Experimental evidence indicates that topoisomerases act locally in DNA allowing two strands of the DNA which are close together to pass through one another (i.e. enabling a “local” strand passage) in order to disentangle the DNA. This has inspired investigation of the following question about self-avoiding polygon (SAP) models: Given a SAP with a fixed knot type, how does the distribution of knots after a local strand passage depend on the initial knot type of the SAP, the length of the SAP, and on the specific details of the strand passage such as where the strand passage occurs and the number of edges altered in the strand passage? In 2000, graduate student M. Szafron introduced a model of unknotted ring polymers in dilute solution for which it is assumed that two segments of the polymer have already been brought close together for the purposes of performing a local strand passage. The conformations of the ring polymer are represented by n-edge unknotted polygons containing a specific pattern (designed to facilitate a strand passage in which exactly two segments of the polygon pass through each other) on the simple cubic lattice. Based on the assumption that each such SAP conformation is equally likely, Soteros and Szafron investigated, both theoretically and numerically, the distribution of knots after a strand passage has been performed at the location of the special pattern. The talk reviewed the theoretical and numerical (via Markov Chain Monte Carlo) results for this model with emphasis on the asymptotic properties as n increases. In addition, results for the extension of the model to other knot types such as the figure-eight knot were presented.

Enzo Orlandini discussed the topological effects of knotting on the dynamics of polymers. Knots are frequent in long polymer rings at equilibrium and it is now well established that their presence can affect the static properties of the polymer. On the other hand, topological constraints (knots) influence also the dynamical properties of a polymer. This has been shown in recent experiments where the motion of a single knotted DNA has been followed within a viscous solution and in the presence of a stretching force. These experiments raise interesting challenges to the theoretical understanding of the problem, an issue that is still in its infancy. As a first step towards the understanding of the mechanism underlying the mobility of a knot, the relaxation and diffusion dynamics of flexible knotted rings in equilibrium under good solvent conditions was investigated by Monte Carlo simulations. By focusing on prime knots and using a knot detection algorithm it was possible to monitor the diffusion in space of the knotted part of the ring, and observe in time the fluctuations of its length along the backbone. This identified a novel, slow topological time-scale, and to show that it is related to a self-reptation of the knotted region. For open chains, knotted configurations do not represent an equilibrium state any more. However, under suitable conditions (for example very tight knots or quite rigid chains), knotted metastable states persist for a very long time and a statistical description of their dynamical properties is then possible. By performing off lattice molecular dynamic simulations of a semiflexible polymer, an estimate was obtained of the average living time and the stability of these states as a function of the initial conditions (size of the initial knot) and of the rigidity of the chain.

Carla Tesi discussed the probability of knotting of polygons under a stretching force. Knots are practically unavoidable in long polymer rings and influence their properties. This has been witnessed by an increasing number of experiments that can nowadays probe the detailed properties of knotted molecules. In particular micro-manipulation techniques enable direct measurements of mechanical properties of a single molecule, and it is also possible to probe the behavior of artificially knotted DNA. It is becoming important to study theoretically how, for example, the presence of topological constraints (knots) can affect the mechanical or elastic responses of knotted molecules under external forces. As a first step in this direction Tesi and colleagues considered first the problem of looking at how the entanglement complexity in ring polymers can be affected by the presence of a tensile or contractile force. A possible experimental realization of this problem could be bacterial (or mitochondrial) DNA in solution with topoisomerases that are subjected to an external force (AFM or optical tweezers) or to flow files (shear flow for example). In this work stretched ring polymers are modeled by polygons in the cubic lattice weighted by a fugacity coupled to its span along a given direction. By performing extensive Monte Carlo simulations on this system they have been able to estimate how the knotting probability and the knot spectra depends on the force strength, both in the extensile and in the contractile regime. These findings were compared with recent rigorous results on similar models of stretched polygons.

Isabel Darcy described the modeling of protein-DNA complexes in three dimensions using TopoICE (Topological Interactive Construction Engine). Protein-DNA complexes have been modeled using tangles. A tangle consists of arcs properly embedded in a 3-dimensional ball. The protein is modeled by the 3D ball while the segments of DNA bound by the protein can be thought of as arcs embedded within the protein ball. This is a very simple model of protein-DNA binding, but from this simple model, much information can be gained. The main idea is that when modeling protein-DNA reactions, one would like to know how to draw the DNA. For example, are there any crossings trapped by the protein complex? How do the DNA strands exit the complex? Is there significant bending? Tangle analysis cannot determine the exact geometry of the protein-bound DNA, but it can determine the overall entanglement of this DNA, after which other techniques may be used to more precisely determine the geometry. KnotPlot, developed by Rob Scharein, is an interactive 3D program for visualizing and manipulating knots. TopoICE-X is a subroutine within KnotPlot for solving tangle equations modeling topoisomerase reactions.

Eric Flapan described the topological faces of the model for DNA knotting and linking developed jointly with Dorothy Buck. Flapan presented a topological model that predicts which knots and links can be the products of site-specific recombination. This is done by describing the topology of how DNA knots and links are formed as a result of a single recombination event, or multiple rounds of (processive) recombination events, starting with substrate(s) consisting of an unknot, an unlink, or a $(2, n)$-torus knot or link. The model relies on only three assumptions and we give biological evidence for each of these assumptions.

Alexander Grosberg described metastable tight knots as a worm-like polymer. Based on an estimate of the knot entropy of a worm-like chain. Grosberg and colleagues predict that the interplay of bending energy and confinement entropy will result in a compact metastable configuration of the knot that will diffuse, without spreading, along the contour of the semi-flexible polymer until it reaches one of the chain ends. The estimate of the size of the knot as a function of its topological invariant (ideal aspect ratio) agrees with recent experimental results of knotted dsDNA. Further experimental tests of these ideas were proposed.

Bertrand Duplantier discussed random linking of curves and manifolds. Duplantier proposed a formalism for evaluating random linking integrals of closed curves in $\mathbb{R}^3$ or, more generally, manifolds in $\mathbb{R}^n$, all in relative motions. It is based on the existence of universal geometric characteristic functions for each closed curve or manifold separately. It allows further averaging over the possible random shapes of those curves and manifolds.

Tetsuo Deguchi discussed the dynamics and statistical mechanics of knotted ring polymers in solution using a simulations approach toward an experimental confirmation of topological effects. Deguchi described how topological effects may give nontrivial results on the macroscopic behavior of ring polymers in solution and how one can confirm them experimentally. Numerical evaluations of some characteristic physical quantities of the solution that can be measured in polymer experiments were presented. This study was strongly motivated by recent experimental developments for synthesizing ring polymers with large molecular weights. Numerical results on dynamical and statistical properties of a dilute solution of ring polymers where topological constraints play a central role were presented. Dynamical quantities such as the diffusion constants of ring polymers in solution and the viscosity of the ring-polymer solution were discussed. These show their difference from those of the corresponding linear polymers with the same molecular weights. Secondly, the osmotic pressure of the ring-polymer solution reflects the topological interaction among ring polymers. It was numerically evaluated in terms of the random linking probability. Thirdly, the mean square radius of gyration of ring polymers under a topological constraint, which is one of the most fundamental quantities in the physics of knotted ring polymers, can be measured in the scattering experiment. The single-chain static structure factor, i.e. the scattering function, can be obtained experimentally for ring polymers with fixed topology, from which one derives the mean square radius of gyration. It is therefore important to evaluate numerically the scattering function of a knotted ring polymer in solution. Some theoretical and simulational results on the scattering functions were discussed.

Kenneth Millett discussed the problem of estimating the number of distinct topological knot types and their proportion in the space of (equilateral) polygonal knots with a fixed number of edges. For very small numbers of edges, one knows the number of knot types and can estimate their proportion but, for larger numbers of edges, only rough estimates are available. Estimates derive from Monte Carlo explorations of the (equilateral) polygonal knot space and an analysis using the HOMFLY polynomial as a surrogate for the topological knot type. As a consequence, one is interested in knowing how large a sample of knots is needed to give a good estimate of the number of topological knot types as detected by distinct HOMFLY polynomials. Some theoretical and experimental efforts concerning this question were discussed.

Rob Kusner discussed the geometric problems for embedded bands in space. Just as one can minimize the ropelength for knotted or linked space curves, one can also minimize the analogous "bandlength" for smoothly framed curves, either within a framed isotopy class, or with a pointwise constraint on the framing (which we view as a normal vector field along the corresponding bands). As a limiting case where the framing for the bands is constant, one gets knotted or linked "raceways" in the plane, a flattened analogue of knotted or linked "ropes" in space. Kusner showed that the bandlength of raceways grows at least as fast as the square root of crossing number (recall that for ropes one had instead the three-fourths power) and that this power is sharp. Kusner also commented on the shapes of length minimizing raceways, and speculated on bands or raceways as models for folded or packed proteins.

Atillio Stella discussed how the probability of realization of configurations with specific knots in closed random chains play a major role in topological polymer statistics and in its applications to macromolecular and biological physics. A problem of considerable current interest is that of comparing the knot spectra obtained for random models with those analyzed by electrophoresis for the DNA extracted from viral capsids. This comparison should help in identifying specific mechanisms of knot formation in the biological context. In the case of collapsed polymer rings, interest in the knot spectrum is also enhanced by the recent discovery that knots are fully delocalized along the backbone. Understanding if, and up to what extent, topological invariants can affect the globular state in such conditions is an intriguing fundamental issue. An analysis of extensive Monte Carlo simulations of interacting self-avoiding polygons on cubic lattice was presented. The results showed that the frequencies of different knots realized in a random, collapsed polymer ring decrease as a power (about -0.6) of the ranking order. This Zipf type of law also suggests that the total number of different knots realized grows exponentially with the chain length. Relative frequencies of specific knots converge to definite ratios for long chains, because of the free energy per monomer and its leading finite size corrections do not depend on the ring topology, while a subleading correction only depends on the minimal crossing number of the knots. This topological invariant appears to play a fundamental role in the statistics of collapsed polymers.

Jon Simon discussed the problem of measuring tangling in a large filament system. Imagine a protein or other polymer filament (or several) entangled in some complicated way, perhaps with tens or hundreds of crossings. Now imagine a second example with similarly large entanglement. Can one say something useful to distinguish the tangling in the two examples? For relatively small systems, topological knotting and linking is a powerful tool, witness the success of "topological enzymology". But for large systems, calculating exact knotting and linking may be computationally impractical; there are uncertainties in how to deal with open filaments; and knowing that one is knot 10.156 and the other 10.157 might not tell us much about the physical properties of the given system. Simon proposed that describing and quantifying tangling in large filament systems should be one of the important next-stage problems for the field of physical knots. To describe shapes of proteins (in static conformations), several researchers have developed numerical descriptors based on variations of Gausss linking-number integrals; these are related to average crossing number. Simon has begun studying another modification of average crossing number called the average bridging number. This is a simple idea, but when taken together with average crossing number, it seems to distinguish nicely between different kinds of packings for long filaments. And there appears to be reasonable stability of the relationship under random perturbations, so this approach may be useful for statistical ensembles as well as for individual conformations.

Jason Cantarella gave a talk intended as an (mostly expository) invitation to the community interested in modeling large molecules to consider an alternate mathematical framework for their work: modeling large macromolecules as divergence-free vector fields instead of as curves, polygons, chains, or tubes. From this point of view, the actual topological knot type of a very large and complicated curve will be seen as less important than its average entanglement complexity. The talk introduced this framework, reviewed some older results about the helicity of vector fields (which measures a kind of average linking number of integral curves), outlined some speculative applications to macromolecules, and introduced some work in progress reformulating the helicity of vector fields from a more modern perspective. Cantarella's reformulation of helicity opens the possibility of constructing a family of "generalized helicity" integrals analogous to finite-type invariants for knots.

Claus Ernst gave a summary of what is currently known about the topological aspects of lattice knots such as their length and curvature. The length as braids is also considered.

Eric Rawdon presented computer simulations to examine the equilibrium length of random equilateral polygons with respect to different spatial quantities, in particular with respect to the total curvature and total torsion of the polygons. Rawdon and colleagues use Markov Chain Monte Carlo methods to determine likely scaling profiles and error bars for the equilibrium length calculations

John Maddocks discussed the optimal packing of tubes in $\mathbb{R}^3$ and $\mathbb{S}^3$, contacts sets in $\mathbb{R}^3$, and connections with sedimentation dynamics.

Henryk Gerlach described the optimal packing of curves on S^2, both families of circles and open curves.

Stuart Whittington reviewed some results about lattice models of ring polymers, focusing on rigorous asymptotic results about the knot probability as a function of length, the topological and geometrical entanglement complexity and the relative frequency of occurrence of different link types. He discussed a number of open questions. For instance, we know that the knot probability goes to unity exponentially rapidly as the size of the lattice polygon goes to infinity but we know almost nothing (rigorously) about the constant appearing in the exponential term. Similarly, although we know that all non-trivial link types where both polygons are knotted grow at the same exponential rate, we know nothing about the sub-exponential terms.

Macromolecules in confined geometries

Javier Arsuaga discussed the topological considerations of the interphase nucleus. During the early phase of the cell cycle (G0/G1) chromosomes are confined to spherical regions within the nucleus called chromosome territories. The position of these territories is important in a number of biological processes (e.g. transcription, replication and DNA repair) and has important implications in human genetic diseases, in cancer and in the formation of chromosome aberrations after exposure to DNA damaging agents. Recently, a model has been proposed for the interface region between territories in which chromosomes overlap and intermingle. This new model naturally raises the question of whether chromosomes are linked or not. Motivated by this problem Arsuaga and colleagues investigated the linking of curves in confined volumes. Arsuaga presented recent results using the uniform random polygon model. First, analytically, they showed that the linking probability between a fixed closed curve and a random polygon of length n increases as $1 - O((\frac{1}{n})^{\frac{1}{2}})$. Next, numerically that the linking probability between two polygons of lengths n and m increase as $1 - O((\frac{1}{nm})^{\frac{1}{2}})$. They extended these results to the case when two polygons have a predetermined overlapping volume (as is the case in experimental observations). Arsuaga concluded with a discussion of potential extensions to other polymer models and biological implications.

Buks Janse van Rensburg discussed the properties of lattice polygons of fixed knot types in a slab of width, w, by using scaling arguments and presented numerical results from Monte Carlo simulations using the BFACF algorithm. If $p_n(K)$ is the number of polygons of length n and of knot type K in the cubic lattice, then it is known that $\lim_{n\to\infty} \frac{[log(p_n(\emptyset))]}{n} = log(\mu_\emptyset)$ exists, where $K = \emptyset$ is the unknot, and $\mu_\emptyset$ is the growth constant of unknotted polygons in the cubic lattice. Suppose that $p_n(K, w)$ is the number of knotted polygons of length n and of knot type K in a slab of width w in the cubic lattice. The generating function of this model is given by $g_K(w;t) = \Sigma p_n$(K,w) t^n, where t is a generating variable conjugate to the length of the polygons. The mean length $< n >_{K,w}$ of polygons of knot type K in a slab of width w may be estimated from $g_{K(w;t)}$ using the BFACF algorithm. The dependence of $< n >_{K,w}$ on w was estimated for $t = \mu_\emptyset^{-1}$, and the results were compared to predictions of scaling arguments. In addition, numerical results for the metric properties of knotted polygons in this ensemble were presented.

De Witt Sumners discussed why DNA knots reveal chiral packing of DNA in phage capsids. Bacteriophages are viruses that infect bacteria. They pack their double-stranded DNA genomes to near-crystalline density in viral capsids and achieve one of the highest levels of DNA condensation found in nature. Despite numerous studies, some essential properties of the packaging geometry of the DNA inside the phage capsid are still unknown. Although viral DNA is linear doublestranded with sticky ends, the linear viral DNA quickly becomes cyclic when removed from the capsid, and for some viral DNA the observed knot probability is an astounding 95%. Sumners discussed comparison of the observed viral knot spectrum with the simulated knot spectrum, concluding that the packing geometry of the DNA inside the capsid is non-random and writhe-directed.

Cristian Micheletti discussed the knotting of ring polymers in confined spaces. Stochastic simulations were used to characterize the knotting distributions of random ring polymers confined in spheres of various radii. The approach was based on the use of multiple Markov chains and reweighting techniques, combined with effective strategies for simplifying the geometrical complexity of ring conformations without altering their knot type. By these means, Micheletti and colleagues extended previous studies and characterized in detail how the probability to form a given prime or composite knot behaves in terms of the number of ring segments n and confining radius R. For $50 \leq$n≤ 450 they showed that the probability of forming a composite knot rises significantly with the confinement, while the occurrence probability of prime knots are, in general, nonmonotonic functions of $\frac{1}{R}$. The dependence of other geometrical indicators, such as writhe and chirality, in terms of R and n was also characterized. It was found that the writhe distribution broadens as the confining sphere narrows

Yuanan Diao discussed the sampling of large random knots in a confined space. Diao proposed 2-dimensional uniform random polygons as an alternative method of sampling large random knot diagrams. In fact, the 2-dimensional uniform random polygons allow one to sample knot diagrams with large crossing numbers that are diagrammatically prime since one can rigorously prove that the probability that a randomly selected 2D uniform random polygon of n vertices is almost diagrammatically prime (in the sense that the diagram becomes a reduced prime diagram after a few third Reidemeister moves) goes to one as n goes to infinity, and that the average number of crossings in such a diagram is on the order of $O(n^2)$. This strongly suggests that the 2- dimensional uniform random polygons are good candidates if one is interested in sampling large (prime) knots. Numerical studies on the 3D uniform random polygons show that these polygons for complicated knots even when they have relatively small number of vertices.

Andrew Rechnitzer talked about the mean unknotting times of random knots and knot embeddings by crossing reversals, in a problem motivated by DNA entanglement. Using self-avoiding polygons (SAPs) and self-avoiding polygon trails (SAPTs) Rechnitzer and colleagues proved that the mean unknotting time grows exponentially in the length of the SAPT and at least exponentially with the length of the SAP. The proof uses Kesten's pattern theorem, together with results for mean first-passage times in the two-parameter Ehrenfest urn model. They used the pivot algorithm to generate random SAPTs of up to 3000 steps, calculated the corresponding unknotting times, and found that the mean unknotting time grows very slowly even at moderate lengths. These methods are quite general—for example the lower bound on the mean unknotting time applies also to Gaussian random polygons. This work was accomplished in collaboration with Aleks Owcarek and Yao-ban Chan at the University of Melbourne, and Gord Slade at the University of British Columbia.

List of Participants

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Chapter 17

Modular Forms: Arithmetic and Computation (07w5065)

Jun 03 - Jun 08, 2007

Organizer(s): John Cremona (University of Nottingham), Henri Darmon (McGill University), Kenneth Ribet (University of California at Berkeley), Romyar Sharifi (McMaster University), William Stein (University of Washington)

Overview of the Field

Modular forms are functions on the extension $\mathbf{H}^* = \mathbf{H} \cup \mathbf{Q} \cup \{\infty\}$ of the complex upper half-plane $\mathbf{H}$ that are holomorphic on $\mathbf{H}$, transform well under the action of a subgroup of $\mathrm{SL}_2(\mathbf{Z})$, and satisfy certain growth conditions at the cusps $\mathbf{Q} \cup \{\infty\}$. In particular, for some level $N \geq 1$ and weight $k \geq 2$, a modular form should satisfy

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbf{H}^*$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N),$$

where $\Gamma_1(N)$ is the congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$ consisting of matrices as above with $a-1$, c, and $d-1$ divisible by N. Modular forms play a central role in number theory, one far larger than that which the scope of any week-long workshop might hope to cover. Our workshop focused on algebraic aspects of their study, especially those amenable to computation.

A modular form f comes endowed with a q-expansion, which is essentially its Taylor series about ∞. One often writes f as its q-expansion:

$$f = \sum_{n=0}^{\infty} a_n q^n,$$

where $q = e^{2\pi i z}$. We say that f is a cusp form if it vanishes at all cusps, which in particular implies that $a_0 = 0$. The modular curve $X_1(N)$ is a Riemann surface that arises as the quotient of $\mathbf{H}^*$ with respect to the action of $\Gamma_1(N)$, with a certain topology. Modular forms of level N and weight k may be also viewed as sections of the line bundle $\Omega^{k-2}_{X_1(N)/\mathbf{C}}$ of higher differentials on the modular curve $X_1(N)$. Certain commuting Hecke operators T_n with $n \geq 1$ act on the space of modular forms of a given weight and level, and one says that f is a (normalized) eigenform if $T_n f = a_n f$ for all such n. Together, these form a commutative algebra known as the Hecke algebra.

One can attach a number of objects to a cuspidal eigenform. For instance, one constructs an L-function attached to f, which is defined for $s \in \mathbf{C}$ with large enough real part by

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

that satisfies a functional equation and has an analytic continuation to all of $\mathbf{C}$. To any modular eigenform, constructions of Shimura [32] and Deligne [13] attach a Galois representation

$$\rho_f \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}_\ell})$$

for some prime ℓ that satisfies $\mathrm{Tr}\,\rho_f(\varphi_p) = a_p$ for any Frobenius φ_p at p for every prime p.

Modular forms are intricately connected with elliptic curves, genus one curves that can be described by equations of the form

$$y^2 = x^3 + Ax + B$$

with $4A^2 + 27B^2 \neq 0$. As with $\Gamma_1(N)$, we have a congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbf{Z})$ consisting of those matrices with lower left-hand entry divisible by N, and we obtain a closed modular curve $X_0(N)$ as the quotient $\Gamma_0(N)\backslash\mathbf{H}^*$. The modular curve $X_0(N)$ may be viewed as a moduli space of (generalized) elliptic curves over $\mathbf{C}$ together with a distinguished subgroup of order N. Conversely, we may define the algebraic curve $X_0(N)$ over any number field F as the moduli space space of elliptic curves defined over F (i.e., with $A, B \in F$) together with such a subgroup of its complex points.

We say that elliptic curve over $\mathbf{Q}$ is modular if it arises as the quotient of a modular curve $X_0(N)$ for some $N \geq 1$. That every elliptic curve over $\mathbf{Q}$ is modular is the famous conjecture of Shimura-Taniyama-Weil proven in most cases in the work of Wiles [37] and Taylor-Wiles [34] and completed in the work of Breuil-Conrad-Diamond-Taylor [4].

There are several equivalent formulations of modularity, two of which can be seen by considering objects attached to elliptic curves analogous to those of modular forms. First, an elliptic curve E defined over the rational numbers also gives rise to an L-function $L(E,s)$. It is determined by the number of points in its reductions modulo all prime numbers p. When this L-function is equal to the L-function of some normalized cuspidal eigenform of weight 2, the elliptic curve is modular. In particular, the modularity theorem yields an analytic continuation of $L(E,s)$ to the entire complex plane and a function equation, a special case of a conjecture of Artin's. Secondly, one has a Galois representation

$$\rho_E \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_\ell)$$

arising from the action of Galois on the first étale cohomology group of E over $\overline{\mathbf{Q}}$ with coefficients in $\mathbf{Z}_\ell$, or essentially equivalently, on the ℓ-adic torsion points of E over $\overline{\mathbf{Q}}$. That the elliptic curve E is modular says that ρ_E is conjugate in $\mathrm{GL}_2(\overline{\mathbf{Q}_\ell})$ to ρ_f for some weight 2 cuspidal eigenform f.

Recent Developments and Open Problems

Today, the central problem in the field is undoubtedly the Birch and Swinnerton-Dyer conjecture, commonly known as BSD. Given an elliptic curve E over $\mathbf{Q}$, it states that the rank r of the Mordell-Weil group $E(\mathbf{Q})$ of rational points of E is equal to the order of vanishing r_{an} of the L-function $L(E,s)$ at $s = 1$. Moreover, its strong form gives a precise description of the leading coefficient of its Taylor expansion about 1:

$$\frac{L^{(r)}(E,1)}{r!} = \frac{\Omega_E^+ \cdot |\mathrm{Sha}(E)| \cdot |R_\infty(E)| \cdot \prod_{p \text{ prime}} c_p}{|E(\mathbf{Q})_{\mathrm{tors}}|^2},$$

where $\mathrm{Sha}(E)$ denotes the Tate-Shafarevich group of E, where $R_\infty(E)$ is a certain regulator attached to E, where the c_p are Tamagawa numbers, where Ω_E^+ is a real period attached to E, and where $E(\mathbf{Q})_{\mathrm{tors}}$ denotes the torsion subgroup of the Mordell-Weil group. The finiteness of $\mathrm{Sha}(E)$ is in and of itself a conjecture of great interest and difficulty: moreover, if $\mathrm{Sha}(E)$ is finite then a result of Cassels tells us that its order is a square.

A proof of BSD appears to be still distant, but some important partial results are known, of which we mention an incomplete sampling. Gross and Zagier [16] gave a formula for $L'(E,1)$ in terms of Heegner points (when $r_{\rm an} \geq 1$) which allowed them to prove that $r \geq 1$ whenever $r_{\rm an} = 1$. Heegner points on elliptic curves are the images of points in an imaginary quadratic field K in the upper half plane that lie in $E(L)$ for an abelian extension L of K. Later work of Kolyvagin [23] on Euler systems of Heegner points then yielded that $r = r_{\rm an}$ if $r_{\rm an} \leq 1$.

Euler systems have also played a major role in Iwasawa theory. In Iwasawa theory, one studies modules over an Iwasawa algebra Λ, which is usually to say the completed $\mathbf{Z}_p$-group ring of the Galois group of the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty$ of $\mathbf{Q}$. One such Iwasawa module is the Selmer group over $\mathbf{Q}_\infty$ of the representation ρ_f attached to a cuspidal eigenform f, which is constructed as a subgroup of a Galois cohomology group attached to the representation, with certain local conditions. This Selmer group is finitely generated as a Λ-module, and it therefore has a characteristic ideal which determines much of its structure. The main conjecture of Iwasawa theory for modular forms states that the characteristic ideal of Selmer group is given by a certain p-adic L-function $L_p(f,s)$ attached to f. This p-adic L-function is a function of the p-adic numbers that interpolates values of the classical L-function $L(f,s)$ at integers s, up to certain prescribed factors. This conjecture is closely related to a p-adic version of BSD [26]. In groundbreaking work, Kato [18] used an Euler system to prove that the characteristic ideal of the Selmer group divides the ideal attached to the p-adic L-function. Since that time, Skinner and Urban have announced a much-anticipated proof of most of the reverse divisibility that uses Galois representations for higher-dimensional automorphic forms.

Classically, elliptic curves with complex multiplication can be used to give an explicit version of class field theory over imaginary quadratic fields. Here, so-called elliptic units play the role that cyclotomic units play in explicit class field theory over $\mathbf{Q}$. In fact, Rubin [29] used this and Kolyvagin's Euler system to prove the main conjecture of elliptic curves with complex multiplication. It was Kronecker's "Jugendtraum" that a similarly explicit theory could be provided for general number fields, and in particular, for real quadratic fields. Gross and Stark had conjectured the existence of special units in abelian extensions of a real quadratic field K that would fill the role of the elliptic units. Darmon gave a conjectural description of these units using a certain multiplicative integral on $\mathbf{P}^1(\mathbf{Q}_p)$ [9]. The major remaining obstacle is that the elements are constructed locally and as of yet only conjecturally arise as global units.

Second only to BSD as an open problem concerning modular forms was Serre's conjecture [30]. It has been proven in very recent work of Khare-Wintenberger [19, 20], together with a result of Kisin [21]. The representation ρ_f attached to a cuspidal eigenform f is given by the action of Galois on a lattice, which allows us to consider its reduction modulo a prime lying over a prime integer ℓ:

$$\rho_{f,\ell} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell).$$

Serre [30] conjectured that every irreducible odd Galois representation into $\mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ is modular in the sense that each is conjugate to $\rho_{f,\ell}$ for some cuspidal eigenform f. Moreover, the now-proven conjecture gives a precise description of an optimal weight and level of a modular form that yields such a representation.

Presentation Highlights

The workshop included reports on a wide variety of major current research on modular forms. By highlighting them in this section, we provide an overview of much of the field as it stands. By design, many of the talks were on computational aspects of the theory.

Several talks at the workshop were given on the structure of Shafarevich-Tate groups of elliptic curves, whose finiteness is predicted by BSD. Amod Agashe [1] spoke on a factor of $L'(E,1)$ in the case that $r_{\rm an} = 1$ that is related to the order of $\mathrm{Sha}(E)$ and presented evidence of a conjecture of Stein's on the structure of Tate-Shafarevich in this case in terms of computations of Cremona and Watkins. Dimitar Jetchev spoke on an improvement of an upper bound of on the Tate-Shafarevich group of E that is the first substantial improvement on the upper bound of Kolyvagin. Jetchev's work [17] was motivated by a computational project that Stein reported on to verify the full Birch and Swinnerton-Dyer conjecture for all elliptic curves with rank at most 1 and conductor at most 1000. Christian Wuthrich [33] also spoke about a project motivated by Stein's work to give upper bounds on the rank and p-primary part of the order of $\mathrm{Sha}(E)$, employing known results coming from Iwasawa theory on the Mazur-Tate p-adic analogue of BSD. Additionally, Neil

Dummigan [14] spoke on his investigation of the critical values of symmetric square L-functions of level one cusp forms and his construction of elements in the associated Shafarevich-Tate groups predicted by the Bloch-Kato conjecture.

Several talks involved the study of Galois representations attached to modular forms. Gabor Wiese [36] discussed his work on the multiplicities of Galois representations attached to modular forms of weight one, settling the final remaining case in the study of the question of multiplicity one for modular Galois representations. In particular, he showed that the multiplicity is always greater than one if Frobenius acts as a scalar. In closely related work, Lloyd Kilford [22] discussed the occurrence of the failure of localizations of Hecke algebras to be Gorenstein in prime weight. He described extensive calculations of the Gorenstein defect, asking whether the muliplicity of the attached Galois representation is always 2 in the case that the Hecke algebra is not Gorenstein. In particular, their work resulted in a major improvement of the algorithm for computing modular symbols over a finite field. In another direction, Aaron Greicius gave a talk on his Ph.D. thesis (with Bjorn Poonen), in which he gave explicit necessary and sufficient conditions for the surjectivity of the global Galois representation into $\mathrm{GL}_2(\hat{\mathbf{Z}})$ attached to an elliptic curve over a number field. In particular, he gave computational examples in which this surjectivity holds.

A number of talks at the workshop were related to the conjectural construction of units of Gross-Stark type units and the Stark-Heegner points used in their construction. Pierre Charollois spoke on his joint work with Henri Darmon on the construction of Stark-type units in abelian extensions of almost totally real extensions of a totally real field F in terms of certain invariants attached to Eisenstein series on GL_2 over F and tori in this group. Samit Dasgupta presented his computations of Gross-Stark units via Shintani zeta functions, discussing work with his student Kaloyan Slavov in implementing algorithms suggested by Dasgupta's work. A paper by Dasgupta [11] on the calculation of Gross-Stark units in the p-adic setting has subsequently appeared. Matt Greenberg presented a more conceptual, cohomological approach to the theory of Stark-Heegner points which has enabled him to vastly extend various definitions, due to Darmon, Dasgupta, Trifkovic, and others, of Stark-Heegner points given previously in the literature (see for instance [15]). In a related direction, Darmon gave a report on work in progress with Bertolini and Prasanna concerned with constructing points on CM elliptic curves using higher dimensional cycles on Kuga-Sato varieties. This work [2, 3] is now almost completed, and will appear in a series of two articles. The first of these, which treats a new a p-adic variant of the Gross-Zagier formula, is about to be submitted for publication.

Frank Calegari and Matthew Emerton gave coordinated talks on their joint work on automorphic forms of cohomological type. Calegari described a bound on the dimension of the space of automorphic forms for a semisimple group over a number field that does not admit a discrete series [7]. Emerton spoke on the cohomology of arithmetic groups and the construction and study of objects such as p-adic representations. They have two joint papers in preparation related to the subject.

Victor Rotger discussed the structure of endomorphism algebras of the modular abelian variety attached to a newform of weight 2. He reported on progress towards a conjecture that there exist only finitely many isomorphism classes of such endomorphism algebras of a given degree over $\mathbf{Q}$. Some of the techniques and results, both theoretical and computational, can be found in [28] and [5]. Noam Elkies discussed a method for determining explicit formulas for genus 2 curves C over $\mathbf{Q}$ for which the endomorphism algebra contains an order in a real quadratic field. Such curves arise, for instance, as the degree two factors of the Jacobian of a modular curve. Armand Brumer reported on joint work with Ken Kramer on the existence of semistable abelian varieties over $\mathbf{Q}$.

Bjorn Poonen spoke on joint work with Ed Schaefer and Michael Stoll [27] on the equation $x^2 + y^3 = z^7$, one of the most difficult to solve equations of Fermat type. The determination of its solutions required a wide variety of sophisticated techniques involving modular curves and their Jacobians, including a sophisticated descent argument and very intensive computational work. Ken Ribet spoke on recent work of his then Ph.D. student, Soroosh Yazdani (also a workshop participant), on elliptic curves of odd modular degree. In particular, he described a relationship between the modular degree and congruences of modular forms that hold with the modular form attached to the elliptic curve.

John Cremona gave an excellent foundational talk on the construction of modular forms over general number fields: discussing the analogues of classical objects such as cusps, Hecke operators, and modular symbols, with an eye towards computing with the latter, as he and his students have already done for many imaginary quadratic fields. Cecelia Busuioc [6] described a construction of a very special modular symbol with values in Milnor K_2 of the ring of p-integers of the pth cyclotomic field, for an odd prime p. She used it to give evidence for a conjecture of Sharifi's [31] relating values of a cohomological pairing on p-units to L-values of cusp forms satisfying congruences with Eisenstein series.

Several talks presented algorithms for computation with objects related to modular forms. Lassina Dembélé presented an algorithm for the computation of Hilbert-Siegel modular forms over real quadratic fields and related it to the modularity of certain threefolds. Gonzalo Tornaría presented his spectacularly fast algorithms for computing Brandt matrices attached to ternary quadratic forms and showed how to use them to compute explicit Shimura correspondences. Ulf Kühn reported on an algorithm developed by his student, Anna Posingies, for computing the first non-vanishing coefficient of L-series that contribute to the constant term of non-holomorphic Eisenstein series, studied in [25].

Beyond the daytime lecture series, there were some evening sessions specifically devoted to discussing mathematical packages devoted to and useful for computing with modular forms and related objects. In particular, Stein gave a tutorial on Sage (`http://sagemath.org`), which is a free open source mathematical software program with substantial new functionality for computing with elliptic curves and modular forms. In addition, Cremona gave another evening session discussing tools for computing with elliptic curves.

Progress Made and Outcomes

The workshop provided an excellent opportunity for the exchange of ideas between researchers and the development of productive collaboration. We report on a small sampling of work that grew out of the conference, collaborations that were formed at the conference or by its participants, and one follow-up meeting that was organized by some of the speakers at the workshop.

In the course of the meeting, Charollois and Darmon found a more elegant interpretation of the invariants discussed by Charollois in his talk as the images of certain (non-algebraic) cycles on a Hilbert modular variety under a map which is formally analogous to the Griffiths Abel-Jacobi map on higher dimensional complex algebraic cycles, which will soon appear.

Also during the meeting, Tornaría pursued his collaboration with Darmon on the connections between the Shimura correspondence and the theory of Stark-Heegner points. The joint work of Darmon and Tornaría that was essentially completed at the BIRS meeting has now appeared [10].

Moreover, Stein and Weise had productive discussions on the details of an article [24] they wrote with Tak-Lun Koo, in which they studied the set of primes for which the pth coefficient of a given CM newform generates its field of coefficients.

The meeting also led to some fruitful exchanges between Dembélé and Greenberg which later led to Dembélé proving the existence of a finite (non-solvable) extension of $\mathbf{Q}$ unramified outside 2, answering a question of Gross and Serre. This work of Dembélé [12], which grew out of exchanges at the BIRS meeting, was solicited by Serre and is to appear.

During the meeting, Dembélé and Weise began to discuss collaboration on a work in progress, and Dembélé has held a position at Universität Duisberg-Essen to work with Weise since that time. In May 2009, he will move to the University of Warwick to work with Cremona. Similarly, Yazdani has held an NSERC postdoctoral fellowship at McMaster University under the supervision of Sharifi since shortly after the meeting.

In August 2008, Kilford, Wiese, and Dembélé organized a follow-up meeting to the BIRS workshop at the Heilbronn Institute in Bristol on Computations with Modular Forms. This meeting was announced at the BIRS workshop, where some of the planning was done by its organizers. The conference emphasized the actual coding of algorithms for computing with modular forms. Links to the code presented can be found at the conference's web page: `http://maths.pratum.net/CMF`.

Conclusion

We are happy to report that the meeting achieved its stated goal of bringing together researchers working with modular forms who employ a wide range of styles in their approaches to computation. Some of the researchers are at the forefront of developing algorithms for modular forms and useful software packages employing these algorithms, others are at the forefront of modern theoretical developments in the field, and quite a few are at both. As conjecture frequently demands computation, the interaction between the researchers in attendance had tremendous potential benefit for the participants, and thereby the field, benefit that has been and will continue to be realized.

List of Participants

Agashe, Amod (Florida State University)
Brumer, Armand (Fordham University)
Busuioc, Cecilia (Boston University)
Calegari, Frank (Northwestern University)
Charollois, Pierre (Institut de Mathématiques de Jussieu)
Cohen, Henri (Universite Bordeaux 1)
Cremona, John (University of Nottingham)
Darmon, Henri (McGill University)
Dasgupta, Samit (Harvard University)
Dembele, Lassina (University of Calgary)
Dummigan, Neil (University of Sheffield)
Elkies, Noam (Harvard University)
Emerton, Matthew (Northwestern University)
Gee, Toby (Imperial College)
Gonzalez Jimenez, Enrique (Universidad Autonoma de Madrid)
Greenberg, Matthew (Harvard University)
Greicius, Aaron (University of California at Berkeley)
Jetchev, Dimitar (University of California at Berkeley)
Johnson-Leung, Jennifer (Brandeis University)
Joyce, Adam (University of Bristol)
Kilford, Lloyd (Oxford University)
Kuehn, Ulf (Universitaet Hamburg)
Poonen, Bjorn (Massachusetts Institute of Technology)
Quer, Jordi (Universitat Politècnica de Catalunya)
Ribet, Kenneth (University of California at Berkeley)
Rotger, Victor (Universitat Politècnica de Catalunya)
Sharifi, Romyar (McMaster University)
Stein, William (University of Washington)
Taylor, Karen (University of Nottingham)
Tornara, Gonzalo (Universidad de la Repblica)
Torrey, Rebecca (King's College London)
Trifkovic, Mak (University of Victoria/Fordham)
Vatsal, Vinayak (University of British Columbia)
Watkins, Mark (University of Bristol)
Wiese, Gabor (University of Duisburg-Essen)
Wuthrich, Christian (Ecole Polytechnique Federale de Lausanne)
Yazdani, Soroosh (University of California at Berkeley)

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Chapter 18

Commutative Algebra and its Interaction with Algebraic Geometry (07w5505)

Jun 10 - Jun 15, 2007

Organizer(s): Anthony Geramita (Department of Mathematics and Statistics, Queens' University), Paul Roberts (University of Utah), Bernd Ulrich (Purdue University)

Overview of the Field

Commutative Algebra and Algebraic Geometry have been closely connected since the early days of both fields. Many of the concepts in Commutative Algebra have their origins in Geometry, and many of the foundations of Algebraic Geometry are based on Algebraic results.

In this conference we brought together mathematicians who work in the interface of these two fields. Many of them are using Geometric methods to solve questions in Algebra, while others are studying Geometry using methods from Commutative Algebra.

The field is very broad, and our original intent was to concentrate on three main areas. The first deals with problems in Positive and Mixed Characteristic, a topic which has been very active in recent years. The second is in Integral Dependence and Integral Closure, a central topic in Commutative Algebra with many applications to Geometry. The third topic is Secant Varieties and Algebraic Statistics, which presents a new and unexpected application of Geometry to Algebra and other fields.

However, as the preparations for the conference progressed, it became clear that it was better to include a wider range of topics, as there are many important developments that do not fit neatly into any of these three areas. As a result, although the majority of the talks were closely related to the topics that we had originally planned, we also had contributions that can be grouped into other areas, as well as a few that were in special topics. The unifying theme was that all of the speakers discussed problems related to both Commutative Algebra and Algebraic Geometry.

The remainder of this report presents the various areas that were represented and abstracts of the talks. In addition to the three areas mentioned above, there is a section on Intersection Theory and Homological Methods, and more general methods of classical Projective Geometry have been included in the section on Secant Varieties.

Problems in positive and mixed characteristic.

The traditional area of study in Algebraic Geometry was spaces defined by equations over the real and complex numbers. However, some problems can be solved by reducing these equations modulo a prime number, and this leads to questions of positive characteristic. A ring R has characteristic p for a prime number p if $pr = 0$ for all r in R. The main advantage this gives is that it allows the use of the Frobenius map, which sends an element r to r^p; it is a ring homomorphism in this case. Most of the theory in positive characteristic involves the Frobenius map in one way or another.

One of the main objects of study in positive characteristic is Hilbert-Kunz multiplicities, which are limits of ordinary multiplicities over iterations of the Frobenius map. A central question is whether Hilbert-Kunz multiplicities are rational, which is addressed in the talk by Paul Monsky.

Hilbert-Kunz functions of singular plane curves

Paul Monsky, Brandeis University

Let G be a homogeneous degree d element of $L[x, y, z]$, where L is the algebraic closure of Z/p. If $q = p^n$, $e_n(G)$ is the colength of the ideal generated by G and the q^{th} powers of x, y and z. Then $e_n(G) = \mu q^2 - R_n$ where μ lies between $3d/4$ and d, and $R_n = O(q)$.

The talk starts with known results about μ and R_n when G is irreducible, and explains ideas in my proof that $R_n = (\text{periodic})q + (\text{eventually periodic})$. It continues with speculation, arising from computer calculations and my work with Teixeira, as to the behavior of R_n when G is reducible. In particular it makes an explicit conjecture as to the value of $e_n(H^j)$ when $p = 2$ and $H = x^3 + y^3 + xyz$. If this conjecture holds the Hilbert-Kunz multiplicity of the 5-variable polynomial $H + uv$ is an irrational element of $Q(\sqrt{7})$—whether Hilbert-Kunz multiplicities can be irrational is an outstanding problem . □

In recent years, another point of intersection of characteristic p methods in Algebra and Geometry has been the study of multiplier ideals. This will be considered further in the next section, but one interesting facet has been a relation with tight closure in positive characteristic. This is examined in the talk of Mircea Mustata.

Test ideals vs. multiplier ideals

Mircea Mustata, University of Michigan

The talk is a report on joint work with Manuel Blickle, Karen Smith and Ken-ichi Yoshida. It compares the behavior of certain invariants of singularities in characteristic zero (namely, the multiplier ideals and their jumping numbers) with invariants in positive characteristic, the so-called generalized test ideals. The multiplier ideals are by now well-established invariants, defined in terms of divisorial valuations, that can be computed using resolutions of singularities.

On the other hand, the generalized test ideals have been introduced by Hara and Yoshida using techniques inspired by tight closure theory. Results of Hara, Takagi, Yoshida and Watanabe, via reduction mod p, relate the multiplier ideals of a singularity in characteristic zero with the corresponding test ideals in positive characteristic. This connection is quite subtle, revealing deep connections with arithmetic, and there are still very interesting open problems in this area.

It was clear from the beginning that several subtle properties of multiplier ideals, that are proved via vanishing theorems (such as Subadditivity or the Restriction Theorem) have analogues in the context of test ideals, and the proofs are much more elementary. The talk discusses joint work with Yoshida emphasizing the different behavior of test ideals and multiplier ideals: roughly speaking, we show that all the algebraic properties of multiplier ideals that follow from the computation in terms of resolutions, fail for test ideals. A surprising result that also highlights this different behavior: we show that every ideal in a regular F-finite local ring can be written as a test ideal.

The talk also covers results with Blickle and Smith about the behavior of the jumping exponents of test ideals: under certain assumptions, these are all rational and form a discrete set (note the analogy with the jumping numbers of multiplier ideals). □

Shunsuke Takagi also talked about test ideals in positive characteristic and presented some new results on jumping numbers.

Rationality of F-jumping numbers on singular varieties

Shunsuke Takagi, Kyushu University

This talk is based on a joint work with Craig Huneke. Let R be an excellent Noetherian ring of prime characteristic p and I be an ideal which is not contained in any minimal prime ideal of R. Then we say that a real number $t > 0$ is an *F-jumping exponent* of I if $\tau(I^t) \neq \tau(I^{t-\epsilon})$ for all $\epsilon > 0$, where $\tau(I^t)$ is the generalized test ideal of I with exponent t (see [1] for the definition of generalized test ideals). Blickle, Mustata and Smith proved that the F-jumping exponents of I are rational and have no accumulation points if R is an F- finite regular ring essentially of finite type over a field or if R is an F-finite regular ring and I is a principal ideal. We generalize their results to the case of strongly F-regular rings. We say that an F-finite reduced ring A of characteristic $p > 0$ is *strongly F-regular* if for every nonzero divisor c of A, there exists $q = p^e$ such that $c^{1/q}A \hookrightarrow A^{1/q}$ splits an A-linear map. The following is our main results. Suppose that R is a strongly F-regular ring of characteristic $p > 0$. Then the set of F-jumping exponents of I have no accumulation points if one of the following conditions holds: (1) $R = \bigoplus_{n \geq 0} R_n$ is a $\mathbb{Q}$-Gorenstein graded ring with R_0 a field and the (Gorenstein) index of R is not divisible by p; (2) R is a $\mathbb{Q}$-Gorenstein ring whose (Gorenstein) index is not divisible by p and I is a principal ideal; (3) $R = \bigoplus_{n \geq 0} R_n$ is a graded ring with R_0 a field and R has finite graded F-representation type (see [2] for the definition of rings with finite graded F-representation type). Every F-jumping exponent of I is a rational number if the condition (1) or (2) holds.

References

[1] Hara, N. and Yoshida, K., *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), no.8, 3143–3174.

[2] Smith, K. E. and Van den Bergh, M., *Simplicity of rings of differential operators in prime characteristic*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 32–62. □

Another topic that was presented at the beginning was the main question in mixed characteristic, namely the well known homological conjectures. Work on these conjectures has led to a deep study of the local cohomology of the absolute integral closure of a local integral domain. This was the topic of the talk by Gennady Lyubeznik.

A Property of the Absolute Integral Closure of an Excellent Local Domain in Mixed Characteristic

Gennady Lyubeznik, University of Minnesota

In this talk we presented a proof of the following theorem:

Theorem. Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic, residual characteristic $p > 0$ and dimension at least 3. Let $\sqrt{pR}$ (resp. $\sqrt{pR^+}$) be the radical of the principal ideal of R (resp. R^+) generated by p. Set $\bar{R} = R/\sqrt{pR}$ (resp. $\overline{R^+} = R^+/\sqrt{pR^+}$). Then

(i) $H^1_{\mathfrak{m}}(\overline{R^+}) = 0$, and

(ii) every part of a system of parameters $\{a, b\}$ of $\bar{R}$ of length 2 is a regular sequence on $\overline{R^+}$.

This theorem suggests the following.

Question. Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic. Is $\overline{R^+}$ a big Cohen-Macaulay $\bar{R}$-algebra, i.e.

(i) is $H^i_{\mathfrak{m}}(\overline{R^+}) = 0$ for all $i < \dim \bar{R}$, and

(ii) is every system of parameters of $\bar{R}$ a regular sequence on $\overline{R^+}$? □

Integral Dependence and Integral Closures

The concepts of integral dependence and integral closure are central to Commutative Algebra, and their connections with Algebraic Geometry are, at present, very active fields of research. Their study is closely related to singularity theory, Rees algebras, and multiplier ideals, all of which were discussed at this conference.

The talk by Brian Harbourne (on symbolic powers) is also related to these topics. Harbourne uses, in an imaginative way, previous work on "fat point ideals" , a very geometric idea.

Comparing powers of ideals with their symbolic powers

Brian Harbourne, University of Nebraska–Lincoln

This talk presents work done jointly with Cristiano Bocci (arXiv:0706.3707v1). Consider a homogeneous ideal $I \subset k[\mathbf{P}^N] = R$, where R is the polynomial ring in $N+1$ indeterminates over an algebraically closed field k of arbitrary characteristic. The underlying question is: for which m and r do we have $I^{(m)} \subset I^r$, where $I^{(m)}$ denotes the m-th symbolic power of I? Our approach to this question is to define a quantity, the *resurgence* $\rho(I)$ of I, this being the supremum of all ratios m/r such that I^r does not contain $I^{(m)}$, and to give bounds on $\rho(I)$ in terms of Hilbert function invariants of I. In particular, if, for any homogenous ideal J, $\alpha(J)$ denotes the least degree t such that $J_t \neq 0$ (i.e., $\alpha(J)$ denotes the M-adic order with respect to the maximal homogeneous ideal M of R), we show that $\lim_{m\to\infty} \alpha(I^m)/\alpha(I^{(m)}) \leq \rho(I)$, and, if I defines a 0-dimensional subscheme, we show that $\rho(I) \leq \text{reg}(I)/(\lim_{m\to\infty} \alpha(I^{(m)})/m)$. We obtain these bounds by applying two principles. The first is that if $\alpha(I^r) > \alpha(I^{(m)})$, then I^r does not contain $I^{(m)}$. The second, which holds if I defines a 0-dimensional subscheme, is that if $\text{reg}(I^r) \leq \alpha(I^{(m)})$, then $I^{(m)} \subset I^r$.

As a consequence, among all homogeneous ideals I for which R/I has Krull dimension $N-d+1$ for a given d, we show that the minimum c such that $m \geq cr$ guarantees $I^{(m)} \subset I^r$ is $c = d$. This shows that the well known results of Ein-Lazarsfeld-Smith and Hochster-Huneke are optimal for every dimension and codimension. We also show that $I^{(3)} \subset I^2$ whenever $I = I(S)$ is an ideal of a finite set $S \subset \mathbf{P}^2$ of generic points. This partially answers a still open question of Huneke: if $I = I(S)$ for a finite set S of points in the plane, is it true that $I^{(3)} \subset I^2$? □

Multiplier ideals and cores

The presentation of Claudia Polini discussed the cores of rings and gives a characterization of certain types of schemes in terms of them. The important interplay between algebra and geometry was also evident in this talk as one of the principal ingredients was the Cayley-Bacharach theorem on finite point sets of projective n-space.

Cayley-Bacharach Schemes and their Cores

Claudia Polini, University of Notre Dame

In the first part of this talk we discuss when the known formulas for cores of ideals are valid in arbitrary characteristic. The core of an ideal I, $\text{core}(I)$, is the intersection of the minimal reductions of I. Being an a priori infinite intersection the core is difficult to compute, and in the last ten years there has been considerable effort to find explicit formulas. There are many reasons to study the core: one is its ties with adjoints and multipliers ideals, another is its connection with Briancon-Skoda type theorems, and last but not least a better understanding of cores could lead to a solution of Kawamata's conjecture on the non-vanishing of sections of line bundles.

In the second part of the talk we study the annihilators of some graded components of the canonical module of a graded ring. We relate them to cores of powers of homogeneous maximal ideals of standard graded reduced Cohen-Macaulay k-algebras. An application of our results characterizes Cayley-Bacharach schemes in terms of the structure of the core of the maximal ideal of their homogeneous coordinate ring, denoted by $\text{core}(X)$. Recall that a set of s points in $\mathbb{P}^n$ is called a Cayley-Bacharach scheme if every subset of $s-1$ points has the same Hilbert function. In particular, we show that a scheme X is Cayley-Bacharach if and only if $\text{core}(X)$ is a power of the maximal ideal. □

The subject of the talk by Dale Cutkosky is pathological behavior of local cohomology; he shows that this can occur even for Rees algebras.

Rees algebras with non tame local cohomology

Steven Dale Cutkosky, University of Missouri–Columbia

Suppose that R_0 is a local ring, $I \subset R_0$ is an ideal, and $R = R_0[It]$ is the Rees algebra of I. Let $R_+ = ItR$ be the irrelevant ideal of R.

The local cohomology module $H^i_{R_+}(R)$ is tame if either $H^i_{R_+}(R)_j \neq 0$ for all $j \ll 0$ or $H^i_{R+}(R)_j = 0$ for all $j \ll 0$.

Brodmann, Hellus, Lim, Rotthaus and Sega have shown that if $\dim(R_0) \leq 2$, then the local cohomology modules of R are tame.

It has recently been shown by Cutkosky and Herzog that tameness can fail for local cohomology of finitely generated modules over standard graded algebras R with $\dim(R_0) = 3$.

Chardin, Cutkosky, Herzog and Srinivasan have found examples showing that tameness of local cohomology fails for Rees algebras. We describe some of their examples below. In all of the examples, R_0 is normal, generalized Cohen Macaulay, and is essentially of finite type over a field k.

The first example has $\dim(R_0) = 3$, and shows periodic failure of tameness. For $j > 0$, $\dim_k(H^2_{R_+}(R)_{-j})$ is 2 if j is even, and is 0 if j is odd.

The second example shows failure of tameness of local cohomology which is not periodic, and is not even a quasi polynomial (in $-j$) for large j. Specifically, we have for $j > 0$,

$$\dim_k(H^2_{R_+}(R)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0 \text{ (mod)}(p+1), \\ 1 & \text{if } j = p^t \text{ for some odd } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the characteristic of k is p. We have $p^t \equiv -1$ (mod) $(p+1)$ for all odd $t \geq 0$. A third example is tame, but

$$\lim_{j\to\infty} \frac{\dim_k(H^2_{R_+}(R)_{-j})}{j^3} = 54\sqrt{2},$$

so $\dim_k(H^2_{R_+}(R)_{-j})$ is far from being a quasi polynomial in $-j$ for large j. □

Lawrence Ein gave a talk about current problems in Algebraic Geometry that are related to multiplier ideals.

Inversion of Adjunction

Lawrence Ein, University of Illinois at Chicago

We discuss the numerical invariant minimal log-discrepancy as a measurement of complexity for singularities occuring in higher dimensional birational geometry. The conjecture of Kollár and Shokurov on inversion of adjunction gives a precise comparison between the minimal log-discrepancies of the variety and its hyperplane sections. We discuss how the recent important work of Birkar, Cascini, Hacon and McKernan on the existence of log-minimal models can be applied to the conjecture. We also discuss the approach using the space of arcs introduced by Ein, Mustata and Yasuda. We state a recent theorem of Ein and Mustata on a precise version of the inversion of adjunction for non-local complete intersection Q-Gorenstein varieties. The theorem is also independently proved by Kawakita. Finally, we discussed an application of these results to bounding regularities of Q-Gorenstein varieties in the projective space defined by degree d equations. □

Intersection Theory and Homology

One of the areas of Algebraic Geometry that has had a strong impact on Commutative Algebra is Intersection Theory, in particular the homological definition of intersection multiplicities given by Serre in the 1950's. On the one hand, this led to a set of conjectures that have been a central part of the subject for many years. In addition, they started an interest in homological methods that is very active today.

The first two abstracts in this section deal with Intersection Theory. This subject has connections to K-theory as well as Algebraic Geometry. The talk by Srinivas had its origin in attempts to study projective modules using invariants from these fields.

Oriented Intersection Multiplicities

V. Srinivas, Tata Institute

Barge and Morel defined a graded "oriented Chow group" of a smooth variety X over a field, which may be viewed as a quotient of a group of "oriented algebraic cycles" modulo a suitable equivalence relation. More formally, they considered certain complexes of abelian groups $GMW^p_\bullet(X)$, the Gersten-Milnor-Witt complexes, and defined the p-th oriented Chow group to be $H^p(GMW^p_\bullet(X))$. The terms in the complexes are obtained by putting together Milnor K-groups, and powers of the fundamental ideal in the Witt rings, of function fields of subvarieties of X. The complexes are modelled after the Gersten complexes in K-theory.

A precursor was the idea of an oriented 0-cycle, suggested by M. Nori, which led to the Euler class group, considered in works of several mathematicians.

J. Fasel constructed an intersection product on the oriented Chow groups of Barge and Morel, leading to an "oriented Chow ring", which admits a graded ring homomorphism to the "usual" Chow ring of (unoriented) cycles.

In this lecture, I'll give an introduction to this emerging area, and discuss joint work with Fasel, about the idea of intersection multiplicities in this context. This leads to the formulation of an oriented analogue of Serre's vanishing conjecture for intersection multiplicities. I'll also discuss results of Morel, and further joint work with Fasel, on bundles with vanishing Euler class (taking values in the oriented Chow group of points).
□

The next results were on the map to the completion of a rings. There had been several questions of whether this was injective on the functors described. The following example of Kurano was also inspired by intersection properties of rings and their behavior under completion.

An example of a local ring R such that $G_0(R)_{\mathbf{Q}} \to G_0(\hat{R})_{\mathbf{Q}}$ is not injective

Kazuhiko Kurano, Meiji University

This is a joint work with V. Srinivas from Tata Institute, India.

For a Noetherian local ring R, let $G_0(R)$ be the Grothendieck group of finitely generated R-modules. Since the completion $R \to \hat{R}$ is injective, it induces the map $G_0(R) \to G_0(\hat{R})$.

In 2001, Kamoi and Kurano proved that the map $G_0(R) \to G_0(\hat{R})$ is injective if R is an excellent local ring that satisfies one of the following 3 conditions, (1) R is henselian, (2) $R = S_{S_+}$, where S is a standard graded ring over a field S_0, (3) R has only an isolated singularity.

However, Hochster gave an example that the map is not injective. In Hochster's example, the ring is non-normal and the kernel is torsion. Recently, Dao gave a new example. In Dao's example, the ring is normal, but the kernel is still torsion.

We constructed an example of a two dimensional (non-normal) ring, that is essentially of finite type over the complex number field such that the map $G_0(R)_{\mathbf{Q}} \to G_0(\hat{R})_{\mathbf{Q}}$ is not injective.

Using the example, we can construct a Noetherian local ring R' such that R' is a Roberts ring, but $\hat{R}'$ is not. □

The talk by Sean Sather-Wagstaff relates the completeness of a ring to vanishing of Ext modules.

Ext-vanishing and ascent of module structures

Sean Sather-Wagstaff, Kent State University

Let $(R, \mathfrak{m}, k)$ be a noetherian local commutative ring. Jensen, Buchweitz and Flenner, and Frankild and Sather-Wagstaff have shown that the $\mathfrak{m}$-adic completeness property for an R-module M is related to the vanishing of the modules $\operatorname{Ext}^n_R(\widehat{R}, M)$. Here $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, viewed as and R-module via the natural local ring homomorphism $R \to \widehat{R}$. The results presented in this talk extend these ideas to include other flat local ring homomorphisms, e.g., the map from R to its henselization R^h or any pointed étale neighborhood $R \to S$.

Theorem. (AJF-SSW-RAW, '07) *Let* $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ *be a flat local homomorphism and* M *a finitely generated* R*-module. The following conditions are equivalent:*

(i) *The* R*-module structure on* M *ascends along* φ.

(ii) *The evaluation map* $\operatorname{Hom}_R(S, M) \to M$ *is bijective.*

(iii) $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over* R *for each* $i \geq 1$.

(iv) $\operatorname{Ext}^{\geqslant 1}_R(S, M) = 0$.

The speaker presented several consequences of this result and discussed examples showing the necessity of the hypotheses on the homomorphism φ.

This is joint work with Anders J. Frankild (University of Copenhagen) and Roger A. Wiegand (University of Nebraska-Lincoln). □

One topic that has become classical by now is the notion of finite Cohen-Macaulay type. The talk by Lars Christensen and Janet Striuli investigates a related concept and its relation to properties of singularities.

Finite Gorenstein representation type implies simple singularity

Lars Winther Christensen and Janet Striuli, University of Nebraska-Lincoln

Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Remarkable connections between the module theory of R and the character of its singularity emerged in the 1980s. They show how finiteness conditions on the category of maximal Cohen–Macaulay R-modules (the finitely generated modules whose depth equals the Krull dimension of R) characterize particular isolated singularities. We report on developments of these connections in several directions.

A local ring with only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules is said to be of finite Cohen–Macaulay (CM) representation type. By work of Auslander, every complete Cohen–Macaulay local ring of finite CM representation type is an isolated singularity.

Specialization to Gorenstein rings opens to a finer description of the singularities; it centers on the simple hypersurface singularities identified in Arnol′d's work on germs of holomorphic functions. By work of Buchweitz, Greuel, and Schreyer, Herzog, and Yoshino, a complete Gorenstein ring of finite CM representation type is a simple singularity.

In the talk we show how to avoid the *a priori* condition that R is Gorenstein by replacing finite CM representation type with a finiteness condition on the category $\mathcal{G}$ of modules of Gorenstein dimension 0. Over a Gorenstein ring, these modules are precisely the maximal Cohen–Macaulay modules, but they are known to exist over any ring, unlike maximal Cohen–Macaulay modules.

Our proof of this result employs a new notion of $\mathcal{G}$*-approximations*, which is close kin to the CM-approximations of Auslander and Buchweitz. Every module over a Gorenstein ring has a $\mathcal{G}$-approximation, and our proof goes via a strong converse: Assume there is a non-free module in $\mathcal{G}$; if the residue field k has a $\mathcal{G}$-approximation, then R is Gorenstein. □

Another talk on homological properties of rings was given by Hailong Dao, who presented new results on some classical homological questions.

On some homological questions over local rings

Hailong Dao, University of Utah

Consider the following classical results:
Theorem. Let (R, m, k) be a regular local ring and M, N be finite R-modules.

1. (Serre 1965) If $l(M \otimes_R N) < \infty$, then $\dim M + \dim N \leq \dim R$.
2. (Auslander 1961, Lichtenbaum 1966) For any integer $i \geq 0$, $\mathrm{Tor}_i^R(M, N) = 0$ implies $\mathrm{Tor}_j^R(M, N) = 0$ for all $j \geq i$.
3. (Auslander-Goldman 1960) Assume that M is a reflexive R-module. If $\mathrm{Hom}_R(M, M)$ is free then M is free.
4. (Auslander 1962) Assume that M is a reflexive R-module. If $\mathrm{Hom}_R(M, M) \cong M^{\oplus t}$ then M is free.
5. (Huneke-Wiegand 1997) Assume that M is a reflexive R-module. If $\mathrm{Hom}_R(M, M)$ satisfies (S_3) then M is free.

In this talk we will discuss some recent attempts to generalize these results to non-regular local rings. We will focus our attention on hypersurfaces and complete intersections, where the questions reveal some surprising connections. One particular result will be discussed:

Theorem. Let R be an admissible hypersurface (meaning $\hat{R}$ is a quotient of an unramified or equicharacteristic regular local ring by a nonzero element) with an isolated singularity. Assume that $\dim R > 2$ and is even. If M is a reflexive R-module such that $\mathrm{Hom}_R(M, M)$ satisfies (S_3), then M is free. □

One of the places where Algebraic Geometry and Commutative Algebra are most closely related is in the study of Castelnuovo-Mumford regularity for resolutions of graded modules. Marc Chardin discussed this topic for Tor modules.

The regularity of Tor over non-regular rings

Marc Chardin, Institut Mathematiques de Jussieu

In this lecture M. Chardin presented results about the behavior of Castelnuovo-Mumford regularity with respect to the functor Tor. One of the first motivations was to provide estimates on the regularity in a geometric context. This is for instance the content of the following result :

Theorem. *Let k be a field, $\mathcal{Z}_1, \ldots, \mathcal{Z}_s$ be closed subschemes of a closed subscheme $\mathcal{S} \subset \mathbf{P}_k^n$. Assume that $\mathcal{S}$ is irreducible with a singular locus of dimension at most 1. If $\mathcal{Z} := \mathcal{Z}_1 \cap \cdots \cap \mathcal{Z}_s \subset \mathcal{S}$ is a proper intersection of subschemes of $\mathcal{S}$ that are Cohen-Macaulay locally at points of $\mathcal{Z}$, then setting $r'_{\mathcal{S}} := \max\{\mathrm{reg}(\mathcal{S}) - 1, 0\}$, one has*

$$\mathrm{reg}(\mathcal{Z}) \leq \sum_{i=1}^{s} \max\{\mathrm{reg}(\mathcal{Z}_i), r'_{\mathcal{S}}\} + \lfloor (\dim \mathcal{S} - 1)/2 \rfloor r'_{\mathcal{S}}.$$

In particular, if $\mathrm{reg}(\mathcal{S}) \leq 1$, then $\mathrm{reg}(\mathcal{Z}) \leq \sum_{i=1}^{s} \mathrm{reg}(\mathcal{Z}_i)$.

Another motivation is the estimates obtained by Eisenbud, Huneke and Ulrich on the regularity of Tor modules over a polynomial ring, and their application to estimate the regularity of powers of an ideal. Their work was inspired by previous results on the regularity of products of ideals and of tensor products of modules by Conca and Herzog, Sidman and Caviglia. They have proved an upper bound for the regularity of $\mathrm{Tor}_i^R(M, N)$ in terms of $\mathrm{reg}(M)$ and $\mathrm{reg}(N)$, when R is a polynomial ring and $\mathrm{Tor}_1^R(M, N)$ is supported in dimension at most 1. M. Chardin presented the following extension :

Theorem. *Let S be a standard graded ring over a Noetherian local ring (S_0, m_0) and M, N be finitely generated graded S-modules. If M or N has finite projective dimension and $\mathrm{Tor}_i^S(M, N) \otimes_{S_0} S_0/m_0$ is supported in dimension at most one for $i \geq 1$, then*

$$\max_i\{\mathrm{reg}(\mathrm{Tor}_i^S(M, N)) - i\} = \mathrm{reg}(M) + \mathrm{reg}(N) - \mathrm{reg}(S).$$

An independent proof of this result in the case where $N = S_0$ is a field was given by Römer in the more general setting of positively graded algebras over a field. The hypothesis on the dimension of the support of all positive Tor modules (in place of only the first, in the polynomial case) is needed since Tor is not rigid when S is singular.

Some properties of multiple Tor modules are presented, in particular the rigidity of multiple Tor modules over a regular ring containing a field and a geometric condition for the vanishing of $\mathrm{Tor}_1^R(M_1, \ldots, M_s)$.

When I is a homogeneous ideal in a polynomial ring R over a field such that $\dim R/I$ is of dimension at most 1, it has been proved by Chandler and Geramita, Gimigliano and Pitteloud that $\mathrm{reg}(I^j) \leq j\mathrm{reg}(I)$ for any j. This has been refined by Eisenbud, Huneke and Ulrich who proved that in this situation $\mathrm{reg}(I^j) \leq \mathrm{reg}(I) + (j-1)(e-1)$ if I is generated in degrees at most $e-1$ and related in degrees at most e. A second refinement of the initial estimate, which holds for a homogeneous ideal I of a Noetherian standard graded ring S is presented :

Theorem. *Let I be an homogeneous S-ideal such that $\dim(S/I) \otimes_{S_0} S_0/m_0 \leq 1$ for any maximal ideal $m_0 \in \mathrm{Spec}(S_0)$. Set $a_i(M) := \max\{j \mid H^i_{S_+}(M)_j \neq 0\}$. Then, for any $m \geq 0$,*

$$\mathrm{reg}(S/I^{m+1}) \leq \max\{a_0(S/I) + b_0^S(I), (a_1(S/I)+1) + \mathrm{reg}_1^S(I)\} + (m-1)b_0^S(I).$$

Notice that $a_1(S/I) + 1$ is the regularity of S/I^{sat}, where I^{sat} is the saturation of I with respect to the positive part of S. $\square$

We include in this section two talks that were on the connections of Algebraic properties with continuous and analytic properties. The talk by Hal Schenk discusses algebraic properties of polyhedral complexes.

Splines on polyhedral complexes

Hal Schenck, Texas A&M University

In mathematics it is often useful to approximate a function f on a region by a "simpler" function. A natural way to do this is to divide the region into simplices, and then approximate f on each simplex by a polynomial function. A C^r-differentiable piecewise polynomial function on a d-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^d$ is called a *spline*. Splines play a key role in geometric modeling, the finite element method for solving PDE's, and in approximation theory.

It is possible to use polyhedra, rather than simplices, to subdivide a region. Splines on a polyhedral complex P have received relatively little attention (compared to the simplicial case), partly because the simplicial case fits very naturally into a homological framework. Billera and Rose observed that for any polyhedral complex, the splines occur as the kernel of a map between free modules; from this they obtain a bound on the Hilbert polynomial; other work on the polyhedral case has been done by Rose, Schumaker, and Yuzvinsky.

This talk describes an approach to the study of splines on a polyhedral complex which uses a certain specialized version of the dual graph of P; in particular, we show that the study of the first three coefficients of the spline module can be reduced to the study of certain subgraphs of the dual graph of P; these subgraphs arise from codimension two linear spaces which arise as intersections of the (linear hull) of the facets of P. This is joint work with Terry McDonald. $\square$

Holger Brenner discussed finding continuous solutions to problems where classically one had looked for algebraic ones. An interesting point is that the solution can be given in terms of algebraic conditions.

Continuous solutions to algebraic forcing equations

Holger Brenner, University of Sheffield

We ask for a given system of polynomials $f_1, ..., f_n$ and f over the complex numbers $\mathbb{C}$ when there exist continuous functions $g_1, ..., g_n : \mathbb{C}^n \to \mathbb{C}$ such that $g_1 f_1 + ... + g_n f_n = f$. This condition defines the continuous closure of an ideal in a polynomial ring and more generally in any ring of finite type over $\mathbb{C}$. This closure sits inside the (weak sub) integral closure. We give inclusion criteria and exclusion results for this closure in terms of the algebraically defined axes closure. Conjecturally, continuous and the algebraically defined axes closure are the same, and we prove this in the monomial case by giving a combinatorial criterion which holds for both. □

Secant Varieties, Statistics, and Classical Projective Geometry

One of the most unexpected applications of Algebraic Geometry, in recent years, has been to the field of Statistics. There are many old and unsolved problems about the secant varieties of Segre varieties, and recently a new impetus for studying these problems has come from the realization that there would be exciting applications for solutions. In this conference we brought together mathematicians working on various aspects of these problems.

We present two talks that deal with the main topics of this section.

The talk by Seth Sullivant gave a striking example of the applications of Classical Geometry, and in particular of Secant Varieties, to Statistics.

Algebraic Geometry of Gaussian Bayesian Networks

Seth Sullivant, Harvard University

Given a directed acyclic graph G, the Bayesian network associated to G is the family of probability density functions that have a factorization of the form

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_i(x_i|x_{\mathrm{pa}(i)})$$

where $\mathrm{pa}(i)$ is the set of parents of the vertex i in the directed acyclic graph G. In the case where we assume that the random vector $X = (X_1, X_2, \ldots, X_n)$ has a multivariate normal distribution (i. e., is a Gaussian random vector) the set of covariance matrices that can arise for this Bayesian network is a rational algebraic variety in the cone of positive definite covariance matrices. We provide an in-depth study of the vanishing ideal $I_G \subset \mathbf{C}[\sigma_{ij} : 1 \leq i \leq j \leq n]$ of this rationally parametrized set of covariance matrices. The parametrization of the set of covariance matrices turns out to be combinatorial in nature and is known in statistics as the trek rule.

Contained in the vanishing ideal of the model I_G, is the subideal CI_G which is generated by polynomial consequences of the conditional independence statements that any distribution in the Bayesian network associated to G must satisfy. The conditional independence ideal CI_G is generated by subdeterminants of the symmetric matrix Σ, that are determined by a combinatorial characterization called d-separation. A basic question is whether or not the conditional independence ideal CI_G is always equal to the vanishing ideal I_G. It turns out that there are already small graphs with only five vertices for which the inclusion $CI_G \subset I_G$ is strict.

Among the results discussed in this presentation is that for any tree T, it is always true that $I_T = CI_T$. The proof exploits the fact that for any tree I_T is a toric ideal and uses tools from the general theory of toric ideals.

The rest of the talk concerns the study of the models and graphs that arise when some of the random variables are hidden. One basic result shows that I_G has a 2-dimensional multigrading induced by a collection of upstream random variables. This, in turn, implies that generators of the ideal of the hidden variable model can be easily related to generators of the ideal I_G.

Finally, it is shown how classical varieties arise as special cases of hidden variable Gaussian Bayesian networks. In particular, it is shown how joins and secant varieties arise when the underlying directed acyclic graph has a partitioned structure in its hidden variables. Among the secant varieties that arise are secant varieties of toric degenerations of the Grassmannian of 2 planes $G_{2,n}$, and secant varieties of the squarefree Veronese variety. □

The topic of Secant Varieties was further discussed later that afternoon by Jessica Sidman, who dealt with them from a computational point of view.

Prolongations and computational algebra

Jessica Sidman, Mt. Holyoke University

In the late '90's Landsberg and Manivel drew intriguing connections between prolongation, a notion which first arose in the context of differential geometry, and the equations vanishing on secant varieties. Work of Sturmfels, Sullivant, et al in algebraic statistics, where secant varieties can be interpreted as statistical models, has renewed interest in secant varieties and their defining equations. Questions of Sturmfels provided the impetus for the work discussed in the talk.

The simplest form of the definition of prolongation follows: Let A be a vector space of homogeneous forms. The r-th prolongation of A, denoted $A^{(r)}$, is the space of all homogeneous forms of degree $d + r$ whose partial derivatives of order r are all contained in A.

The focus of the talk is to explain the connection between secant varieties and prolongation. In particular, we will see how the definition of prolongation can be reformulated algebraically in terms of polarization and how this version of the definition is related to forms vanishing on a secant variety.

This is joint work with Seth Sullivant.

A preprint can be found online: arXiv:math/0611696v2. □

In addition to the statistical applications, there has been a lot of work on the classical problems themselves. The group headed by Chiantini and Ciliberto (and mostly centered in Italy) has reexamined the work of the Italian 'masters' of the late 19th and early 20th century. They have recast the main classical results in modern terms and pushed them to unforseen levels. This Workshop was a perfect opportunity to have these ideas explained and discussed by experts.

The talk by Luca Chiantini considered the question of finding certain types of subvarieties in a general hypersurface and had, as its main ingredient, the study of secant and join varieties to the varieties of reducible forms.

Complete intersection subvarieties in hypersurfaces

Luca Chiantini, University of Siena

Which subvarieties Y does one find in a general hypersurface X of the complex projective space? The Noether-Lefschetz theorem describes the situation when codim$(Y, X) = 1 < \dim(X)$. For higher codimension, the problem is wide open.

In a joint research with E. Carlini and A. Geramita, we consider the problem of finding, in a general hypersurface X of degree d, a subscheme which is complete intersection of type $a_1, \ldots, a_s$ (the only relevant case being clearly $a_i < d$ for all i). Since complete intersections are arithmetically Gorenstein, when codim$(Y, X) = 2$ the problem is related with the theory of rank 2 vector bundles without intermediate cohomology on X, as well as to the pfaffian representation of general forms.

The problem has a nice interpretation in terms of secants and joins of some subvarieties of the variety of forms, namely the varieties of reducible forms. Using Terracini's lemma, the problem is then translated in a problem on the structure of certain Artinian algebras.

With this reduction, we are able to prove results on the subject. For example, we prove that for any choice of positive integers $a, b < d$, a general plane curve of degree d contains a complete intersection set of points of type a, b. The case of surfaces in the projective space turns out to be completely different. In general, one has no complete intersection of type a, b, c in a general surface of high degree. We are able to classify completely the (few) triples (a, b, c) such that a general surface of any degree $d > a, b, c$ contains a complete intersection set of points of type a, b, c. □

Another classical topic that has had impact in both Algebra and Geometry is that of varieties defined by determinantal ideals. Winfried Bruns described the variety defined by exterior powers.

The variety of exterior powers of linear maps

Winfried Bruns, University of Osnabrück

The lecture is based on joint work with Aldo Conca.

Let V and W be vector spaces of dimension m and n over a field K of characteristic 0. We investigate the Zariski closure X_t of the image Y_t of the map $\mathrm{Hom}_K(V,W) \to \mathrm{Hom}_K(\bigwedge^t V, \bigwedge^t W)$, $\phi \mapsto \bigwedge^t \phi$. In the case $t = \min(m,n)$, $Y_t = X_t$ is the cone over a Grassmannian, but for $1 < t < \min(m,n)$ one has $X_t \neq Y_t$. We analyze the $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$-orbits in X_t. It turns out that they are classified by two numerical invariants, one of which is the rank and the other a related invariant that we call *small rank*. Surprisingly, the orbits in $X_t \setminus Y_t$ arise from the images Y_u for $u < t$ and simple algebraic operations.

The classification of the orbits is based on explicit normal forms on the one hand, and a determination of the G-stable prime ideals in the coordinate ring A_t of X_t, the algebra generated by the t-minors of a generic $m \times n$-matrix in the polynomial ring $K[X]$ generated by the entries of the matrix. We investigate this algebra by means of its standard monomial basis.

In previous work with Conca we have shown that A_t is always normal and Cohen-Macaulay. For $t = 1$, one has the trivial case $A_t = K[X]$. The algebra A_t is also well-understood in the Grassmannian case $t = \min(m,n)$. If $t = m-1 = n-1$, then A_t is again isomorphic to a polynomial ring over K. Apart from these exceptional cases, in which A_t is a factorial domain, it has class group **Z** and is Gorenstein if and only if $1/t = 1/m + 1/n$. The singular locus of X_t is then formed by all elements of rank ≤ 1. □

The talk by Kuroda was on another very classical problem: the finite generation of algebras defined in certain ways. The original problem due to Hilbert asked whether certain subrings of polynomial rings, which included rings of invariants of algebraic groups, were finitely generated. The first counterexample was due to Nagata and was an application of Algebraic Geometry to this problem in Algebra. Since then there have been attempts to get simpler examples, of which the following talk is a culmination.

How to construct counterexamples to Hilbert's 14th problem easily

Shigeru Kuroda, Tokyo Metropolitan University

Let R be the polynomial ring in n variables over a field k for $n \in \mathbf{N}$, and K the field of fractions of R. Then, Hilbert's 14th problem asks whether the k-algebra $L \cap R$ is finitely generated whenever L is a subfield of K containing k. In 1950's, Zariski showed that the answer to this problem is affirmative if $\mathrm{trans.deg}_k(L) \leq 2$, while Nagata gave the first counterexample in case of $\mathrm{trans.deg}_k(L) = 4$ and $n = 32$. Here, $\mathrm{trans.deg}_k(L)$ denotes the transcendence degree of L over k. In 1990, Roberts found a different kind of counterexample. Following Roberts, we have made various kinds of new counterexamples. For example, we gave one having $\mathrm{trans.deg}_k(L) = 3$, one for which K/L is an algebraic extension, and a variety of derivations whose kernels are counterexamples to Hilbert's 14th problem.

In the talk, we give a simple method of converting a graded k-subalgebra of R with some conditions into a counterexample to Hilbert's 14th problem. As an application, we demonstrate how to construct (i) a counterexample with $[K : L] = d$ for each $d \geq 3$ when $n = 3$; (ii) a counterexample which is realized as the invariant field for an action of $\mathbf{Z}/2\mathbf{Z}$ on K for $n = 4$; (iii) a counterexample which is realized as the kernel of a derivation of K for $n = 4$. There commonly exist graded k-subalgebras which satisfy our conditions, so that we can get a large number of counterexamples by this method. □

Another active topic in this area is the existence of curves with given multiplicities through given points. Rick Miranda talked about his recent advance in that field. This study is strongly related to open problems involving secant varieties of the classically studied Segre and Segre-Veronese varieties.

Curves of degree 174 with ten points of multiplicity 55

Rick Miranda, Department of Mathematics, Colorado State University

Fix general points $p_1, \dots, p_n$ in the plane, and multiplicities $m_1, \dots, m_n$. Let

$$\mathcal{L} = \mathcal{L}_d(m_1, \dots, m_n)$$

be the linear system of plane curves of degree d having multiplicity at least m_i at p_i for each i. The *virtual dimension* of $\mathcal{L}$ is $v(\mathcal{L}) = d(d+3)/2 - \sum_i m_i(m_i+1)/2$ and the *expected dimension* is $e(\mathcal{L}) = \max\{-1, v\}$.

It is easy to see that if there exists a (-1)-curve C (on the blowup of the plane) such that $\mathcal{L} \cdot C \leq -2$ and $\dim(\mathcal{L}) \geq 0$, then $\mathcal{L}$ does not have the expected dimension.

Gimigliano-Harbourne-Hirschowitz Conjecture: This is if and only if: If no such (-1)-curve exists, then $\mathcal{L}$ has the expected dimension.

Gimigliano-Harbourne-Hirschowitz is true for $n \leq 9$. (Castelnuovo, 1891; Nagata, 1960; Gimigliano, Harbourne, 1986)

The virtual dimension of $\mathcal{L}_d(m^{10})$ is equal to -1 (the most delicate case) for (d, m) in the following table:

d	m	empty
3	1	easy: cubic through ten general points
19	6	posed by Dixmier, solved by Hirschowitz early 80s
38	12	Gimigliano's thesis
174	55	?
778	246	?
1499	474	?
6663	2107	?
$\vdots$	$\vdots$	?

For these linear systems, one expects there to be no such curves ($H^0 = 0$) and because $v = -1$, this is equivalent to having $H^1 = 0$ (for the line bundle on the ten-fold blowup of $\mathbb{P}^2$).

In this talk, the author explained the proof of the following:

Theorem: $\mathcal{L}_{174}(55^{10})$ is empty.

Theorem: $\mathcal{L}_d(m^{10})$ has the expected dimension if $d \geq (174/55)m$.

The proof is by an explicit degeneration of the plane to a configuration of nine surfaces. The degeneration is constructed by a sequence of blowups and blowdowns related to (-1)-curves on the components in the central fiber. $\square$

Finally, we present two talks on Hilbert schemes. First, Greg Smith described some geometric properties of multi-graded Hilbert schemes, a generalization of the classical singly graded Hilbert schemes.

Multigraded Hilbert schemes

Greg Smith, Queens University

There is a parameter space for all ideals in the polynomial ring $S := \mathbb{C}[x_1, \dots, x_n]$ with a fixed Hilbert function. What are the geometric properties of these spaces?

To be more precise, fix an abelian group A. An A-*grading* of S is induced by a group homomorphism $\deg\colon \mathbb{Z}^n \to A$. This map provides a decomposition $S = \bigoplus_{a \in A} S_a$ where S_a is the span of all the monomials of degree a in S. A homogeneous S-ideal I is *admissible* if $\dim_{\mathbb{C}}(S/I)_a < \infty$ for all $a \in A$; its *Hilbert function* $h_{S/I}\colon A \to \mathbb{N}$ is $h_{S/I}(a) := \dim_{\mathbb{C}}(S/I)_a$. M. Haiman and B. Sturmfels construct a quasiprojective scheme Hilb^h parametrizing all admissible S-ideals with Hilbert function $h\colon A \to \mathbb{N}$.

In general, the geometry of a multigraded Hilbert scheme Hilb^h is complicated. For example, R. Vakil shows that every singularity type appears in certain Hilb^h and F. Santos shows that there exists disconnected Hilb^h. In contrast, J. Forgarty proves that Hilb^h is smooth and irreducible when $n = 2$ and $A = 0$. Similarly, L. Evain proves that Hilb^h is smooth and irreducible when $n = 2$, $A = \mathbb{Z}$, both $\deg(x_1)$ and $\deg(x_2)$ are positive integers and h has finite support. Building on these results, M. Haiman and B. Sturmfels conjecture that Hilb^h is smooth and irreducible when $n = 2$. By extending L. Evain's methods, D. Maclagan and G. Smith prove this conjecture. □

The talk by Irena Peeva was also on the topic of Hilbert schemes and their relation to homological properties of rings.

Hilbert schemes and maximal Betti numbers

Irena Peeva, Cornell University

Throughout, S stands for the polynomial ring $k[x_1, \ldots, x_n]$ over a field k of characteristic 0. The ring S is graded by $\deg(x_i) = 1$ for each i. If J is a graded ideal, then the Hilbert function $h : \mathbf{N} \longrightarrow \mathbf{N}$ defined by $i \mapsto \dim_k J_i$ is an important numerical invariant. Lex ideals are special monomial ideals, defined in a simple combinatorial way. They play an important role in the study of Hilbert functions and syzygies.

Theorem 1.1. (over the polynomial ring S)

1. (1) (Macaulay) For every graded ideal J in S there exists a lex ideal L_J with the same Hilbert function.
2. (2) (Hartshorne) The Hilbert scheme $\mathcal{H}_S^h$, that parametrizes all graded ideals in S with a fixed Hilbert function h, is connected. More precisely, every graded ideal in S with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.
3. (3) (Bigatti, Hulett, and Pardue) Every lex ideal in S attains maximal Betti numbers among all graded ideals with the same Hilbert function.

Analogues of these results are proved over an exterior algebra by Kruscal-Katona, Peeva-Stilman, Aramova-Herzog-Hibi, and Mermin-Peeva-Stilman. Gasharov and Peeva prove the following analogues over a class of projective toric rings, which has received a lot of interest in Commutative Algebra and Algebraic Geometry: Veronese rings.

Theorem 1.2. Let $R = S/I$ be a Veronese toric ring.

1. (1) For every graded ideal J in R there exists a lex ideal L_J with the same Hilbert function.
2. (2) The Hilbert scheme $\mathcal{H}_R^h$, that parametrizes all graded ideals in R with a fixed Hilbert function h, is connected. More precisely, every graded ideal in R with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.
3. (3) Every lex ideal in R attains maximal Betti numbers among all graded ideals with the same Hilbert function.
4. (4) Every lex-plus-I ideal in S attains maximal Betti numbers among all graded ideals containing I with the same Hilbert function. □

The abstract by Adam Van Tuyl is in the area of classical geometry but discusses a purely algebraic idea, the property of being Arithmetically Cohen-Macaulay.

ACM sets of points in multiprojective space

Adam Van Tuyl,Lakehead University

Let $R = k[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{r,0}, \ldots, x_{r,n_r}]$ with $\deg x_{i,j} = e_i$ denote the $\mathbb{N}^r$-graded coordinate ring associated to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. If $\mathbb{X}$ is a finite set of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, and if $I_{\mathbb{X}}$ is the multi-homogeneous ideal of forms vanishing at $\mathbb{X}$, then it can be shown that the coordinate ring of $\mathbb{X}$, that is $R/I_{\mathbb{X}}$, has the property that $\dim R/I_{\mathbb{X}} = r$, but $1 \leq \operatorname{depth} R/I_{\mathbb{X}} \leq r$. A set of points $\mathbb{X}$ is called arithmetically Cohen-Macaulay (ACM) when $R/I_{\mathbb{X}}$ is Cohen-Macaulay, or in other words, when $\operatorname{depth} R/I_{\mathbb{X}} = r$. When

$r = 1$, then a set of points $\mathbb{X}$ in $\mathbb{P}^n$ is always ACM. However, when $r \geq 2$, it is possible that a set of points may fail to be ACM, so one is naturally lead to ask whether ACM sets of points can be classified.

For sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, three such classifications exist. The first classification, due to S. Giuffrida, R. Maggioni, and A. Ragusa (1992), classified ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ via their Hilbert functions. A second classification, based upon the geometry of the points, was later developed independently by the author (2003) and E. Guardo (2001). More recently, L. Marino (in progress) has shown how to use the notion of a multihomogeneous separator of a point (a multihomogeneous form F that pass through all but one of the points of $\mathbb{X}$) to classify ACM points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In this talk, I will begin by recalling each classification. I will then show that the natural extension of each classification in $\mathbb{P}^1 \times \mathbb{P}^1$ to a general multiprojective space no longer holds. However, some new necessary and sufficient conditions for a set of points to be ACM will be presented. This talk is based upon joint work with Elena Guardo (University of Catania). $\square$

Outcome of the Meeting

As we had intended, the meeting brought together researchers in many areas that connected Commutative Algebra with Algebraic Geometry, with the hope of promoting interaction between researchers in various different but related fields. The results were even better than we had hoped. There is a tremendous amount of research being carried out in these areas, and the participants profited greatly from hearing about and discussing the interactions between them. We expect this to lead to new developments and further expansion of the field.

List of Participants

Brenner, Holger (University of Sheffield)
Bruns, Winfried (University of Osnabruck)
Catalisano, Maria Virginia (University of Genoa)
Caviglia, Giulio (University of California at Berkeley)
Chardin, Marc (CNRS & Université Pierre et Marie Curie)
Chiantini, Luca (University of Siena)
Christensen, Lars Winther (University of Nebraska-Lincoln)
Corso, Alberto (University of Kentucky)
Cutkosky, Steven Dale (University of Missouri - Columbia)
Dao, Hai Long (University of Utah)
Ein, Lawrence (University of Illinois at Chicago)
Geramita, Anthony (Department of Mathematics and Statistics, Queens' University)
Harbourne, Brian (University of Nebraska-Lincoln)
Kurano, Kazuhiko (Meiji University)
Kuroda, Shigeru (Tokyo Metropolitan University)
Li, Jinjia (Syracuse University)
Lyubeznik, Gennady (University of Minnesota)
Migliore, Juan (University of Notre Dame)
Miller, Claudia (Syracuse University)
Miranda, Rick (Colorado State University)
Monsky, Paul (Brandeis University)
Mustata, Mircea (University of Michigan)
Peeva, Irena (Cornell University)
Polini, Claudia (University of Notre Dame)
Roberts, Paul (University of Utah)
Rosenschon, Andreas (Fields Institute)
Sather-Wagstaff, Sean (Kent State University)
Schenck, Hal (Texas A&M University)
Schwede, Karl (University of Michigan)

Sidman, Jessica (Mt. Holyoke)
Singh, Anurag (University of Utah)
Smith, Gregory G. (Queens University)
Srinivas, Vasudevan (Tata Institue of Fundamental Research)
Stillman, Mike (Cornell University)
Striuli, Janet (University of Nebraska)
Sullivant, Seth (Harvard University)
Swanson, Irena (Reed College)
Takagi, Shunsuke (Kyushu University)
Ulrich, Bernd (Purdue University)
Van Tuyl, Adam (Lakehead University)
Veliche, Oana (University of Utah)

Chapter 19

Geometric Inequalities (07w5503)

Jun 17 - Jun 22, 2007

Organizer(s): Mark Ashbaugh (University of Missouri-Columbia), Almut Burchard (University of Toronto), Bernd Kawohl (University of Koeln), Robert McCann (University of Toronto)

Overview of the Field

Analytical methods are playing an ever increasing role in geometry. Sharp inequalities for integral functionals and for eigenvalues of the Laplacian contain much information about the geometry of the underlying space. Conversely, geometric ideas are crucial for understanding optimization problems, specifically problems with symmetry. In recent years, a web of new inequalities and surprising relationships with known inequalities have been discovered. In spectral geometry, recent results have included progress on some long-standing conjectures. New applications of geometric inequalities in kinetic theory and statistical mechanics are emerging. Some of these developments have been described in the recent surveys of Gardner [47], Villani [86] and Ashbaugh [5, 6].

Optimal transportation

Nowhere are geometry and analysis more tightly linked than in the Monge-Kantorovich theory of optimal transportation. Originally formulated as the practical problem of transporting one given mass distribution to another in a cost-minimizing matter, it has become a major tool in geometric analysis, while simultaneously giving rise to new geometric problems. Many of the geometric applications are based on theorems of Brenier [21] and McCann [69], which imply that any pair of probability measures can be connected by a transportation map with desirable monotonicity properties. In the late-1990's, a one-line proof of the isoperimetric inequality based on the Brenier-McCann map became part of the folklore in the area. With this map many classical and new inequalities with can be proved with nothing more than the arithmetic-geometric mean inequality, integration by parts, and change of variables. Among the results that have been obtained in this way are generalizations of Young's inequality by Barthe [13], a family of sharp Gagliardo-Nirenberg-Sobolev inequalities by Cordero-Erausquin, Nazaret and Villani [41], sharp Brezis-Lieb trace inequalities by Maggi and Villani [66], and some variants of the Gaussian correlation conjecture by Caffarelli [31] and Cordero-Erausquin [39]. Besides providing new geometric insight, optimal transportation proofs have led to the discovery of previously unknown "dual" versions of many classical inequalities [13, 3].

Optimal transportation also lends geometric meaning to the Wasserstein distances on probability measures. The function spaces defined by these distance functions have proved to be useful for the study of entropy inequalities and dissipative equations. Displacement convexity, which was discovered by McCann [68] and extended to Riemannian settings by Otto and Villani [73] and Cordero-Erausquin, McCann and Schmuckenschläger [40], has turned out to be the right notion for defining Ricci curvature bounds on a class of metric-measure spaces [63, 79]. This resolved a long-standing problem in synthetic geometry.

Symmetrization

The symmetric decreasing rearrangement has long been known to improve the value of physically relevant functionals such as the Coulomb energy of a charge distribution, the kinetic energy of a single particle in quantum mechanics, and the fundamental tone of a membrane. Originally devised for analytical proofs of the isoperimetric inequality by Steiner [77], rearrangements appeared in the 1970's and 1980's as key ingredients of Talenti's identification of the sharp constants in the Sobolev inequalities [81] and Lieb's corresponding results on the Hardy-Littlewood-Sobolev inequalities [60] Since then, much of the research in rearrangements has focused on technical questions, such as the continuity results of Almgren and Lieb [4], the search for a "master inequality" by Baernstein [11], and the characterization of equality cases in various rearrangement inequalities by Brothers and Ziemer [26], Burchard [27], and Chlebík, Cianchi, and Fusco [36]. Rearrangements have found many applications to geometric inequalities for eigenvalues of elliptic and higher order operators, starting with the Faber-Krahn inequality for the fundamental tone of a membrane, and continuing to the recent proof of the Payne-Pólya-Weinberger conjecture by Ashbaugh and Benguria [7, 8].

Nonlinear heat flows

Very recently, nonlinear heat flows have appeared independently in results of Carlen, Lieb and Loss [32] and Bennett, Carbery, Christ and Tao [17] on Young-type inequalities for multiple integrals on $\mathbf{R}^n$ and $\mathbf{S}^n$, and in Perelman's spectacular proofs [74] of the Poincaré and geometrization conjectures. The basic idea is that for a functional whose extremals include simple objects such as Gaussians in $\mathbf{R}^n$ or constant functions on a manifold, one should be able to construct a nonlinear diffusion that drives the functional monotonically towards its extremum while preserving the relevant side conditions. In other words, the problem is to construct a flow for which the functional in question acts as a Lyapunov function. This method also has a long history — we are aware of correlation inequalities that were obtained in this way by Pitt [75] and Herbst and Pitt [34], and suspect that there are earlier instances. Heat flow ideas have also been used for the Gaussian isoperimetric inequality by Bobkov [20] and Bakry and Ledoux [3].

Applications and Open Problems

Connections of the eigenvalues of the Laplacian (and related operators) with the geometry of the underlying domain have attracted the interest of mathematicians for at least the last 50 years, and are likely to do so for many years to come. A particularly stubborn but very interesting open geometric eigenvalue problem which might be amenable to proof via rearrangements is the Pólya-Szegö conjecture for the first buckling eigenvalue of a clamped plate. This problem involves the biharmonic operator and the result would be the analog of the Faber-Krahn inequality in this setting, that is, that among all domains of a given area, the first eigenvalue is minimized at the disk. In physical terms this says that among homogeneous plates of the same material and of a fixed area, with all possible shapes, the circular one buckles first when subjected to compressive loading of its edges. As a mathematical problem, the conjecture makes sense in any dimension. It has remained open since it was stated by Pólya and Szegö around 1950, despite attempts by various mathematicians to prove it. Under the assumption that a first eigenfunction of the problem exists which does not change sign, Szego was able to prove the conjecture in the 1950s. However, it is known that this assumption is rather restrictive and not at all the general case. Without the sign assumption, weaker estimates of the right general type have been obtained by Bramble and Payne [22], for n dimensions, by Ashbaugh and Laugesen [10]. A related conjecture for the vibration of a clamped plate (due, in the classical, two-dimensional case, to Rayleigh) is open in all dimensions above three, the cases of dimensions two and three having been established affirmatively by Nadirashvili [70, 71] and Ashbaugh and Benguria [8], respectively.

New applications for geometric inequalities are appearing in statistical mechanics. Mass transportation ideas have been used to study dissipative equations from kinetic theory since the late 1990's, and the area is evolving rapidly. Displacement convexity is being used in the work of Otto [72], Agueh [1] and others to prove logarithmic Sobolev inequalities and rates of convergence to equilibrium for nonlinear diffusion equations. There is reason to expect that displacement convexity will prove useful to estimate rates of decay for correlations in other models from statistical mechanics, with the goal of establishing phase transitions. The hope is that, in contrast with current methods that take advantage of the specific structure of models, convexity methods will be robust under small changes to the model.

For questions of convergence, stability, or robustness, a lower bound on the difference between the two sides of a geometric inequality in terms of some geometric quantity can be very valuable. For a long time, the best available result was a quantitative isoperimetric inequality due to R. R. Hall [52], which bounds the difference between a body in $\mathbf{R}^n$ and a suitably translated ball of the same volume (a measure of "asymmetry") from above in terms of the difference between their perimeters (the "isoperimetric deficit"). Fusco, Maggi, and Pratelli [45, 46] have sharpened Hall's result and obtained corresponding quantitative versions of Sobolev inequalities. Hall's inequality was used by Sznitman [80] and Povel [76] to understand the long-time behavior of Brownian motion with obstacles in dimensions three and above. Related asymmetry results have been used for dynamical stability results in Vlasov-Poisson and Vlasov-Maxwell systems citeGR,BG. It is a challenge to find quantitative inequalities for integral functionals that involve convolutions.

A larger challenge is posed by inequalities for functions defined on manifolds taking values in $\mathbf{R}^n$. Recently, Goldshtein and Troyanov [14] have proved Sobolev inequalities for differential forms on manifolds. Nothing is known about the sharp constants, which presumably encode geometric information about the manifold. Rearrangements do not apply to vector-valued functions, and it is hard to obtain inequalities on manifolds other than $\mathbf{S}^n$ and hyperbolic spaces. Heat flow methods may be the most promising approach here.

The regularity of optimal transportation plans has been the subject of many studies since the early 1990's. A major open problems are the precise geometric obstacles for the regularity of an optimal transportation plan for two given measures on differnt Rimennian manifolds. In the standard setup on $\mathbf{R}^n$, where the cost is given by the square of the Euclidean distance, The key geometric assumption is that the target measure should be supported on a convex set. Here, the regularity of the Brenier-McCann map was resolved (in the context of boundary-value problem for the Monge-Ampére equation) by Caffarelli [30] Delanoe [43] and Urbas [82]. Concave cost functions can also give rise to discontinuities even for smooth measures [67]. Finally, negative curvature of the underlying manifold can cause discontinuities, but positive curvature by itself does is not sufficient to guarantee regularity [57]. Recent results of Ma, Trudinger and Wang [65], Loeper [62] and Kim and McCann [58, 59] have opened the door towards a comprehensive regularity theory of optimal transportation.

Presentation Highlights

Optimal transportation

Alberto Bressan opened the workshop with a lecture on optimal transportation metrics for nonlinear wave equations. He discussed some examples where the solution does not depend continuously on the data in any of the natural Sobolev norms. However, the flow can be rendered Lipschitz continuous with respect a new distance function, which is determined by a problem of optimal transportation. This gives rise to a Riemannian structure in the problem. [4, 25].

Stamatis Dostoglou explained how to approximate solutions of the Navier-Stokes equation, using ides from optimal transportation. This gives rise to a new concept of weak solutions [9].

Franck Barthe extended the optimal transportation method for log-Sobolev type inequalities and isoperimetric inequalities. He described sufficient conditions for a measure to satisfy such an inequality that assume strong integrability, rather than convexity. As a result, he recovered precise concentration inequalities for log-concave measures, extended Bobkov's isoperimetric inequalities in this case, as well as Wang's extension of the Bakry-Emery criterion [14].

Dario Cordero-Erausquin talked about interpolation and geometric inequalities, discussing joint work with Bo'az Klartag. His goal was to find various ways of interpolating between norms to obtain sharpened versions of the Brunn-Minkowski and Santal$\hat{o}$ inequality [38].

Young-Heon Kim presented new results on curvature and the continuity of optimal transportation maps. He reported on recent results with Robert McCann of a semi-Riemannian metric associated with the cost function when transporting between two manifolds [58, 59]. This gives a general geometric framework for the regularity theory of Ma, Trudinger and Wang for optimal transportation plans [65, 62].

Guillaume Carlier gave an elementary variational proof of the classical theorems of Minkowsi and Alexandrov that guarantee the existence (and uniqueness up to translations) of a closed convex hypersurface with a given Gaussian curvature or with a given surface function. He emphasized the analogy with the classical optimal transportation problem and considered some applications to shape optimization [33].

Geometric flows

Tony Carbery discussed recent work on the Brascamp-Lieb inequalities for multilinear integrals of products of functions in several dimensions. He described the main tool, a monotonicity formula for positive solutions to heat equations in linear and multilinear setting [17].

Stefan Valdimarsson described all optimizers for this Brascamp-Lieb inequality. His proof combines the heat flow method with a careful analysis of the multilinear structure of the functional [83].

Aaron Smith presented a family of special solutions of Ricci flow on two-dimensional asymmetric cigars. These asymmetric cigars converge under the flow towards the standard symmetric cigar soliton, in the sense that they are bi-Lipschitz equivalent and the Lipschitz constant approaches one as time becomes large. He also described the precise exponential rate of convergence [29].

Sharp Gagliardo-Nirenberg and Sobolev inequalities

Rafael Benguria discussed optimal Gagliardo-Nirenberg inequalities for fourth order elliptic equations in one dimension. He compared minimization problems of Gagliardo-Nirenberg type on finite intervals with periodic boundary conditions with the corresponding problems on the whole real line. The main result is that the infimum is not achieved on the whole real line, but that it agrees with the minimum that is achieved on any finite interval [5].

Martial Agueh presented a method giving the sharp constants and optimal functions of all the GagliardoNirenberg inequalities involving the L^p-norm of the gradient. Optimal functions are explicitly derived from a specific non-linear ordinary differential equation which appears to be linear for a subclass of the GagliardoNirenberg inequalities or when the space dimension reduces to 1. The analysis includes also the sharp L^p-Nash inequalities [2].

Nicola Fusco discussed a quantitative version of the Sobolev inequality. Here, the difference between the two sides of the inequality is expressed in terms of the distance from the family of optimizers. In the proof, the inequality is proved first for radially decreasing functions, and then extended successively to n-symmetric functions and general functions in $W^{1,p}$, using symmetrization arguments [42].

Hardy inequalities

Several speakers addressed improved Hardy-type inequalities. Since these inequalities have no optimizers, it is a natural question whether the inequality can be improved by adding a suitable positive term to the left hand side.

Adele Ferone described such improved Hardy inequalities both in the classical and the limiting case. The relevant positive remainder terms depend on the distance from the family of “virtual” extremals [37].

Amir Moradifam gave criteria for a radially symmetric potential such that the remainder term in Hardy’s inequality for a domain can be bounded from below by a potential term. His approach clarifies the issue behind the lack of an optimal improvement, while yielding other interesting “dual” inequalities. These results have immediate applications to the corresponding Schrödinger equations [48].

In the special case of a three-dimensional half-space, Rupert Frank showed that the sharp constant in the Hardy-Sobolev-Maz’ya inequality is given by the Sobolev constant. This is shown by a duality argument that relates the problem to a Hardy-Littlewood-Sobolev type inequality whose sharp constant is determined as well. [16]

Francesco Chiacchio reported on improved Hardy inequalities for the Gaussian measure, and explained their connection with Gross’ log-Sobolev inequality. He also presented a family of factorized measures that enjoy isoperimetric inequalities, and used them to get sharp estimates for elliptic problems with degeneracy at infinity [23, 34].

Isoperimetric inequalities for eigenvalues

Marcello Lucia presented an inequality for the isoperimetric profile of a compact connected Riemannian manifold. This inequality can be used to bound the principal eigenvalue of the Laplacian on the manifold from below. As an application of the inequality, he obtains new uniqueness results for two-dimensional semilinear equations. [64]

Lotfi Hermi discussed how to use trace identities of the type derived by Harrell and Stubbe to produce universal bounds on the eigenvalues of the Dirichlet Laplacian. He then showed how to use these identities to produce new Weyl-type bounds for averages of eigenvalues, as well as an alternate proof of the Berezin-Li-You inequality as viewed by Laptev and Weidl [54].

Calculus of Variations and elliptic equations

Benjamin Stephens reported on the thread-wire problem, which is a minimal surface problem with a fixed boundary (given by a "wire") and a free boundary (given by a "thread"). He used isoperimetric arguments to show that if the length of the thread is close to the length of the wire, then a minimizing surface will remain close to the wire. An isoperimetric argument plays a key role in the proof [78].

Juncheng Wei considered a nonlinear elliptic eigenvalue problem with a supercritical negative exponent on a domain in $\mathbf{R}^n$. He showed that this problem as a unique minimal solution, provided that $\lambda > 0$ is small enough, and the branch of solutions must undergo infinitely many bifurcations or turning points [51].

Tobias Weth presented a result on radial symmetry of positive solutions to a class of semilinear polyharmonic Dirichlet problems in the unit ball. The result was obtained via a new variant of the moving plane method. In some special cases the result implies uniqueness of positive solutions [18].

Andrea Cianchi discussed isocapacitary inequalities that relate the relative capacity of a subset of an open domain $\Omega \subset \mathbf{R}^n$ to its Lebesgue measure. As a consequence, he obtained a priori estimates for nonlinear elliptic Neumann problems [7].

Antoine Henrot derived isoperimetric inequalities for the product of some moments of inertia on convex sets. As an application, we demonstrates an isoperimetric inequality for the product of the n first nonzero eigenvalues of the Stekloff problem in $\mathbf{R}^n$ [34].

Outcome of the Meeting

A meeting between experts in optimal transportation, rearrangements, geometric flows and spectral geometry was long overdue. This workshop came at a time that saw an explosion of interest in geometric flows in response to Perelman's reslts on Ricci flow. At the same time, optimal transportation methods were just becoming accessible to non-experts through Villani's books [84, 85] available at www.umpa.ens-lyon.fr/ cvillani). The workshop brought together mathematicians working on geometric inequalities with colleagues interested in current and potential applications. The audience included specialists in optimal transportation, rearrangements, nonlinear heat flows, and spectral geometry, as well as researchers with interested in applications of optimal transportation to dissipative PDE and fluid mechanics, and a few participants with broader interests in geometric inequalities, geometric flows and calculus of variations.

Several subgroups had worked on closely related problems with different methods, sometimes with equivalent results, which suggested deeper deeper connections waiting to be explored. Arguably, rearrangements are just particular transportation plans, but the relationship with optimal transportation has yet to be made explicit and put to use. For instance, to our knowledge, optimal transportation methods have not been used in spectral geometry. Connections with the theory of Ricci curvature and Ricci flow, especially in non-smooth settings, have begun to emerge in preprints by Lott and Villani, Sturm, and Topping and McCann, while potential applications for geometric inequalities in statistical mechanics are emerging. By making these connections, we believe that the meeting has accelerated the rate of progress and has opened directions for future research.

List of Participants

Agueh, Martial (University of Victoria)
Ashbaugh, Mark (University of Missouri-Columbia)
Baernstein, Al (Washington University in St. Louis)
Barthe, Franck (Universite Toulouse III)
Benguria, Rafael (Universita Catolica de Chile)
Bressan, Alberto (Pennsylvania State University)
Burchard, Almut (University of Toronto)
Carbery, Anthony (University of Edinburgh)
Carlier, Guillaume (Universite Paris Dauphine)
Chiacchio, Francesco (Università degli Studi di Napoli)
Cianchi, Andrea (Università di Firenze (Italy))
Cordero-Erausquin, Dario (University of Marne-la-Vallee (France))
Cox, Graham (Duke University)
Denzler, Jochen (University of Tennessee, Knoxville)
Dostoglou, Stamatis (University of Missouri)
Ferone, Adele (Seconda Universita di Napoli)
Frank, Rupert (KTH Stockholm (Sweden))
Fusco, Nicola (Universita di Napoli)
Ghoussoub, Nassif (BIRS)
Goldman, Dorian (University of Toronto)
Henrot, Antoine (Institut Elie Cartan (France))
Hermi, Lotfi (University of Arizona)
Hillion, Erwan (Université Paul Sabatier)
Kawohl, Bernd (University of Koeln)
Kim, Young-Heon (University of British Columbia)
Kraiem, Mouna (University of Cergy-Pontoise)
Loss, Michael (Georgia Institute of Technology)
Lucia, Marcello (Universitat zu Koln)
McCann, Robert (University of Toronto)
Moradifam, Amir (University of British Columbia)
Pass, Brendan (University of Toronto)
Smith, Aaron (Queens University / Stanford)
Stephens, Benjamin (University of Toronto)
Trokhimtchouk, Maxim (University of California, Berkeley)
Valdimarsson, Stefan (University of California, Los Angeles)
Wei, Jun Cheng (Chinese University of Hong Kong)
Weth, Tobias (University of Giessen)

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The three methods also share an intuitive geometric appeal. A useful property for applications to variational problems is that they demand little a priori regularity, even for the characterization of equality cases. Applying these tools requires no insight into their construction. In particular, though much recent work on rearrangements relies on geometric measure theory, the resulting theorems can be applied naively, and the Brenier- McCann map can be used without expertise in the Monge-Kantorovich theory of optimal transportation. Similarly, heat flows can provide deceptively simple proofs of inequalities; however, finding the right flow can be difficult, and currently no general construction principle is known.

Chapter 20

Statistical methods for High-throughput Genetic Data (07w5023)

Jun 24 - Jun 29, 2007

Organizer(s): Jiahua Chen (University of British Columbia), Yuejiao Cindy Fu (York University), Mary Lesperance (University of Victoria), David Siegmund (Stanford University), Heping Zhang (Yale University), Hongyu Zhao (Yale University)

Overview of the Field

Recent years have seen the rapid accumulation of various types of genomic information due to concerted efforts by the scientific community and advances in molecular technologies. The publication of the human genome sequences and the sequences of many other species represent a great milestone in our scientific history. In addition, a large number of genetic variants responsible for the diversity seen in a given organism have also been identified. For example, more than 10 million "common" single nucleotide polymorphisms (SNPs) are estimated to be present in humans, and a many of these have been discovered and documented in the literature and public databases. Parallel to SNP discovery, microarrays have made it possible to examine gene expression levels at the genome level, to study genomic-wide DNA copy number changes, which are ubiquitous in human DNA, but especially in cancer tissue, to identify essentially all the binding targets of a transcription factor under different conditions, to evaluate epigenetic controls of genetic regulation, and to collect other types of information across the whole genome. All of these great successes in knowledge and data acquisition have created opportunities and challenges for statisticians, mathematicians, computer scientists, engineers, physicists, and other quantitative scientists to work closely with biologists and biomedical researchers. Mathematical scientists can assist biologists to utilize efficiently such enormous amounts of data, to identify genetic variants underlying human diseases, to dissect biological pathways and address many other scientific inquiries.

One immediate and apparent challenge in statistical genetics is the computational load of any statistical method when applied to high-throughput data set. One particular problem, for example, is that human error can occur at any stage of data collection. To identify inconsistency and sometimes make necessary corrections can greatly reduce the potential bias in the subsequent statistical analysis. This is an important area still under intensive research. An important research product is computer software that provides convenient tools to biological researchers.

Another area to which this workshop devoted substantial attention is the variable selection problem for the large parameter spaces that are most relevant in statistical genetics. A surge of publications has started to appear in many statistical journals. This workshop foresaw this tread and had many presentations in this hot research area.

Finite mixture models have long found applications in statistical genetics. There has been substantial progress in developing statistical methodology in recent years. A vast proportion of them have implications to the way some genetic data should be analyzed. Some provide completely new means of analysis. This workshop seized the opportunity to present these new statistical ideas to biologically oriented researchers who may not yet be following these developments due to their different research focus.

Outline of the Workshop

The workshop has invited active statistical researchers with diverse expertise in statistical genetics to meet and exchange ideas. We strived to invite scientists from all over the world, scientists working on a wide variety of genetic problems, and scientists that are in all stages of their careers. The workshop participants include scientists from the National University of Singapore, University of Hong Kong, Hebrew University, as well as a large concentration of scientists from North America. Many participants are already well known in statistical genetics such as Professor David Siegmund who is a member of the National Academy of Sciences in the USA and renowned for his advanced research in statistical genetics, Professor Jun Liu who was the winner of the prestigious award of the Committee of Presidents of Statistical Societies (COPSS) and who has made outstanding contributions in Bayesian methods among others. Both Professor Heping Zhang and Professor Hongyu Zhao from Yale University have their research strongly supported by research grants from the National Science Foundation, and supervise large research labs on statistical genetics. Professor Shelly Bull from the University of Toronto is one of the leading researchers in statistical genetics in Canada and directs the Samuel Lunenfeld Research Institute of Mount Sinai Hospital. Dr. Dongsheng Tu, from Queens University, Canada, and Professor Benny Zee from Chinese University of Hong Kong and many other participants have close collaborative relationship with clinical geneticists. They discussed their first hand experience on the relevance of research progress in genetics, drug development, and treatment improvements.

This workshop also includes researchers who excelled in other areas of statistics, and look forward to extending their research impact to statistical genetics. Some participants are at the early stage of their research career but are attracted to this very promising area and its abundant opportunities to to make a significant impact. The workshop also provided opportunities to many post-doctoral scholars and current graduate students. For many, this is their first time to have distinguished listening to their research ideas and results in detailed presentations and in after presentation social activities.

The workshop takes the form of inviting participants to submit their latest research for possible presentation and discussion. Over 28 researchers presented their work in either one-hour plenary talks or in 30-minute invited talks. Lengthy discussions followed each plenary talk, and sessions of invited talks. Many researchers exchanged their email addresses for more specific future one-to-one correspondence.

Presentation Highlights

Linkage Mapping and Association Studies

Gene mapping, either by linkage or by association, is concerned with identifying genomic regions that harbor genetic polymorphisms that contribute to phenotypes of interest. It is practiced by scientists studying plants, animals, animal models of human genetics, and humans. Because changes in the technology of high throughput genotyping introduce new experimental possibilities, the subject is in a state of rapid change.

The opening plenary talk by Professor D. Siegmund posits an intriguing question to all statisticians working on linkage analysis in experimental or in human genetics: “Do complex statistical methods help in mapping complex and quantitative traits?” Striving to find a statistical model that match various characteristics of genetic data, statisticians may want to develop more and more sophisticated models. One may choose to use (i) standard statistical methods for genome scans, or (ii) more complex methods designed to take advantage of the possibilities of gene-gene and gene-environment interactions. The trade-off is between not taking necessary factors into proper consideration, and paying the price of devoting a portion of data (in some sense) to capture complex modeling features. The presentation included enlightened theoretical developments and well thought-out computer simulations.

The presentation by Professor Daniel Weeks introduces "linkage statistics that model relationship uncertainty." In linkage analysis, reported family relationships are assumed to be correct, and thus misspecified relationships can lead to erroneous results. Often, studies either discard individuals with erroneous relationships or use the best possible alternative pedigree structure. The linkage statistics developed by Professor Weeks and his collaborators model relationship uncertainty by properly weighing different possible true relationships. Using simulated data containing relationship errors, statistical methods are compared for maximum likelihood statistic and non-parametric LOD scores for small pedigrees and for large pedigrees.

A second plenary talk by Professor Warren Ewens: "The transmission-disequilibrium test (TDT) and its generalizations" addressed family based association tests, The transmission-disequilibrium test was developed as a test of linkage between a marker locus and a purported disease susceptibility locus. It is however more frequently used today as a test of association between the alleles at these respective loci. Complications to the test arise with the current availability of hundreds of thousands of marker loci. There hence have been substantial new developments to handle this and other novel situations. Professor Ewens' presentation provided a timely update and provoked participants to think carefully about the test.

The presentation of Professor Josee Dupuis ("Mapping quantitative trait genes using high density SNP scans in extended families: challenges and partial solutions") also addresses the challenge in statistical analysis of high-throughput data with familial relationship. High density SNP scans are currently being performed on several family-based population samples with multiple phenotypes available. There are several statistical challenges related to finding genes influencing quantitative traits in such rich datasets. Is there any value to performing linkage analysis using dense SNPs, when it is believed that genome-wide association (GWA) analysis is the answer to many questions? With the huge number of tests that result from a GWA scan with multiple phenotypic outcomes, how does one control type-I error and still retain some power to detect effects of modest size? Should one perform analyses to safe-guard against false positive results arising from population stratification, at a cost of a possible reduction in power, or should one rely on replication studies to limit false positive associations? This presentation provided some partial solutions to these questions in the context of the Framingham Heart Study and urged other researchers to think hard for further improvements.

While many new statistical methods presented in this workshop deals with straightforward association studies, Professor Shelly Bull reminded us that genome-wide association studies(GWA) are typically designed with multiple stages. Whether the second stage involves an independent sample of individuals or a family-based design, genetic effect estimates at a second stage will be less optimistic that those obtained at the first stage, due to selection bias arising from genome-wide screening. Motivated by a GWA study of the genetics of complications of type I diabetes, Professor Bull and her research team presented their research result on evaluating implications for power and bias in alternative designs and analytic strategies for detection and mapping of gene regions using high-density SNP arrays. The bias of genetic effect estimates depends on sample size, true effect size, minor allele frequency, and screening stringency. They found that the application of computationally-intensive bootstrap estimation yields less biased effect estimates, and hence more realistic specifications for replication in a subsequent stage.

Statistical Genomics and Computational Biology

Information on the parental origin of each of two alleles at each locus on chromosome cannot be measured directly but must inferred by statistical means. Such haplotype information is very useful in genetic research. Haplotypes have hence become a central topic in genetics analysis in recent years.

It is a rule rather than an exception, that the data do not contain the enough information to completely determine haplotypes. Utilizing information from genotypes of a good sized neighborhood loci, a probability distribution with good certainty is possible. Yet the sheer size of the number of possible configurations makes complete numeration beyond the power of modern computers. Instead, a Bayesian approach via stochastic exploration or some other cleverly designed approaches can provide a solution. Dr. Jun Liu from Harvard University gave a review of his recent work on Bayesian Inference of Haplotypes and Epistasis. These included Bayesian models that have been developed over the past few years for haplotype inference and included a new hierarchical Bayes model and a Bayesian approach to detect multi-locus interactions (epistasis) for case-control association studies.

The talk by Dr. Fei Zou from the University of North Carolina at Chapel Hill focused on a very timely topic, the identification of genetic variants affecting gene expression levels (eQTL). She investigated use of Bayesian methods and focused on the computation of posterior distributions. A common feature of her research and other work presented at the workshop is a massive data set that poses a serious numerical challenges. In her talk titled “Fast Bayesian eQTL Analysis,” Dr. Zou discussed a Bayesian linkage model that offers highly interpretable posterior densities for linkage. Instead of direct numerical computation the likelihood functions, which is very costly, she has developed Laplace approximations that are highly accurate and efficient in numerical implementation so that the computation of posterior densities for over 30,000 transcripts becomes feasible.

In another eQTL talk, Dr. Liang Chen from University of Southern California focused on “Considering Dependency among Genes for the False discovery Control of eQTL mapping. In most studies, the dependency among genes is largely ignored in consideration of multiple comparison adjustments. However, such dependency may be strong for eQTL data, and it can have significant impact on the outcome of the data analysis and its interpretation. Dr. Chen and colleagues introduced a weighted version of false discovery control to improve the statistical power to identify eQTL. The relative performance of the new method in eQTL studies was illustrated through simulations and data analysis.

Motivated by high throughput genotyping platforms, Dr. Ingo Ruczinski from Johns Hopkins University spoke on “An integrated approach for the assessment of chromosomal abnormalities using SNP chip estimates of genotype, copy number, and uncertainty measurements”. Copy number variations have become a focus of human genetics research in the past several years, yet the detection and quantification remain a challenging statistical problem. Dr. Ruczinski and colleagues developed Hidden Markov Models (HMMs) based on SNP array data. Their approach can simultaneously integrate gene copy number estimates, genotype calls, and the corresponding confidence scores when available. They have further implemented their methods in the R programming language.

Data integration is an intensively researched area. In his talk on “Bayesian methods for reconstructing transcriptional regulatory networks” Dr. Hongyu Zhao described a Bayesian error analysis model to integrate protein-DNA binding data and gene expression data to reconstruct transcriptional regulatory networks. There are two unique aspects to this proposed model. First, transcription is modeled as a set of biochemical reactions, and a linear system model with clear biological interpretation is developed. Second, measurement errors in both protein-DNA binding data and gene expression data are explicitly considered in a Bayesian hierarchical model framework. Model parameters are inferred through Markov chain Monte Carlo. The usefulness of this approach was demonstrated through its application to infer transcriptional regulatory networks in the yeast cell cycle.

With a network focus, Dr. Steve Horvath from the University of California at Los Angeles discussed “Weighted Gene Co-Expression Network Analysis and Other Systems Genetic Approaches for finding Complex Disease Genes”. His talk covered several theoretical topics, including network construction, module definition, network-based gene screening, and differential network analysis. The usefulness of these methods were illustrated using several applications, including i) screening for biomarkers of kidney transplantion success ii) finding obesity related genes in mice, and iii) complex disease gene mapping in humans.

Variable Selection and Data Mining in Genomics

One goal of the analysis of high throughput data is to identify genetic variations that are responsible for phenotypic variations such as cancer and diabetes. As the name suggests, the high throughput data provides hundreds of thousands candidate genetic variations to be screened. Paradoxically, the large volume of data has the potential to make the statistical inference more powerful, but it poses serious challenges to develop statistical methods that are solid in theory and efficient in computation. Genetically, it is not realistic to chase after hundreds of suspects. Further, with so many candidates, many subsets of these genetic variations may appear to be strongly associated with the phenotypic variation. How do we identify a small set of true culprits, quantify the associated uncertainty, and statistically measure the degree of success?

This workshop provides an ideal platform for participants to report their latest achievements in developing novel statistical methods in identifying candidate genes or biomarkers for diseases.

The first presentation in this area was given by Professor Heping Zhang from Yale University, entitled “Tree and forest based approaches to genomewide association studies”. He compared the statistical analysis in this area to the practice of gold mining. The key is not how pure the final product obtained in the initial screen, but concentration of gold for further purification. In this presentation, Professor Zhang presented tree and forest based analyses in genomewide association studies to identify high risk genes and gene-gene interactions. Simulation studies were used to demonstrate the potential of the proposed method. A re-analysis of existing data from a genetic study of age-related macular degeneration also revealed the advantages of the proposed method in identifying a possibly protective variant.

Professor Jiahua Chen from University of British Columbia spoke on "Extended Bayesian Information Criteria for Model Selection with Large Model Space". It is observed that the model space for analyzing high throughput data sets is extremely large. The special terminology of "large-n-small-p" has recently been coined to describe statistical models of this nature. Many participants of the workshop have make research contribution in this area, and it is generally recognized that the ordinary Bayes information criterion is too liberal for model selection when the model space is large. In this talk, Professor Chen re-examined the Bayesian paradigm for model selection and proposed an extended family of Bayes information criteria. Unlike the original Bayes information criterion, which balances the log likelihood by a penalty on the number of unknown parameters, the extended Bayes information criteria take into account both the number of unknown parameters and the complexity of the model space. The consistency of the extended Bayes information criteria is established, and its performance in various situations is evaluated by simulation studies. It is also compared with the original Bayes information criterion in terms of positive selection rate and false discovery rate in problems of variable selection. It is demonstrated that the extended Bayes information criteria incurs some loss in positive selection rate but tightly controls false discovery rate, a desirable property in many applications. The extended Bayes information criteria are extremely useful for variable selection in problems with moderate sample size but very large numbers of covariates, especially in genome-wide association studies.

"Penalized Methods for Variable Selection and Estimation with High Dimensional Data" delivered by Professor Jian Huang from the University of Iowa, discussed some non-Bayesian penalized approaches for variable selection, including "lasso" and bridge penalties. The new type of variable selection represented by lasso is one of the latest techniques developed in the statistical literature. It suggests to fit models according to some likelihood or least squares criteria, but penalizes complex models with a non-smooth function. By adjusting a tuning parameter, the severity of the penalty can be increased. A very interesting property, as a consequence of the non-smooth penalty, is that the fitted values of many regression coefficients are likely to be exactly zero, rather than simply small. When the number of variable subjects to selection remains low compared to the sample size, the procedure has been found to have desirable properties, as well as being consistent in general. Since this type of procedure can be directly applied to the "small-n-large-p" problems and is numerically efficient, it is of great interest whether these methods also possess desirable statistical properties under new circumstances. Due to its importance in genetic applications, there have been intensive research on the properties of this kind of variable selection procedures. Professor Huang is one of the first few made significant contributions in this respect. He presented some preliminary results concerning variable selection consistency and asymptotic oracle properties of lasso and bridge methods in high-dimensional settings. Under some sparsity assumptions, he showed that the majority of the true variables will survive the first round of screening under the lasso and other methods. He also illustrated applications of these methods to the analysis of binary and censored outcomes with high-dimensional genomic covariate data.

The importance of the variable selection procedures for small-n-large-p situations is further discussed in other presentations. Professor Jie Peng from University of California-Davis presented results on "Model Selection for QTL Mapping," where she considered the lasso, but her application is to QTL mapping. In addition, she also proposed another method called the "fence method," which is useful for model selection in both regression and random effects models and enjoys some computational advantages over the BIC and AIC type of methods in searching of the model space. The idea is first to build a statistical fence to eliminate incorrect models and then among the correct models, to select optimal one in a defined sense.

The workshop concluded with a presentation by Professor Zehua Chen from the National University of Singapore, entitled "A tournament approach to model selection with applications in genome-wide association studies". Ultimately, no matter how innovative a statistical model selection procedure is, to make an impact in genetic application, it must be computationally feasible. His presentation echoes the fact that the sheer amount of the covariates (genetic markers) and a relatively small sample size make many existing model selection methods infeasible. Even for newly developed methods that are renowned for their computational efficiency, additional measures make be taken to enhance their applicability. To this end, he presented a novel tournament approach to model selection for the situation that the number of covariates under consideration far exceeds the sample size. The approach consists of a stage-wise screening procedure, which mimics the rounds of competitions in a tournament, and an extended Bayes information criterion.

Although the speakers considered different methods, they all recognized one of the common challenges in genomic data analysis, traditionally called variable selection in statistics. Whatever the name, the methods have a defined pool of models. The selection of a model or models from the pool is then based on a penalization criterion that considers the size of the model pool as well as the complexity of the selected model. While the progress is promising, it is also evident from all presentations that the challenge has not been fully resolved. The participants were very pleased to find so much common ground, and surprised to see so many others are working on the same problem. Needless to say, heated discussions and exchange of references and emails were carried out all the time.

Mixture Models

A unique feature of this workshop is to explore the use of specific, advanced statistical methodologies in analysis of high-throughput genetic Data. Mixture models have a long history of being applied to statistical genetics. In classical linkage analysis, it is often suspected that only a subgroup of patients have a disease gene which is linked to the marker. Detecting the existence of this subgroup amounts to the acceptance of a mixture model description of the whole population. Consider the analysis of high-throughput genetic data where hundreds of thousands gene expression levels are obtained, it is plausible to assume that only a proportion of these levels are elevated. Again, finite mixture models can be naturally employed.

This workshop invited several researchers working on statistical theory of finite mixture models. Bruce Lindsay (Distinguished Professor and head of the Department of Statistics, Penn State University) gave a plenary talk on “Modes, mixtures, and diffusion kernels - Building a three way theory”. Lindsay’s talk was a report on a new type of model for genomic sequence data as well as inference for that model. Interest is focused on data sampled from some population, and a tree of relationships for those sequences is created. Each sampled sequence is modelled as having first been sampled from a population of ancestral sequences; that population is modelled by an unknown distribution Q on sequence space. The chosen sequence then undergoes T time units of evolution using Markov Chains that describe mutation and recombination processes before it is observed. The goal is to go backward in time T units and estimate the ancestral sequence distribution Q, as well as which modern sequence maps to each ancestor (or ancestors in the case of recombination). One methodology involves using maximum likelihood inference on the model for each fixed T on a grid, then linking the estimated ancestors together over T to create an ancestral tree. A second method is to use modal inference. This new method, which can be motivated by a reverse diffusion argument, seems to give results similar to maximum likelihood with much less programming and computation time.

Professor Ji-Ping Wang from Northwestern University presented his research entitled “Statistical method for nucleosome DNA sequence alignment and linker length preference prediction in Eukaryotic cells”. Eukaryotic DNAs exist in a highly compacted form known as chromatin. The nucleosome is the fundamental repeating subunit of chromatin, formed by wrapping a short stretch of DNA, 147bp in length, around four pairs of histone proteins. Nucleosome DNA obtained by experiments however varies in length due to imperfect digestion. Wang developed a mixture model that characterizes the known dinucleotide periodicity probabilistically to improve the alignment of nucleosomal DNAs. To further investigate the chromatin structure, he obtained experimentally cloned and sequenced di-nucleosome sequences from yeast, chicken and human. Each dinucleosome sequence roughly covered two nucleosomes (located toward the two ends) with a linker DNA in between. A Hidden Markov Model model was trained based on the nucleosome sequence alignment for prediction of nucleosome positioning. Results show that Eukaryotic cells do favor periodic linker length in chromatin forming on a roughly 10 bp basis, however with two different forms, i.e. with peaks around 5 bps or 10bps.

A Ph.d student, Pengfei Li from the University of Waterloo, presented yet another new development in finite mixture models. As pointed out by Professor Hangfeng Chen, it is highly desirable to develop a theory that does not rely on the artificial compactness assumption. At the same time, it is also noticed that some widely used finite mixture models in statistical genetics, have infinite Fisher information, which are also excluded from most existing theory for finite mixture models. To overcome these shortcomings, a new class of hypothesis tests is proposed. It is show the new method not only has broader range of applications, but also provide a more efficient procedure.

The analysis of directional data occupies a special place in statistics. Such data not only raise as measurement of directions, but also as measurement of phases for physical processes with periodic behaviors. They are useful in genetic in modeling the circles of gene expression levels among others. The presentation by Professor Yuejiao Fu from York University discusses inference problems related to finite mixtures of von Mises distributions and its applications to statistical genetics. Fu proposed the use of the modified likelihood ratio test and the iterative modified likelihood ratio test in general two-component von Mises mixture with a structural parameter. Two accuracy enhancing methods are developed. The limiting distributions of the resulting test statistics are derived. Simulations show that the test statistics have accurate type I errors and adequate power.

Outcome of the Meeting

We would like to thank BIRS for the excellent facilities and great support. Our workshop is a big success. It provided a forum for statisticians to mix with biologists allowing both groups to identify important scientific questions. The talks were of high quality. Participants expressed their great interest for further discussion. The continuation of already existing collaborations and the creation of new ones is an important outcome of the 5-day workshop. We also received a lot of feedback from the participants:

"It's a wonderful meeting and I enjoyed it very much!"

"Thank you all for organizing such an excellent workshop! I really enjoyed it and was very stimulated by many of the talks."

"I wanted to thank you all for the fantastic job you did in organizing the workshop. I have learned a lot."

"I have no doubt that this workshop is very beneficial and rewarding experience. Thank you very much!"

"Overall this is one of THE BEST workshops I have attended."

Press Release

¿From Mendel's agricultural experiments on peas to human genome projects on chromosomes, geneticists as well as the general public have been fascinated by factors that inherently define the vastly diverse characters of the living world. While only most obvious traits such as the color of flowers were observed in the old days, modern techniques enable geneticists to measure hundreds of thousands of genes of an organism on a single microarray chip. A high amount of high-throughput data are thus generated routinely. The task of identifying important genes out of tens of thousands, which are associated with traits such as cancer and diabetes, demands serious effort in designing effective statistical analysis procedures.

¿From June 24 -29, 2007, a group of statisticians/geneticists from all over the world will come to Banff International Research Station to exchange ideas and report their advances on the analysis of high-throughput data. This event is co-organized by Professor Jiahua Chen from the University of Waterloo, who has recently been awarded the tier I Canada Research Chair in Statistical Genetics at the University of British Columbia. Other organizers include Professor Mary Lesperance from the University of Victoria, Professor Yuejiao Fu from York University, Canada; Professor David Siegmund from Stanford University who is a member of the National Academy of Sciences in the USA and renowned for his advanced research in statistical genetics; and Professors Heping Zhang and Hongyu Zhao from the School of Public Health, Yale University. Professor Heping Zhang is director of the Collaborative Center for Statistics in Science, and Professor Hongyu Zhao is the director of the Center for Statistical Genomics and Proteomics.

List of Participants

Allison, David (University of Alabama at Birmingham)
Baglivo, Jenny (Boston College)
Bryan, Jennifer (University of British Columbia)
Bull, Shelley (University of Toronto)
Chen, Jiahua (University of British Columbia)
Chen, Zehua (National University of Singapore)

Chen, Hangfeng (Bowling Green State University)
Chen, Liang (University of Southern California, Los Angeles)
Dupuis, Josee (Boston University School of Public Health)
Ewens, Warren (University of Pennsylvania, Philadelphia)
Fan, Guangzhe (University of Waterloo)
Fu, Yuejiao Cindy (York University)
He, Xuming (University of Illinois at Urbana-Champaign)
He, Wenqing (University of Western Ontario)
Horvath, Steve (University of California, Los Angeles)
Huang, Jian (University of Iowa)
Lesperance, Mary (University of Victoria)
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Song, Peter (University of Waterloo)
Tu, Dongsheng (Queen's University)
Wang, Ji-Ping (Northwestern University)
Wang, Huixia (North Carolina State University)
Wang, Hsiao-Hsuan (York University)
Weeks, Daniel (Univesity of Pittsburgh)
Yakir, Benjamin (Hebrew University Mount Scopus)
Zee, Chung-Ying (Chinese University of Hong Kong)
Zhang, Heping (Yale University)
Zhang, Meizhuo (Yale Univesity, School of Public Health)
Zhao, Hongyu (Yale University)
Zou, Fei (University of North Carolina at Chapel Hill)

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Chapter 21

Bioinformatics, Genetics and Stochastic Computation: Bridging the Gap (07w5079)

Jul 01 - Jul 06, 2007

Organizer(s): Arnaud Doucet (University of British Columbia), Raphael Gottardo (University of British Columbia), Christian Robert (Ceremade, Universite Paris Dauphine)

The meeting that took place at BIRS, Banff, on July 1-6 2007, gathered 33 people from America (Canada and USA), Europe, Japan and Australia. It was quite successful, thanks to the superb organisation of the Center and the friendly help of the staff, and all invited participants showed up for the meeting. The talks were uniformly well-attended by an overwhelming number of the participants who stayed the whole week, with two exceptions only. There was no mountain accident or bear encounter to deplore, and the organisers were not aware of any complaint from the participants, during or after the meeting, but instead gathered many thanks and requests for a follow-up meeting. Several collaborations were initiated during the meeting as well.

Objectives

On the one hand, there is an explosion of complex statistical models appearing in genetics and bioinformatics. These typically highly structured systems are unfortunately very difficult to fit. On the other hand, there has been recently significant advances on MC methods but most of these methods are unknown to the applied community and/or formulated in ways that are too theoretical for direct application. By providing scientists with improved inferential methods it would allow them to consider richer models, which are more realistic than those dictated by computational constraints. The exchanges between the applied and methodological communities remain surprisingly limited.

We believe that this workshop would be an ideal place to (1) Gather people from different research communities and foster links between these communities (applied, methodological and theoretical). (2) Expose the applied community to novel statistical methodologies and advanced MC methods, and expose the MCMC community to the specifics of the complex modelling problems met in bioinformatics and genetics. (3) Classify topologies of computational problems met by bioinformatics and genetics, and equip all participants to the workshop with benchmark problems, if possible before the workshop.

We anticipate this workshop to create an exceptional opportunity for exchanging ideas between the communities; and help to shape the future of stochastic computation within bioinformatics and genetics. Input from statisticians working in bioinformatics and genetics is absolutely crucial for development of appropriate statistical methodologies. We will encourage researchers to bring data from their own field, which could be used to implement methods and try new algorithms. We have targeted areas of stochastic computation that are of great interest to practitioners such as automatic algorithms, computational issues and parallel implementations. Bioinformatics and genetics are relatively new fields that enjoy a high representation of young talents, including many women. This workshop would be a great learning/training environment for these new talents.

With regards to the goal of the meeting replicated above, the meeting was clearly centred on the statistical aspects of *the complex models appearing in genetics and bioinformatics*, namely on the statistical methodology that allowed the participants to tackle the statistical analysis of those models. Most talks were therefore at the interface between statistical modelling, statistical methodology, and computational statistics. The new advances on Monte Carlo methods were indeed at the forefront of most talks, which mostly developed new tools and produced new results to handle their complex models, more realistic than those dictated by computational constraints. We believe that the presentations at the meeting enhanced the *exchanges between the applied and methodological communities*, thanks to the open schedule adopted by us. We indeed *gathered people from different research communities and foster links between these communities (applied, methodological and theoretical)*: D. Balding, M. Beaumont, R. Gottardo, M. de Iorio, S. Keles, S. Schmidler, and D. Stephens are primarily working in Genomics and, for some of them, are not statisticians, R. Craiu, A. Dobra, A. Doucet, P. Fearnhead P. Green, A. Jasra, J.-M. Marin, P. Müller, S. Richardson, C. Robert, C. Sabatti, M. Stephens, M. Vanucci, M. West and D. Wilkinson are mostly focussing on the theory of computation and of Bayesian inference, with forays into genetic and biological applications, while L. Bornn, A. Brockwell, F. Caron, M. Gupta, C. Holmes, J. Keith, A. Lewin, T. Matsumoto, K. Mengersen, K. Murphy, S. Schmidler, and E. Thompson, centre their research on the specific development of statistical methods for biological and genetic models, therefore being truly at the interface. Obviously, this classification in three classes is somehow arbitrary. The *exposure [of] the applied community to novel statistical methodologies and advanced MC methods,* was clear since five of the keynote speeches were dealing with methodological topics, mostly related to computational Statistics, and the *exposure [of] the MCMC community to the specifics of the complex modelling problems met in bioinformatics and genetics* was operated via most than half of the talks. We however failed short of *classify[ing] topologies of computational problems met by bioinformatics and genetics* in a coherent manner, and definitely did not *equip all participants to the workshop with benchmark problems,* due to the difficulty of collecting sufficiently challenging datasets that would appeal to every attendant. Although we did *encourage researchers to bring data from their own field, which could be used to implement methods and try new algorithms*, this alas did not happen. In retrospect, this aspect would have required a smaller number of participants and would have thus restricted the field processed by the meeting.

Similarly, the workshop did constitute *an exceptional opportunity for exchanging ideas between the communities*, as, again, shown by the involvement of all participants in every session of the meeting, despite outdoor temptations all around!. We cannot tell at this stage how much the workshop help in *shaping the future of stochastic computation within bioinformatics and genetics* but we believe major actors in this field took part in the meeting, including young talented researchers for whom this workshop truly was *a great learning/training environment*.

For instance, Ajay Jasra (Imperial College, London) presented an important advance for the processing of stochastic trees, which are so prevalent in (population) Genetics. The difficulty in handling the likelihood function was solved in this joint work with Maria de Iorio and Marc Chadeau from Imperial using importance sampling and sequential Monte Carlo techniques. In order to handle the computational difficulty with simulating backward in time, Ajay Jasra introduced controlled approximations where the bias remained under control. Similarly, Luke Born (University of British Columbia) introduced sequential Monte Carlo methods towards computational gains in prior sensitivity, even in cases when the distribution of interest is not available analytically. Francois Caron (University of British Columbia) considered the problem of identifying novel RNA transcripts using tiling arrays. Standard approaches to this problem rely on the calculation of a sliding window statistic or on simple changepoint models. These methods suffer from several drawbacks including the need to determine a threshold to label transcript regions and/or specify the number of transcripts. He thus proposed a Bayesian multiple changepoint model to simultaneously identify the number of transcripts, the transcript boundaries and their associated levels. In addition, he presented a computationally efficient on-line algorithm which allows to jointly estimate both the changepoint locations and the model parameters. Using two publicly available transcription data sets, he compared his method to a common sliding window approach and a simple changepoint model. establishing that his on-line estimation procedure provides good estimates of transcript boundaries and model parameters. Alex Linwin presented a Bayesian hierarchical model for detecting differentially expressed genes using a mixture prior on the parameters representing differential effects. He formulated an easily interpretable 3-component mixture to classify genes as over-expressed, under-expressed and non-differentially expressed, and model gene variances exchangeably to allow for variability between genes. He showed how the proportion of differentially expressed genes, and the mixture parameters, can be estimated in a fully Bayesian way, extending previous approaches where this proportion was fixed and empirically estimated. Good estimates of the false discovery rates were also obtained. Different parametric families for the mixture components can lead to quite different classifications of genes for a given data set. Using Affymetrix data from a knock out and wildtype mice experiment, he showed how predictive model checks can be used to guide the choice between possible mixture priors. These checks showed that extending the mixture model to allow extra variability around zero instead of the usual point mass null fits the data better.

These talks were linked with the keynote talk of Matthew Stephens that very broadly set the challenges met in this area, as well as the directions for their resolution. Another related keynote speech was given by Peter Green (University of Bristol) on clustering gene expression profiles using Dirichlet process models. He introduced a Bayesian mixture model that allowed to express a gene expression profile across different experimental conditions as a linear combination of covariates characterising those conditions, plus error. In a standard Bayesian nonparametric formulation, the expectations and the error precisions of the expression measurements would jointly follow a Dirichlet process (DP). In this set-up the clusters generated by the process are a priori exchangeable. However in the gene expression context, it commonly occurs that some genes are not influenced by the covariates, but fall into a ‘background’ class. This calls for an extension to the DP model generating a background cluster that is not exchangeable with the others, and he also built regression on covariates characterising experimental conditions into the expectation structure. He defined a particular heterogeneous Dirichlet process as a mixture of a random point mass and a Dirichlet process. The location of the point mass has a partially degenerate distribution, allowing some regression coefficients to be fixed at zero for the background cluster. Standard posterior sampling methods for DP models can be extended to make use of this heterogeneous prior model. In particular, in the case of conjugacy, he thus generalised the partition Gibbs sampler/weighted Chinese restaurant process to this situation. The background or ‘top-table’ cluster can be identified in the posterior sample. He used a loss function approach following Lau and Green (2008) to derive a point estimate of the remaining clusters.

Another of the keynote talks was given by Mike West (Duke University) where he presented a wide ranging survey of the recent results he and his team obtained on statistical inference for dynamic cellular networks in systems biology. Advances in bioengineering technologies are generating the ability to measure increasingly high-resolution, dynamic data on complex cellular networks at multiple biological and temporal scales. Single-cell molecular studies, in which data is generated on the levels of expression of a small number of proteins within individual cells over time using time-lapse fluorescent microscopy, is one critical emerging area. Single cell experiments have potential to develop centrally in both mechanistic studies of natural biological systems as well as via synthetic biology – the latter involving engineering of small cellular networks with well-defined function, so providing opportunity for controlled experimentation and bionetwork design. There is a substantial lag, however, in the ability to integrate, understand and utilise data generated from single-cell fluorescent microscopy studies. The talk highlighted aspects of this area from the perspective of Mike West’s forays in single cell studies in synthetic bacterial systems that emulate key aspects of mammalian gene networks central to all human cancers. The most relevant aspects were about

1. The data in those studies come as movies of colonies of cells developing through time, with a need for imaging methods to estimate cell-specific levels of fluorescence measuring mRNA levels of one or several tagged genes within each cell. This is complicated by the progression of cells through multiple cell divisions that raises questions of tracking the lineages of individual cells over time.

2. In the context of their synthetic gene networks engineered into bacterial cells, they have developed discrete-time statistical dynamic models inspired by basic biochemical network modelling of the stochastic regulatory gene network. These models allow the incorporation of multiple components of noise that is ”intrinsic” to biological networks as well as approximation and measurement errors, and provide the opportunity to formally evaluate the capacity of single cell data to inform on biochemical parameters and ”recover” network structure in contexts of contaminating noise.

3. Last and not least, in their approaches to model fitting, they have developed Bayesian methods for inference in non-linear time series. This involves MCMC methods that impute parameter values coupled with novel, effective Metropolis methods for what can be very high-dimensional latent states representing the unobserved levels of mRNA or proteins on nodes in the network as well as contributions from ”missing” nodes.

In connection with the keynote talk of Sylvia Richardson (Imperial College London) on the Bayesian and computational tools required to run models selection in “large p small n” linear models, namely models with many more covariates than observations, which requires the use of default priors across all models like Zellner’s g-priors, Jean-Michel Marin (Universit Paris Sud) presented a talk on an hierarchical extension of the g-prior that allowed for less informative inputs as well as computational gains, avoiding the recourse to complex techniques like reversible jump. In the same spirit, Marina Vanucci (Rice University) addressed in her talk methods for Bayesian variable selection for high-dimensional data. While initially dealing with the simple linear regression model, she extended the setup to probit models for classification and to clustering settings, as well as survival data. Her talk included many examples from genomics, in particular DNA microarray studies. In addition, the analysis of the high-dimensional data generated by her studies challenges standard statistical methods and she discussed the performances of those methods both on simulated and real data. Additional talks in this quite important area included Anthony Brockwell’s, Adrian Dobra’s, Sundunz Keles’, Alex Lewin’s, Peter Mueller’s and Chiara Sabatti’s.

In his keynote talk, Paul Fearnhead considered Bayesian analysis of a class of multiple changepoint models. While there are a variety of efficient ways to analyse these models if the parameters associated with each segment are independent, there are few general approaches for models where the parameters are dependent. Under the assumption that the dependence is Markov, he proposed an efficient online algorithm for sampling from the an approximation to the posterior distribution of the number and position of the changepoints. In a simulation study, Paul Fearnhead showed that the approximation introduced is negligible. He illustrated the power of his approach through fitting piecewise polynomial models to data, under a model which allows for either continuity or discontinuity of the underlying curve at each changepoint. This method is competitive with, or out-performs, other methods for inferring curves from noisy data; and uniquely it allows for inference of the locations of discontinuities in the underlying curve.

Conclusion

While quantifying the impact of a workshop always is a delicate task, we are convinced that this meeting has had an influence on the community of computational statisticians working in Bioinformatics and Genomics. The recent rise of ABC (standing for Approximate Bayesian Methods) methods can for instance be partly connected to the debate about this method initiated during the meeting after the talk of David Balding. Similarly, the current revival of Bayesian model choice evaluation has links with the talks of Jean-Michel Marin, Sylvia Richardson, and Scott Schmidler. The fact that many of us keep referring to this meeting as a highlight, even two years later, is also a significant indicator that some alchemy took on during the workshop, even though putting a finger on exactly what happened is not possible. Once again, we are immensely grateful to PIMS for its support and trust, as well as to the BIRS centre and its friendly staff for a superb organisation that let us concentrate 110% on scientific issues.

List of Participants

Balding, David (Imperial College)
Beaumont, Mark (University of Reading)
Bornn, Luke (University of British Columbia)
Brockwell, Anthony (Carnegie Mellon University)
Caron, Francois (University of British Columbia)
Craiu, Radu (University of Toronto)
de Iorio, Maria (Imperial College)
Dobra, Adrian (University of Washington)
Doucet, Arnaud (University of British Columbia)
Fearnhead, Paul (Lancaster University)
Gottardo, Raphael (University of British Columbia)
Green, Peter (University of Bristol)
Gupta, Mayetri (University of North Carolina at Chapel Hill)
Holmes, Chris (Oxford University)
Inoue, Lurdes (University of Washington)
Jasra, Ajay (Imperial College London)
Keith, Jonathan (Queensland University of Technology)
Keles, Sunduz (University of Wisconsin, Madison)
Lewin, Alex (Imperial College, Centre for Biostatistics)
Marin, Jean Michel (Project Select INRIA Futurs)
Matsumoto, Takashi (Waseda University)
Mengersen, Kerrie (Queensland University of Technology)
Mueller, Peter (The University of Texas M. D. Anderson Cancer Center)
Murphy, Kevin (University of British Columbia)
Richardson, Sylvia (Imperial College)
Robert, Christian (Ceremade, Universite Paris Dauphine)
Sabatti, Chiara (University of California, Los Angeles)
Schmidler, Scott (Duke University)
Stephens, Matthew (University of Chicago)
Stephens, David (McGill University)
Thompson, Elizabeth (University of Washington)
Vannucci, Marina (Texas A&M University)
West, Mike (Duke University)
Wilkinson, Darren (Newcastle University)

Chapter 22

L-functions, ranks of elliptic curves, and random matrix theory (07w5114)

Jul 08 - Jul 13, 2007

Organizer(s): Brian Conrey (American Institute of Mathematics), Michael Rubinstein (University of Waterloo), Nina Snaith (University of Bristol)

Overview of the Field

The group of rational points on an elliptic curve is one of the more fascinating number theoretic objects studied in recent times. The description of this group in terms of the special value of the L-function, or a derivative of some order, at the center of the critical strip, as enunciated by Birch and Swinnerton-Dyer is surely one of the most beautiful relationships in all of mathematics; also it's understanding carries a $1 million dollar reward!

Random Matrix Theory (RMT) has recently been revealed to be an exceptionally powerful tool for expressing the finer structure of the value-distribution of L-functions. Initially developed in great detail by physicists interested in the statistical properties of energy levels of excited nuclei, RMT has proven to be capable of describing many complex phenomena, including average behavior of L-functions.

The most important invariant of an elliptic curve is the rank of its (Mordell-Weil) group of rational points; it is a non-negative integer, believed to be 0 or 1 for almost all elliptic curves. The beginnings of the subject is a conjecture (see [1]) about how often the rank is greater than or equal to 2 for the family of quadratic twists of a given elliptic curve. Each elliptic curve has an L-function associated with it; this is an entire function which satisfies a functional equation. The Birch and Swinnerton-Dyer conjecture asserts, among other things, that the order of vanishing at the central point of the L-function associated with an elliptic curve is equal to the rank. It is generally conjectured that almost all elliptic curves have rank zero or one according to whether the sign of the functional equation of the related L-function is $+1$ or -1. Rank two curves should occur with L-functions that have a $+1$ sign of their functional equation but vanish nevertheless at the central point. These are expected to be rare; the question of how rare is the subject here.

If the elliptic curve is given by $E : y^2 = x^3 + Ax + B$, and if d is a fundamental discriminant, then the quadratic twist of E by d is the elliptic curve $E_d := dy^2 = x^3 + Ax + B$. The conjecture, derived from RMT and number theory, is that E_d will have rank 2, or greater, for asymptotically $c_E x^{3/4}(\log x)^{b_E+\frac{3}{8}}$ values of d with $|d| \le x$; here b_E is one of four values (see [7]):

- $b_E = 1$ when E has full rational 2-torsion,
- $b_E = \frac{\sqrt{2}}{2}$ when E has one rational 2-torsion point,
- $b_E = \frac{1}{3}$ when E has no rational 2-torsion and the discriminant is a perfect square,

- $b_E = \frac{\sqrt{2}}{2} - \frac{1}{3}$ when E has no rational 2-torsion and the discriminant is not a square.

The constant c_E is yet to be determined, but depends on a mix of RMT, number theory, and probabilistic group theory; see [10].

This conjecture, while interesting, is not as compelling as it might be because of our ignorance of c_E. However, an absolutely convincing case for RMT can be given by considering curves of rank 2 or higher, as above, but divided into *arithmetic progressions* of d modulo some prime p.

Using RMT arguments combined with a number theoretic discretization of the problem, one is led to predict that if a is a quadratic residue mod p and b is a quadratic non-residue then the ratio of rank 2 or higher twists among $d \equiv a \bmod p$ to $d \equiv b \bmod p$ is, in the limit,

$$R_p := \sqrt{\frac{p+1-a_p}{p+1+a_p}}$$

where $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the L-function associated with E. Those familiar with the conjecture of Birch and Swinnerton-Dyer might not be surprised to see the ratio

$$\frac{p+1-a_p}{p+1+a_p}$$

show up; however, it is the square-root, contributed by RMT, that is the surprise. Here is some data on quadratic twists of the elliptic curve of conductor 11 where the absolute value of the discriminant is at most $T = 333605031$ (here $R_p(T)$ is the computed ratio of positive even ranks amongst twists by discriminants in square residue classes modulo p to those in non-square residue classes modulo p, i.e. the empirical approximation to R_p):

p	a_p	R_p	$R_p(T)$
3	-1	1.2909944	1.2774873
5	1	0.8451542	0.84938811
7	-2	1.2909944	1.288618
13	4	0.74535599	0.73266305
17	2	1.118034	1.1282072
19	0	1	1.000864
23	-1	1.0425721	1.0470095
29	0	1	0.99769402
31	7	0.80064077	0.78332934
37	3	0.92393644	0.91867671
41	-8	1.2126781	1.2400086
43	-6	1.1470787	1.1642671
47	8	0.84515425	0.82819492

The basic calculation to obtain this result involves a ratio of conjectures for

$$\sum_{\substack{d \equiv a \bmod p \\ d \leq x}} L_{E_d}(1/2)^{-1/2};$$

the reason that one takes the -1/2 power here is due to the fact that the s'th moment of characteristic polynomials of even orthogonal matrices has its rightmost pole at $s = -1/2$. The description of this calculation and the compelling numerical evidence is in the paper [1]. The calculation is taken a step further in the paper of Conrey, Pokharel, Rubinstein, and Watkins [3] where lower order terms for the moments are incorporated and lead to an even more precise evaluation of these ratios.

The conjectures about quadratic twists can be generalized to cubic twists in two different ways. One involves the frequency of rank 2, or greater, elliptic curves within the classical family $E_m := x^3 + y^3 = m$. Here we restrict attention to m for which the sign of the functional equation of the elliptic curve associated with E_m is $+1$. One expects that amongst such m for primes $p \equiv 2 \bmod 3$ the m for which E_m has at least rank 2 will be evenly distributed among the residue classes modulo p. For primes $p \equiv 1 \bmod 3$ the conjecture is more interesting. Here we define $a_c(p)$ to be one of the three solutions of $a \equiv 2 \bmod 3$ with $4p = a^2+3b^2$. For example, we have $a_c(7) = 5$ for $c = 3, 4$; $a_c(7) = -1$ for $c = 1, 6$; and $a_c(7) = -4$ for $c = 2, 5$. Then, the ratio of the frequency of curves of at least rank 2 amongst $m \equiv c_1 \bmod p$ compared with $m \equiv c_2 \bmod p$ is conjecturally

$$\sqrt{\frac{p+1-a_{c_1}(p)}{p+1-a_{c_2}(p)}}.$$

Notice that

$$\sqrt{\frac{7+1-a_{2,5}(7)}{7+1-a_{3,4}(7)}} = 2 \qquad \sqrt{\frac{7+1-a_{1,6}(7)}{7+1-a_{3,4}(7)}} = \sqrt{3} \qquad \sqrt{\frac{7+1-a_{2,5}(7)}{7+1-a_{1,6}(7)}} = \frac{2\sqrt{3}}{3}$$

so that, conjecturally, curves of rank two or higher come up twice as often for $m \equiv 2 \bmod 7$ as for $m \equiv 3 \bmod 7$. Here is some data for the number of curves with rank r greater than zero for each residue class $c(\bmod 7)$:

c	$\#r > 0$	# curves	ratio
1	109569	595982	0.184
2	125728	595952	0.211
3	59440	595912	0.100
4	58759	595903	0.099
5	125714	595963	0.211
6	110125	595937	0.185

The other way to do a cubic twist is to take a fixed elliptic curve E and a Dirichlet character χ of order 3 and consider the twisted L-function, $L_E(s,\chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$. David, Fearnley, and Kisilevsky [5] have shown, very surprisingly, that such twists vanish for about $x^{1/2}$ cubic twists of modulus $\leq x$, and have given precise conjectures, based on RMT, for the asymptotic frequency of this event. They also consider quintic twists and conclude that there are (barely!) infinitely many order five characters for which the twisted L-function vanishes at the central point. These predictions are based on considerations with groups of unitary matrices, whereas the previously mentioned conjectures arise from calculations on groups of orthogonal matrices.

It is interesting to begin with a weight 4 modular newform f, with integer Fourier coefficients, and similarly ask about vanishing of, say, quadratic twists of the associated L-function. In this case it is expected that there will be asymptotically $c_f x^{1/4} (\log x)^{b_f}$ vanishings at the central point of the quadratically twisted L-functions . The possible values of b_f have not been worked out here; however, if one restricts to prime discriminants, then the power on the log is expected to be $-5/8$ in both this case and the case of twists of elliptic curve L-functions. If one considers weight 6 or higher, it is expected that there will only be finitely many vanishings of quadratic twists of the associated L-functions. It is not clear what happens if one considers all such weight 6 forms and all of their twists if one then accumulates infinitely many vanishings. There is an arithmetic significance to the vanishings of the twists of the weight 4 modular forms: it is related to the rank of an associated Chow group, about which we hope to say more at a later time.

In the twists mentioned here, of course we only consider the twists for which there is a plus sign in the functional equation.

The evidence for many of the above conjectures has been accumulated by a combination of forces: Tornaria, Rodriguez-Villegas, Rosson, Mao, and Rubinstein. Much of it is based on an algorithm of Gross for finding the half-integral weight form, as a theta series involving ternary quadratic forms, whose Fourier coefficients yield the values of the twisted L-series at the central point. Prior to a workshop at the Newton Institute in February 2004, only a handful of such theta series were known. During that workshop, the first four people above worked out the obstacles to further progress and provided literally thousands of examples to the last named person who computed hundreds of millions of values for each; this provides a nice data bank for testing conjectures.

All of the above discussion has been focused on curves of even rank two or higher. The question of modeling rank 3 or higher members of a family is much more difficult; in fact it is not at all satisfactorily addressed. In the case of trying to determine rank 2 quadratic twists the Random Matrix model is based on a discretization arising from the beautiful formula, due in this form to Kohnen and Zagier [14]:

$$L_{E_d}(1/2) = \kappa_E \frac{c_E(|d|)^2}{\sqrt{|d|}}$$

where $c_E(|d|)$ is an integer and $\kappa_E > 0$. In the case of trying to determine the frequency of odd rank > 1 amongst quadratic twists, we consider the conjectural formula of Birch and Swinnerton-Dyer for the value of the derivative of an odd $L_{E_d}(s)$:

$$L'_{E_d}(1/2) = \kappa_E \frac{h_{E_d}|\text{Sha}_{E_d}|}{\sqrt{d}}$$

where h_{E_d} is the height of a generating point and $|\text{Sha}_{E_d}|$ is the order of the Tate-Shafarevich group. Now we don't know what kind of discretization to give h_{E_d}. It could conceivably be as small as $\log |d|$ but statistically this does not seem to be the correct model. By the work of Snaith [27], the right-most pole of the derivative of the sth moment of characteristic polynomials of odd orthogonal matrices occurs at $s = -3/2$. This might suggest, if one uses the discretization $(\log |d|)/\sqrt{|d|}$, that there are only about $x^{1/4}$ rank 3 curves among the family of twists with conductor smaller than x. However, Rubin and Silverberg [25] give examples of E which have many more rank 3 quadratic twists. In examining the limited data we have about rank 3's, an interesting phenomenon seems to appear: it looks as though $L'_{E_d}(1/2)$ cannot be as small as $(\log |d|)/\sqrt{|d|}$. Is it possible that when Sha is small then the height of a generating point is big and vice-versa? This linkage does not seem unnatural if one compares for example to the situation of the class number of a real quadratic field. There one finds that the product of the regulator times the size of the class group is always about the size of the square root of the discriminant. However, this analogy may not be correct, since this involves L-functions at the edge of the critical strip whereas we are discussing values at the center. Much more data is needed to make an informed conclusion.

New Directions

The initial goals for this workshop were to:

- Give a good statistical model for rank three quadratic twists.
- Find a fast algorithm to compute which curves from a quadratic twist family have rank 3.
- Understand why the size of the regulator times the size of Sha is so big for rank one curves.
- Produce theta series for thousands of weight 4 and 6 forms.
- Find an algorithm to give the theta series for odd weight form.
- Refine our understanding of the value of c_E.
- Investigate the constants $b_3(E)$ and $c_3(E)$ which arise in the case of cubic twists.

But as the workshop developed it transformed into being much more about RMT and statistics of higher rank L functions. And so a big motivation for the workshop was to see if arithmetical applications of Random Matrix Theory could be found for more exotic or higher rank L-functions. For example, there are conjectures due to Böcherer and Schulze-Pillot about the special values at the central point of L-functions associated with Siegel modular forms. One wonders whether Random Matrix models can be used to predict the frequency with which these values vanish and in general what their value distribution is. The discussion of these topics has led to a desire to be able to do explicit calculations with these L-functions. Almost nothing has been done in this direction, but a specific outcome of the workshop is the resolution to compute these.

Another example of the investigation higher rank is the work of Martin and Watkins [18] on central values of symmetric powers of L-functions of elliptic curves. For example, they find many suspected double zeros of the symmetric cubes of these. RMT should be able to use an appropriate discretization based on conjectural formulas for the special values to predict the frequency with which this happens.

Watkins [32] also reported on the implementation of an algorithm using Heegner points to check for rank 3 curves elliptic curves in families of quadratic twists in essentially the same amount of time that it now takes to do rank 2 curves. He counts rank 3 curves from a family of quadratic twists and gives evidence that they are distributed as

$$\left(\frac{p+1-a_p}{p+1+a_p}\right)^{-3/2}.$$

Watkins also has conjectures based on RMT about how many elliptic curves of conductor up to X will have rank 2, namely $cX^{\frac{19}{24}}(\log X)^{\frac{3}{8}}$.

The families mentioned above are all families of quadratic twists of a given curve. David, Fearnley and Kisilevsky have been investigating families of odd order twists, especially twists by Dirichlet characters of orders 3, 5, and 7. They predict $c_3(E)x^{1/2}(\log x)^{b_3(E)}$ rank two curves among cubic twists, while about $(\log x)^c$ rank two curves among quintic twists.

Another area of interest concerns the generalization wherein the modular form of weight 2 associated with the elliptic curve is replaced by a modular form of weight 4. In this case the arithmetic question is about the rank of a Chow group. Some of these forms are associated with Calabi-Yau threefolds (see for example the work of Helena Verrill) and so are of interest to a wide community of scientists. A small amount of data has been gathered here; more would be extremely valuable. For weight 6 newforms, only a finite number of twists are expected to vanish. Investigating how this finite number varies with the level is of interest.

Considering vanishing for twists of odd weight rational new forms would also be very interesting. No one knows any theta series to assist with numerical experiments here.

Another application for these ideas is in the context of Siegel modular forms which also have formulas for the special values of their twists.

Presentations

The workshop brought together people from the Random Matrix side of things with number theorists. Therefore, it was decided to feature some introductory lectures from both fields. The lecture of Rubinstein and two lectures by Keating provided this introduction on the Random Matrix Theory side of things, while Kohnen gave a series of three lectures introducing Siegel modular forms. In addition, there was a lengthy discussion session on Friday morning; a very loose transcript of that is given below.

- Monday
  - 9:00 Rubinstein: Probability models for elliptic curves Corbett Hall
  - 10:30 Darmon: Survey of special values of L-functions, BSD, and Heegner points
  - 14:30 Keating: Random matrix theory I
  - 16:00 Kohnen: Siegel modular forms I
- Tuesday
  - 9:30 Mao: Shimura correspondence and computation of L-values
  - 11:00 Darmon: Shintani lifts, p-adic families, and derivatives of quadratic twists
  - 14:00 Kohnen: Siegel modular forms II
  - 15:30 Keating: Random matrix theory II
- Wednesday
  - 9:00 Watkins: Non-trivial vanishings of odd quadratic twists
  - 10:30 Delaunay: Odd rank quadratic twists of elliptic curves
- Thursday

- 9:30 Kisilevsky: Ranks of Elliptic Curves in Families of Cubic Extensions
- 11:00 Rubin: Ranks of elliptic curves in families of quadratic twists
- 14:00 Kohnen: Values of spinor zeta functions at the central point
- 15:30 Miller, Duenez, and Huynh: Finite conductor models for zeros of elliptic curves

- Friday
  - 9:00 - 12:00 Discussion and wrap-up

Discussion Topics

Statistics of N consecutive eigenvalues

Mike Rubinstein: Instead of looking at $U(N)$ for finite N, let M tend to infinity and look at N consecutive normalized eigenvalues of $U(M)$.

David Farmer: There's no way that this will be what we want. In the large M limit we have a characteristic function of whether or not overlap. Nina and Jon's picture: hasn't converged to the Gaussian, won't appear. Will throw away all the lower order terms.

Mike R: Is that a theorem?

David F: Take a huge matrix, segment of its characteristic polynomial will be exactly Gaussian and not skewed Gaussian.

Mike R: So then David thinks we can easily check wi' a single statistic, numerical statistic.

David F: yes.

Mike R: Eigenvalues will still repel. Jon?

Jon Keating: You said 2×2 matrices. There are some statistics where answer is extremely close to infinite. Local statistics largely independent of size of matrix block. Long range statistics.

David F: leading order close, not lower order. Lower order will fall apart instantly if M is not the right size.

Mike R: this is the better model to fit what we're doing with L-functions. (The mixed Hadamard and primes model).

Weyl Measure: Is it RMT or the underlying distributions that are significant for L-functions

David Farmer: In Keating's talk he showed calculations involving Weyl integration formula. Everything done in subject only uses that measure. Now, if had been told that measure but not where it came from, would you think RMT is the right thing as to where it came from, or something else? Only the *measure* on the eigenvalues we use; might be missing something (only using measure).

Mike R: What about function field analogue, different families (unitary, symplectic, orthogonal). How are the L-functions choosing?

David F: sounds natural, but not necessarily right.

Jon Keating: famous: arrival times of bus in Mexican town, spacing b/w parked cars in London. Same measure, no matrix. Comes from an entropy formula. Appears elsewhere.

David F: take N points at the origin, Brownian motion to the unit circle without allowing crossing. Unitary statistics. What if someone offered this as the hypothesis as the behavior of the L-functions.

Packaging twists by cubic Dirichlet characters

Hershy Kisilevsky: L-values from cubic twists. Envious of Rubinstein who can use half-integral weights to get millions of computations. Find some way to package things.

Henri Darmon: Might be a paper in some special situations.

Mike Rubinstein: Maybe fast Fourier technique for the ternary calculations. Maybe D^ϵ for a single computation on average.

Brian Conrey: to calculate one value is $\sqrt{T}$ steps, but to calculate T of them it takes about $T^{1+\epsilon}$ (O-S). Maybe if we organize it to recognize things being repeated we can save a $\sqrt{D}$.

Many (10^{12}) quadratic twists

Mike R: This would be another project, in the back of my mind. Today might be possible to do 10^{10} to maybe 10^{12} (pushing it).

Jack Fearnley: we're at about 10^6.

Mike R: It's on a logarithm scale. There are some secondary terms which we will see a bit of an improvement on.

Brian C: certain asymptotics that we just won't be able to see. Many problems trying to modify $X^A \log^B X$, tough to get at our level of data.

Interpolating integer moments

Brian C: Paul has a formula for

$$\int_{U(N)} |\Lambda(1)|^{2k-r} |\Lambda'(1)|^{2r} d\mu \tag{22.1}$$

for integers r and k: analytic continuation? (Note: may have the wrong exponents for the Λ's.)

Paul: If you fix k, can only evaluate for a finite number of r's. Integer points under a line in the (r, k)-plane. Know it is not the correct formula if continue. Use finitely many points, analytically continue above the line, but can get to that point by analytically continuing horizontally (which know is correct), but get two different answers.

David F: there is at least one example where replacing factorials with Gamma functions doesn't give the right thing (when going from integers to non-integers). Always say there are formulas that interpolate. There are so many of them that surely some are wrong. Here is an example. I would like to understand when it is okay to interpolate,

Mike R: related to the question: can we develop an analytic theory for the full moments. Leading term is just $a_k g_k$. Can we develop heuristics (proofs are beyond us) for the analytic theory of the full moment:

$$\int_0^T |\zeta(1/2+it)|^{2k}\, dt \;=\; \int_0^T P_k\left(\log \frac{t}{2\pi}\right) dt + O\left(T^{1/2+\varepsilon}\right). \tag{22.2}$$

Here we can do it for k an integer, want to interpolate for all k. We need a difference approach to the moments that will allow us to work it out. Maybe a refined version of the hybrid formula and the independence hypothesis.

Independence in the hybrid formulas

Mike Rubinstein: Hybrid formulas for L-functions, use these formulas to do the moments, assume the truncated Euler product and the truncated Hadamard product are independent in the sense that when you do the moments you can do each piece separately. Only gives the leading term but not the lower terms. How independent are they (statistical tests)? The fact that it gives the moments is a confirmation. Can we attach a statistical value to how independent on the two factors.

Jon Keating: independence is proved for the first few moments. Can prove it splits because we know what the moments are.

Mike R: doesn't mean it is independent, just an identity consistent with being independent. Maybe try and correct for the fact that they are not completely independent, maybe would help with the previous question.

Brian C: use the hybrid model to find lower order terms.

Constructing certain Siegel modular forms

Brian Conrey: Construct Siegel modular forms of degree 2 and weight 2 or 3 whose spinor L-function is primitive (means doesn't split into a product of lower L-functions).

Audience and Conrey: Upper bound for dimension computed for these spaces up to 23 and we have enough examples to exhaust (???). In principle can produce all the examples and test. As soon as given a bound can try to meet it. David F: for which levels do we know the space of cusp forms? Answer: don't have exact formula, have upper bound, generate lifts (more or less experimentally), as soon as have as many linearly independent lifts as allowed we are done. In some cases there are known formulas ($\Gamma_0(p)$ and higher weights, starting at weight 5). Hope is for p large. Conrey: what if someone told you one exists for $p = 101$: can you find it? Answer: don't know, think would be hard. David F: If it factors, as it's degree 4 it factors into either two of degree 2 or one of degree 1 and one of degree 3. Know bounds of conductors: test whether or not it factors into degree 1 or degree 2 (know candidates, check by brute force).

Special values of quadratic twists D of Rankin-Selberg on $\mathrm{GL}_2 \times \mathrm{GL}_2$

Brian Conrey: This is another degree 4 thing. This could possibly produce a bunch of degree 4 things that vanish.

Mark Watkins: have with symmetric powers. Doesn't seem to match with RMT. It seems that the ones for which RMT produces more twists that vanish actually have less twists that vanish.

Abelian surfaces

Henri Darmon (and audience and Brian Conrey): Look at twists by Dirihclet characters of order ℓ (a prime) of the L-function of an abelian surface. DFK (David, Fearnley and Kisilevsky) conjectures for elliptic curves that $\ell \geq 7$ implies only finitely many twists vanish. What about abelian surfaces?

Brian Conrey (and audience): need to know size of L-function, size of D in special value. Degree 4 thing. Are these standard L-functions? Are they primitive? Can compute the value of the L-function, if small enough declare it to be zero.

Henri Darmon: when twist pick up a D^2?

Brian Conrey: weight k get $1/|D|^{(k-1)/2}$. This goes to $1/|D|^{(k-1)/4}$ when take the square root. If $\frac{k-1}{4} \leq 1$ expect infinitely many non-trivial vanishings, and if $\frac{k-1}{4} > 1$ then only finitely many. If can answer the question as to what the power of D, would lead to the conjecture.

Miscellaneous questions

- Can one predict how often one will get vanishing of a triple product L-function?
- If one looks at an elliptic curve over say a real quadratic number field, can one effectively do a discretization and give an RMT model which predicts the frequency of vanishing? Or does the fact that there are very small integers prevent this?
- Can one predict the frequency with which various finite groups appear when studying the group of points modulo a prime p in the family of quadratic twists of a fixed elliptic curve?
- For a fixed cusp form of weight 6, we expect that for only finitely many quadratic twists the L-function will vanish. What if one varies the cusp form? How large a finite number can one get?

Results at the workshop

Interestingly there were a few real-time developments that occurred at the workshop.

One is that we carried out an experiment to help us understand the asymptotic number of elliptic curves with even positive rank in the family if quadratic twists of a given elliptic curve. We had previously understood that Delaunay's heuristics for Tate-Shafarevich groups are relevant for this problem, yet some subtle behaviour seems to complicate the answer. This depends on certain exceptional primes, and, at the workshop, we formulated and numerically tested a hypothesis that these exceptional primes are governed by classical Cohen-Lenstra heuristics for class groups. This gave an important step towards fully understanding the asymptotics.

Another is that some of the computational experts at the workshop (Tornaria, Watkins, and Rubinstein) figured out how to do some computations of interest to David, Fearnley, and Kisilevsky and generated lots of data for their experiments that at first seemed to be beyond reach.

Conclusion

The main achievement of the workshop was to bring to the forefront many new opportunities for interactions between RMT and number theory. There is now a motivation to develop algorithms for counting and computing L-functions from higher rank groups, especially for the L-functions for Siegel modular forms. Certainly many more questions were raised than were solved. This bodes well for the future of the subject.

List of Participants

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Chapter 23

Quadrature Domains and Laplacian Growth in Modern Physics (07w5008)

Jul 15 - Jul 20, 2007

Organizer(s): Darren Crowdy (Imperial College London), Bjorn Gustafsson (Royal Institute of Technology, Stockholm), Mark Mineev (Los Alamos National Laboratory), Mihai Putinar (University of California at Santa Barbara)

Introduction. A great many physical processes involving moving boundaries can be reduced, after various idealizations, to the so-called Hele-Shaw problem. These are also known in the modern physics literature as Laplacian growth processes since the field equation governing is Laplace's equation and the subsequent interface motion is given by some surface derivatives of this field. Numerous physical phenomena that fall into this category: they include (but are not limited to) solidification processes [24], electrodeposition [10], viscous fingering [4], bacterial growth and the modelling of cancer cells [3]. The meeting in Banff in June 2007 focussed on the investigation of the dynamics and growth of unstable interfaces that appear as a result of such growth processes.

This field of research arguably originated in the 1940s [31, 12] and has attracted a great deal of attention recently in both the mathematical and physical communities due to newly found connections and applications to areas of classical physics and mathematical physics. These include integrable models, 2-dimensional quantum gravity and matrix models, the dynamics of quantum hall droplets, transport of 1-dimensional fermions, propagation of crystallization fronts and lightning propagation. The list of applications is clearly very wide-ranging.

The common mathematical background for these developments is the study of dynamics of the interface growth known as the Laplacian growth. The simplest model for this process is the displacement of the viscous fluid (say, oil) by a non-viscous one (referred to as *air* or, *water*, between two closely spaced horizontal plates (the Hele-Shaw cell). Due to the high viscosity of the oil, its incompressibility and the assumption that the flow takes place between two closely-spaced glass plates, the flow is governed by Darcy's law and the normal velocity of the interface is proportional to the normal gradient of the pressure. The zero viscosity of the air allows one to assume that the pressure is constant across the air domain.

If an interface develops a bump at some point in time the pressure gradient on that bump will be much larger than on the remaining "flatter" parts of the boundary of the interface. The bump therefore grows faster and will be quickly amplified during the growth process. This process is thus very unstable and can produce asymptotic shapes such as a "finger" [35], [5] or, within a discrete model, a fractal with characteristics similar to those obtained in Diffusion-Limited Aggregation processes, cf., e.g., [24, 10, 3]. The study of this growth phenomenon is currently enjoying a resurgence.

The mathematical model governing these and many other similar processes reduces to the following "simple" equation for the moving boundaries:

$$V(\xi) = \partial_n G_{D(t)}(\xi, a). \tag{23.1}$$

Here V is the normal component of the velocity of the moving boundaries $\partial D(t)$ of the time dependent domains $D(t) \subset \mathbf{R}^d$, $\xi \in \partial D(t)$, t is time, ∂_n is the normal component of the gradient, and $G_{D(t)}(\xi, a)$ is the Green function of the domain $D(t)$ for the Laplace operator with a unit source located at the point $a \in D(t)$.

Up to this point, the most rewarding theory from the applications perspective has been developed in two dimensions (2D). In two dimensions, the above equation can be rewritten as the area-preserving diffeomorphism identity

$$\Im\left(\bar{z}_t z_\phi\right) = 1, \tag{23.2}$$

where $z(t, \phi) := \partial D(t)$ is the moving boundary parameterized by $\phi \in [0,\, 2\pi]$ and conformal when analytically extended into the region $\Im\phi := \operatorname{Im}\phi \leq 0$ [12, 31]. The equation (23.2) possesses many remarkable properties among which the most noticeable one is the existence of an infinite set of conservation laws:

$$C_n(t) = \int_{D(t)} z^n \, dx \, dy = C_n(0), \tag{23.3}$$

where n runs over all non-negative [32] (non-positive [27]) integers in the case of a finite (infinite) domain $D(t)$, and an impressive list of exact time-dependent closed form solutions [37]. For a rather pleasing interpretation of conserved quantities C_n as coefficients of the multi-pole expansion of the fictitious Newtonian potential induced by the matter uniformly occupying the domain $D(t)$ see, e.g., [37].

It was established in [29] that the interface dynamics described by (23.2) is equivalent to the dispersionless integrable 2D Toda hierarchy [38], constrained by a so-called string equation. Remarkably, this hierarchy, being one of the richest existing integrable structures, describes an existing theory of 2D quantum gravity (see the comprehensive review [38] and references therein). The paper [29] generated a great deal of activity in apparently different mathematical and physical directions revealing profound connections between Laplacian growth and random matrices [20], the Whitham theory [21], and quadrature domains [30], [13, 11].

In short, Laplacian growth encapsulates a remarkable interconnection between mathematics, physics, and engineering. This means that any noticeable advance in one of these three branches of the subject often (if not always) produces subsequent discoveries in another. The BIRS meeting of 2007 gathered experts from all these fields, and this has reflected the highly interdisciplinary nature of the subject. The interaction among the participants was intense and the results, outlined below, are impressive.

The following publications reflect, if only partially, recent progress in the field obtained in a few fresh collaborations started at the BIRS 2007 workshop: [1, 2, 7, 14, 18, 25], see also the volume [6] and the survey [30].

1. Potential theory and Riemann surfaces. Recall that Laplacian growth is an interface dynamics where the boundary velocity equals the normal derivative of the Green function of the moving domain. Remarkably, this non-linear complex dynamics with infinitely many degrees of freedom possesses a complete set of conserved quantities (namely, the Richardson harmonic moments). Consequently, wide classes of *generalized quadrature domains* are preserved during the evolution. This implies in particular that so-called algebraic domains ("classical quadrature domains") remain algebraic.

A classical quadrature domain is an open subset Ω of the complex plane, which satisfies the quadrature identity

$$\int_\Omega f \mathrm{dArea} = u(f), \quad f \in L^1(\Omega, \mathrm{dArea}),$$

and u is a distribution with finite support (contained in Ω). By a doubling procedure such domains can be identified with a class of symmetric Riemann surfaces. Thus, Laplacian growth corresponds to certain kinds of dynamics of Riemann surfaces or, equivalently, of algebraic curves. A notable link between classical function theory on algebraic curves, elimination theory and quadrature domains was discovered in the article [14].

It is worth mentioning that an exact reconstruction algorithm of quadrature domains from finitely many power moments, or equivalent data, exists, see [11].

Much less understood is the "negative" Laplacian growth, under which the bounded domain either shrinks down to a potential theoretic skeleton of its original configuration, or breaks down due to singularity development on the interface. This process has a common fluid dynamics interpretation, namely a water/oil interface motion in a Hele-Shaw cell, where a fingering instability discovered by P. G. Saffman and G. I. Taylor in 1958, occurs [33]. Despite many efforts during subsequent years there are still unanswered questions in formulating a complete mathematical theory of it and quite a few contributions to the workshop were aimed in this direction.

2. Elliptic growth and the Beltrami operator. The occurence of the Laplace operator in mentioned above physical processes stems from the continuity and incompressibility conditions satisfied by a fluid involved in the potential flow. Specifically $\mathbf{v} = -\lambda \nabla p$, where $\mathbf{v}, \lambda$, and p are the fluid velocity vector field, conductivity, and scalar velocity potential, respectively:

$$\nabla \cdot \mathbf{v} = -\nabla \cdot \lambda \nabla p = -\lambda \nabla^2 p = 0.$$

As a first approximation, the conductivity λ was supposed to be constant, while generally it is not; therefore the major equation for growth has to be reconsidered as $\nabla \cdot \lambda(\mathbf{x}) \nabla p = 0$.

In [18] the authors present a natural extension of the Laplacian growth, where the Green function of $D(t)$ for the Laplace operator ∇^2 is replaced by the Green function of a linear elliptic operator,

$$L = \nabla \cdot (\lambda(\mathbf{x}) \nabla) - u(\mathbf{x}), \qquad \lambda(\mathbf{x}) > 0, \qquad \mathbf{x} \in \mathbf{R}^d. \tag{23.4}$$

Such a process, which was naturally cristened *an elliptic growth*, is clearly much more common in physics than the Laplacian growth. Consider, for instance, viscous fingering between viscous and inviscid fluids in the porous media governed by Darcy's law

$$\mathbf{v} = -\lambda \nabla p, \tag{23.5}$$

where λ is the filtration coefficient of the media and p is the pressure (equal to the Green function, $G_{D(t)}$). One can easily imagine a non-homogeneous media where the filtration coefficient λ is space-dependent. Such examples of elliptic growth, where the elliptic operator L has the form of the Laplace-Beltrami operator, $L = \nabla \cdot \lambda \nabla$, and λ is a prescribed function of $\mathbf{x}$, are called an elliptic growth *of the Beltrami type*. It is clear that all moving boundary problems other than viscous fingering with a non-homogeneous kinetic coefficient λ fall into this category.

From a mathematical point of view this process is the Laplacian growth occurring on curved surfaces instead of the Euclidean plane. In this case the Laplace equation is naturally replaced by the Laplace-Beltrami equation, and λ (that can also be a matrix instead of a scalar) is related to the metric tensor. There are several works addressing the Hele-Shaw problem on curved surfaces.

Another major source of examples of elliptic growth is related to screening effects, when $u \neq 0$, while λ is constant in (23.4). The simplest example of this kind is an electrodeposition, where the field p is the electrostatic potential of the electrolyte. It is known that in reality electrolyte ions are always locally surrounded by a cloud of oppositely charged ions. This screening modifies the Laplace equation for the electrostatic potential by adding to the Laplace operator the negative screening term, $-u(x)$, that stands for the inverse square of the radius of the Debye-Hukkel screening in the classical plasma. For the homogeneous screening u is a (positive) constant, so the operator L becomes the Helmholtz operator, while for the non-homogeneous case, when u is not a constant, L is a standard Schrödinger operator. Motivated by this example, the moving boundary problem for $L = \nabla^2 - u$ may be called an elliptic growth of *Schrödinger type*.

As was shown in [18] these rather general types of elliptic growth still retain remarkable mathematical properties, similar to those possessed by the Laplacian growth. A mixed case with a non-constant λ and non-zero u also shares similar properties but is less representative in physics and can always be reduced to one of the two former types of elliptic growth by a simple transformation.

It must be mentioned that in few prior works on elliptic growth an infinite number of conservation laws, regarded as extensions of (23.3), were identified as well, cf. [37, 28]. An integrable example in 2D, corresponding however to a very special choice of the conductivity function, $\lambda(\mathbf{x})$, was explicitly constructed in [25]. The elliptic growth there essentially reduces to the well-known Calogero-Moser integrable system.

3. Stochastic analysis and fractal growth. Very recently, an *integrable model* for the stochastic Laplacian growth with finite-size deposited particles within the framework of so-called *Loewner chains* was developed. As a consequence, it is expected to recover universal geometric characteristics, such as the multifractal spectrum of the growing clusters. Notably, this process retains integrability, despite its randomness.

In another important work combining stochasticity with integrability [16], the authors have obtained a list of surprising results connecting random entities and the tau-function - a powerful concept in the theory of integrable systems. Taken together with the above mentioned results in Laplacian growth, such nontrivial interconnections between integrability and randomness provide a constant source of new ideas.

4. Laplacian growth as a large N limit of random matrices spectra. Some deep links between the stochastic Laplacian growth and the theory of random matrices are discussed in the survey [30]. As it is often the case in other applications of random matrices, this connection sheds new light on older classical problems. To give a single example, an important observation was made in [19] and developed in [36] (see also review [39]) that the Laplacian growth can be simulated by the evolution of an averaged spectrum of normal random matrices as a function of a re-scaled size of matrices from the statistical ensemble, when the size of a matrix, N, goes to infinity.

The main observation is surprising: the evolution of the support of the eigenvalues can be treated as the Laplacian growth of the domain. Namely, it behaves exactly as an air bubble in the Hele-Shaw cell (with zero surface tension). Considering random matrices and the more general 'beta-ensembles' with the probability measure

$$\prod_{j<k}^{N} |z_j - z_k|^{2\beta} \prod_{l=1}^{N} d\mu(x_l, y_l),$$

(here $d\mu(x, y)$ is a smooth measure in the plane), a natural framework is calling to be developed, with the aim at solving the stochastic version of Laplacian growth. Clearly, the large N approximation is by no means enough for this purpose, so one should properly take into account $1/N$-corrections and understand the structure of the whole $1/N$-expansion. This direction is now under intense development, and it can be fair to state that it has started at the BIRS 2007 workshop.

5. Complex orthogonal polynomials. As mentioned above, the (renormalized) eigenvalues of ensembles of random normal matrices constrained by simple external field potentials are known, in the limit as the size of the matrix tends to infinity, to occupy regions that are generalized quadrature domains. In this way, the methods of statistical physics intersect with ideas from function theory and approximation theory with very surprising results (cf. [1, 30]). In particular, it has been proved that the geometry of the limiting domain, encoded in its Schwarz function, determines the cluster of zeros of some canonically associated complex orthogonal polynomials. The resulting potential theoretic skeleton of the limiting domain remains quite mysterious, and it is currently under intense investigation by a number of researchers.

6. Applications to classical physics. The same mathematics of Laplacian growth, involving conformal mapping theory, analytical/numerical uniformization and function theory on compact Riemann surfaces, also arises in a rich array of quite separate problems in classical physics: in fluid mechanics, for example, it arises in the study of free surface Stokes flows and in vortical solutions of the Euler equations. A review of many different physical problems, all arising just within the field of classical fluid dynamics, where quadrature domains arise has recently been compiled [9]. Most recently, Laplacian growth models have been found to be relevant to describing ionization processes in electrical streamers [26] – a physical problem where Maxwell's equations govern the physics. Such cross-disciplinary applications of the mathematics of Laplacian growth are many and varied and new instances are continually being uncovered.

7. Numerical simulations and industrial applications. It is well-known that, in the continuous Laplacian growth problem, the initial value problem can, under certain conditions, be ill-posed. Owing to this ill-posedness, when small regularization effects *are* included – for example, by including surface tension effects – any numerical method for resolving the subsequent dynamics encounters a variety of challenges and much research has gone into resolving these numerical issues over recent years. Other challenging mathematical problems arise in the asymptotic analysis of such problems. For example, a long-standing problem that was eventually solved in the 1980's involved the *selection mechanism* for the Saffman-Taylor viscous fingering problem. The solution to this problem gave birth to a new area of asymptotic analysis now known as *asymptotics beyond all orders* [34] owing to the role of exponentially small terms in picking out allowable solutions. Many challenges in both the numerical and asymptotic analysis of Laplacian growth problems remain and are the subject of ongoing work.

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Chapter 24

Mentoring for Engineering Academia II (07w5030)

Jul 22 - Jul 27, 2007

Organizer(s): Robert Gray (Stanford University), Sheila Hemami (Cornell University), Eve Riskin (University of Washington), Rabab Ward (University of British Columbia)

Preface

The presentations and discussions of the workshop were distilled into a softcover book [1] providing details, supporting information, resources, and a large bibliography. In this report the introduction and overview is extracted along with the table of contents of the book to provide a description of the workshop and a summary of the topics treated.

Introduction and Overview

The percentage of women holding academic positions in most branches of engineering and related fields in mathematics and science is far below their representation in the overall population. Often a department of 60-70 professors will have only one or two women, and there may be none in smaller departments. While the numbers and the environment have improved during the past century in some professions, such as biology and law, with a few exceptions engineering has remained well behind in the gender and ethnic diversity of its faculty and academic administration. The situation is even worse for underrepresented minorities. This lack of diversity means that the available pool for both academic and nonacademic jobs in high technology areas important to industry and government is significantly lower than it should and could be. The problem tends to propagate itself because the number of role models and mentors able to help new generations of students is limited. While increasingly government and academia have encouraged diversity, the results have, unfortunately, often amounted to little more than lip service with little demonstrable improvement in the actual numbers. There is also sometimes a backlash against efforts to increase diversity: search committees for engineering faculty might not buy into the goal of diversifying the pool, and female candidates who do make it to campus for an interview are met with suspicion — the assumption is that the invitation is based on pressure rather than qualifications. A familiar litany of excuses has often been put forward to explain the failure of institutions to make genuine progress, including public statements by officials that reinforce the myths regarding the effects of gender and ethnicity on mathematical and technical abilities. Happily some institutions have made progress, and even small numbers of sympathetic faculty and administrators have made a notable impact on the recruiting, hiring, mentoring, and advancing of women and underrepresented minorities in engineering faculties; these few have in turn had an impact on the larger student population.

The reasons and catalysts for such successful changes are not generally known and are often highly specific to the institutions and people involved. Furthermore, many of the problems have stubbornly resisted solution. However, the pooling of experience and ideas can contribute to their eventual resolution.

During 22– 27 July 2007 a workshop was held at the Banff International Research Station (BIRS) in Alberta, Canada, to bring together students, faculty, and representatives of academic administration to collect, invent, discuss, develop, and document ideas on how individuals and groups within academic environments can effectively promote awareness and progress regarding mentoring underrepresented groups in their institutions in general, and among their students and colleagues in particular. The workshop was loosely modeled on the successful workshop "Mentoring for Engineering Academia," held in June 2004 at Stanford University with the support of a Presidential Award for Excellence in Science, Mathematics, and Engineering Mentoring (PAESMEM) — a program sponsored by the White House and administered by the US National Science Foundation — and the Stanford University School of Engineering. That workshop brought together 70 students, faculty, and administrators for presentations, panels, and discussions on mentoring graduate students and both junior and mid-career faculty and provided an enthusiastic and stimulating exchange of ideas. Proceedings were produced in both a Web format — available along with the presentation slides at `http://paesmem.stanford.edu/` — and a paperback book, which was distributed to participants and more widely through the US NSF and by request (copies are available while they last from `rmgray@stanford.edu`.

The BIRS workshop was envisioned as a followup to the original PAESMEM/Stanford workshop and had an organizing committee with a strong overlap with the original workshop. However, the structure, specific topics treated, and location changed. Topics treated in the original workshop and revisited at BIRS included the following:

- Fundamentals of mentoring for engineering academia
  - Graduate students
  - Junior faculty
  - Post-tenure faculty
- Mentoring for academic leadership
- Faculty and family: strategies for time-sharing profession and parenting

New topics considered at the BIRS workshop included

- Mentoring for undergraduate students and for non-tenure track faculty (both demographics were represented in the participants)
- Promoting fairness and openness in search committees
- Measuring sustainable progress for women in academia
- Outreach
- Building a mentoring system from the ground up: a one-stop shopping guide for deans/chairs to put things in place.

Each of these topics became the focus of a session at the workshop and forms a chapter in the workshop proceedings [1], which may downloaded as a pdf from `http://birs07.stanford.edu/`. Hard copies of the softcover book may be requested from `rmgray@stanford.edu` (while they last). The *Proceedings* include detailed articles distilled from the presentations and discussions along with supporting information, resources, and photos of the Banff area taken by and of the participants. The slides used by the presenters may also be found at the website.

The BIRS workshop shared the two primary goals of the earlier workshop:

1. helping the participants be more effective both locally and globally in improving the environment and diversity of faculty in engineering and related disciplines, and

2. producing a book distilling the ideas generated at the workshop that will be useful to colleagues as well as to review panels and visiting committees charged with evaluating institutional progress and recommending potential improvements.

The relevance and importance of this initiative are amply illustrated by the pressing national need for trained technical talent and the implicit need for enlarging the pool of trained, talented members of the profession. The issues will remain timely until the population in the engineering professions better reflects the population in general, as has happened in biology and law.

The primary objective of the workshop was the development and documentation of ideas on how to mentor students, colleagues, and academic administrators on issues relating to academic careers in engineering and related disciplines, with an emphasis on issues relavent to women faculty in electrical engineering and computer science. Many forms of mentoring were considered, including

- mentoring of students on pursuing a successful academic career involving teaching, research, and leadership
- mentoring academic colleagues on
  1. seeking genuinely open and fair searches that actively seek and recruit a wide diversity of applicants
  2. working for a supportive and cooperative environment in which junior faculty can thrive and advance
  3. helping recently tenured mid-career faculty plan the next stages of their career, and
  4. encouraging and assisting junior and mid-career faculty to consider roles in academic administration
- mentoring academic administrators on providing adequate support for individual students and student organizations
- two-way mentoring, explicitly encouraging senior faculty to incorporate feedback from their protégés regarding the effectiveness of their mentoring skills and the accuracy of their perception of their protégé's environment.

Most sessions began with brief presentations followed by discussion, a break, and more discussion. These proceedings represent a distillation of the presentations, the slides used by presenters, notes taken by scribes, and audio recordings of the sessions. The editors comprise the workshop organizers along with others who assumed responsibility for drafting the chapters for each topic. The proceedings were produced by iteratively revising the original draft using comments, corrections, and suggestions from the presenters and other participants.

Following the workshop, two student participants — Lydia Contreras and Jamie Walls — put together a survey as a means of gathering feedback from the participants in order to formally evaluate the impact of the workshop upon them.

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Chapter 25

Topological and Geometric Rigidity (07w5094)

Jul 29 - Aug 03, 2007

Organizer(s): Jim Davis (Indiana University), Shmuel Weinberger (University of Chicago)

Introduction

This conference discussed a wide variety of problems motivated directly and indirectly by Mostow rigidity and a wide variety of approaches to these problems using tools from geometry, algebra, quantitative topology, and index theory. We shall give here a brief survey of these problems by mentioning at the appropriate places where talks in this conference fit in providing context for the work of the conference and an indication of that work. In particular, these notes are variants of the talks that Davis and Weinberger gave at the meeting.

At the end of each section we will give a few references for additional information. We will often reference a survey rather than an original source. Also, our mention of the topic discussed by a speaker is not intended to indicate that it was not an exposition of joint work.

Mostow rigidity (an example of geometric rigidity)

It appropriate to begin our survey with Mostow Rigidity:

Theorem 1 *Suppose M and N are closed hyperbolic manifolds of dimension $n > 2$, and $f : \pi_1 M \to \pi_1 N$ is an isomorphism, then there is a unique isometry $F : M \to N$, inducing f.*

For $n = 2$ this is false, with there being a contractible space of hyperbolic structures on any surface, Teichmuller space. The theorem is true for noncompact manifolds with finite volume (Prasad) and for irreducible lattices in semisimple Lie groups (Mostow). Later we will discuss Margulis's superrigidity which is a far-reaching extension of Mostow's theorem.

Without semisimplicity, isometry is too much to hope for: indeed even the torus has a large, yet contractible, space of flat structures SL_n/SO_n. Nevertheless, even this can be thought of as a form of rigidity (and the nonpositive curvature of that space does indeed give rise to a number of important applications.)

Belegradek's talk at the conference, among other things, give more examples of these kinds of classical rigidity phenomena.

References for this section include: [31] and [34].

Borel conjecture (an example of topological rigidity)

Borel, in a letter to Serre, after hearing Mostow talk about rigidity (actually in the solvable setting where a smooth, rather than geometric rigidity, occurred) made the following far-reaching conjecture:

Conjecture 2 (Borel conjecture) *Suppose M and N are closed aspherical manifolds, and $f : \pi_1 M \to \pi_1 N$ is an isomorphism, then there is a homeomorphism $F : M \to N$, inducing f.*

Borel singled out asphericality, that is, having contractible universal cover, as the appropriate topological analog of being hyperbolic (or a lattice). Being aspherical already leads to a homotopy rigidity: any two aspherical CW complexes with isomorphic fundamental groups are homotopy equivalent.

At the time, the difference between the smooth and topological categories was not yet understood, but this conjecture is definitely at the core of topological topology rather than differential topology, despite its smooth origin.

We will only discuss the situation of manifolds of dimension > 4; the low dimensional problem is closely tied to the geometrization conjecture in dimension 3, and topological surgery in dimension 4: subjects which we not the focus of this workshop.

The first progress on this conjecture was its proof by Hsiang–Shaneson and Wall for tori using Farrell's thesis[1]. Wall pointed out that essentially the same argument proves it for Poly-**Z** groups. Using Cappell's splitting theorem in place of Farrell's thesis one can handle many other manifolds inductively (such as surfaces, and many Haken 3-manifolds) but this approach definitely fails for any group with property T (explained below) and therefore, in particular, any higher rank lattice in a simple Lie group.

A real change in perspective came with the work of Farrell and Hsiang on flat manifolds that used a mix of algebraic techniques with ideas of controlled topology, most notably the α–approximation theorem of Chapman and Ferry. The subsequent revolutionary work of Farrell and Jones introduced foliated control in place of metric control and used dynamics to prove many results related to the Borel conjecture, for example proving the Borel Conjecture for a class of groups which includes more groups that the original work of Mostow. (Note that the Borel Conjecture applies in the more general context of topological rigidity while Mostow rigidity gives strongly results in a more restricted setting and is geometric rigiditiy.)

The talks by Bartels and Reich at the conference described excellent recent progress on the Borel conjecture for hyperbolic groups.

References for this section include: [12], [13], [15], [16], [36], and [43].

Novikov's conjecture (and theorem)

Another important problem that enters this mix is a conjecture of Novikov's. The original statement of the problem is quite simple. (See the references for several surveys; there are related surveys of the Baum–Connes conjecture.)

Recall first the Hirzebruch signature theorem:

Theorem 3 *Let M^{4k} be a closed oriented smooth manifold, and let $\mathrm{sign}(M)$ denote the signature of its cup product form on H^{2k}. There are explicit polynomials in the Pontrjagin classes of M, denoted by $L_k(M)$, so that*

$$\mathrm{sign}(M) = \langle L_k(M), [M] \rangle .$$

Noticing that the Hirzebruch formula gives an example of a homotopy invariant combination of Pontrjagin classes, Novikov suggested that for nonsimply connected manifolds there might be additional homotopy invariant classes.

Conjecture 4 (Novikov conjecture) *If $f : M \to B\pi$ is a map and $\alpha \in H^r(B\pi; \mathbf{Q})$, then $\langle f^*(\alpha) \cup L(M), [M] \rangle$ is a homotopy invariant.*

[1] Khan's talk at this meeting explained some low dimensional variants of Farrell's fibering theorem.

Here $L = 1 + L_1 + L_2 + L_3 + \cdots$.

Nowadays it is common to Poincaré dualize and phrase the problem in terms of other homology theories. This will be discussed in the next section.

Novikov himself used the homotopy invariance properties for simply connected manifolds to lay the foundations of the topological category, by showing the Pontrjagin classes are rationally independent of the smooth structure of the defining manifold. From a modern point of view, Novikov's theorem can be viewed as a metric analogue of his conjecture (see [19]).

References for this section include: [9], [17], [19] and [11].

Reformulation

Surgery theory enables one to reformulate the Borel conjecture algebraically and connect it to the Novikov conjecture. If one believes the Borel conjecture in full generality, then it actually follows that one is committed to believing it for aspherical manifolds with boundary, if one deals with maps of pairs that are already homeomorphisms on the boundary. (The most direct way to see this uses M. Davis's reflection group method.) That version then allows us, by thickening any finite-dimensional $B\pi$ in a high dimensional Euclidean space, enables us to discuss "the Borel conjecture for the group π" – that a more general class of groups than the fundamental groups of closed aspherical manifolds. In fact, below, we extend the Borel conjecture to a more general class yet, that of torsion-free groups.

Note for example that if the Borel conjecture were to be true for an aspherical manifold V, and for $V \times [0,1]$, then the Whitehead group of $\pi_1(V)$ would have to vanish.

Conjecture 5 *If π is torsion free, then $Wh(\pi) = 0$.*

This relates to the geometric problem only for countable groups of finite cohomological dimension (or, if we insist compactness, of type FP). Nevertheless, the above is compatible with all of the evidence and with the conjectures of Farrell and Jones discussed in section 25.

Conjecture 6 *If π is torsion free, then the assembly map*

$$A : H_n(B\pi; \mathbf{L}(e)) \longrightarrow L_n(\pi)$$

is an isomorphism.

This map arises in the modern (post–Wall) formulation of surgery theory as the critical ingredient arising in the classification of manifolds simple homotopy equivalent to a given one. If one tensors with $\mathbf{Q}$, then the left hand side becomes $\oplus H_{n-4i}(B\pi; \mathbf{Q})$. The injectivity of this map then is a restatement of the Novikov conjecture for manifolds:

Conjecture 7 *If π is a group, then the assembly map*

$$A : H_n(B\pi; \mathbf{L}(e)) \longrightarrow L_n(\pi)$$

is an injection after tensoring with $\mathbf{Q}$.

Below we will discuss statements for all groups that include both the conjectures of Novikov and Borel. However, for the meantime, we shall also state an analogue of this conjecture in the setting of C^*-algebras, that was initially introduced by Mischenko and Kasparov as a tool for the Novikov conjecture. It is sometimes called the (strong) Novikov conjecture (although its integral form does not imply the integral form of the Novikov conjecture, e.g. that for torsion free groups, the assembly map is injective).

Conjecture 8 *If π is a group, then the assembly map*

$$A : K_n(B\pi) \longrightarrow K_n(C^*\pi)$$

is an injection after tensoring with $\mathbf{Q}$..

(If π is torsion free, then this map is conjectured to be an isomorphism, if one is careful to use the reduced C^* algebra. This is the torsion free case of the Baum–Connes conjecture.)

References for this section include: [9], [11], [22], [35], [36], and [44].

Other operators

The main advantage of topological formulations of the problem is that the problem (and the variants that it generates) are well adapted to a range of topological problems. The analytic formulation has the advantage that it is well adapted to other elliptic operators.

One of these is the following:

Conjecture 9 (Gromov-Lawson Conjecture) [2] *If M is a Riemannian manifold with positive scalar curvature whose universal cover is spin, then for any map $f : M \to B\pi$ and $\alpha \in H^r(B\pi; \mathbf{Q})$, one has $\langle f^*(\alpha) \cup \hat{A}(M), [M] \rangle = 0$.*

In particular, no aspherical manifold should admit a metric of positive scalar curvature.

Here $\hat{A}$ is the $\hat{A}$-genus studied by Borel and Hirzebruch, and identified with the index of the Dirac operator by Atiyah and Singer. As a conjecture, it has the same relation to the theorem of Atiyah–Lichnerowicz–Singer on positive scalar curvature, as the Novikov conjecture has to the Hirzebruch Signature theorem.

It follows, as was proved by Rosenberg, from the strong Novikov conjecture. We leave its survey to other writers. We remark, though, that the ideas involved in the study of the positive scalar curvature problem have had important influence on the study of the Novikov conjecture as well.

Rosenberg had also suggested that the classical birational invariance of the Todd class, should have a higher analogue involving the fundamental group. This is true (not merely conjectural) as was proven by several groups of researchers[3].

The algebra–geometric side of this area appeared in the talks of both Block and Cappell where ideas of non-commutative algebraic geometry were applied to complex manifolds which are not varieties, and maps between varieties that are far from birational were studied respectively. Roe's talk developed the analogy between surgery theory and invertible elliptic operators.

References for this section include: [5], [38], [39], and [40].

The L^2-index theorem

One of the ideas implicit in the previous section is the use of the universal cover as a tool in understanding the topology and geometry of a compact manifold. This is indeed one of the important ideas in most proofs of the Novikov conjecture.

A tool in this development is Atiyah's index theorem. It takes off from the multiplicativity of the index of an elliptic operator under finite sheeted covers. Thus, the "G-index" for a finite group acting freely on (M, D), meaning ind$(D)/|G|$, is the same as ind$(M/G, D/G)$. Atiyah defined, using von Neumann traces, the "G-index" even if G is an infinite group acting only properly discontinuously. If the action is free, he showed the expected equality. (Note that in the non-free case, even for finite groups, there are contributions from the fixed point sets of elements of G in a formula for the G-index.)

It is also possible to define purely topological invariants by this method—which gives the L^2-betti numbers. Note, for example, that M has finite fundamental group G, then its 0^{th} (and top) L^2-betti number is (are) $1/|G|$. If G is infinite then this number is 0. In any case, it is easy to give examples where these numbers have fractional part when G contains torsion[4], but as far as we know, this is impossible if G is torsion free. This impossibility is called the Atiyah conjecture—although Atiyah had only asked this as a question.

The vanishing of Whitehead groups in the torsion free case hypothesized in the previous section has a spiritual connection to the idea that $\mathbf{Z}\pi$ has only the obvious units $\pm\pi$ and no zero divisors if π is torsion free. For the precise connection between these conjectures and the Atiyah conjecture and the Baum–Connes conjecture, we recommend Lück's book.

At the conference, Linnell gave a talk showing extremely strong results on denominators present in L^2-betti numbers for linear groups. Sauer gave new methods via measurable equivalence relations and more sophisticated algebra to compute L^2-betti numbers.

References for this section include: [1], [8] and [30].

[2] There is a stronger conjecture, made by Rosenberg, that includes integral information and a converse as well.

[3] Block–Weinberger, Brasselet–Schuermann–Yokura and Borisov–Libgober.

[4] It is not known whether these numbers are always rational. There are no examples where an *index* is irrational by an orbifold index theorem.

Groups with torsion

It is time to discuss groups with torsion. A geometric way to generalize the Borel conjecture is to assert the rigidity of e.g. hyperbolic orbifolds. In the differential geometric setting there is no difficulty in extending the theorem from manifolds to orbifolds. (Note that the isometry in Mostow's theorem is unique, so it is automatically equivariant with respect to any group of isometries of domain and range that induce the same outer automorphisms of fundamental groups.)

However, in the topological setting the presence of singularities in the orbifold does change the situation a great deal. Indeed there are counterexamples to the Equivariant Borel conjecture from different sources (see Weinberger's book referred to above); we will discuss some presently.

Note that in the geometric setting, singularities are all aspherical manifolds, so that inductively, assuming the ordinary Borel conjecture, it is reasonable to assume that our map is a homeomorphism on the lower strata. The relative structure set is (aside from algebraic K-theoretic decoration issues) the fiber of an assembly map that has a purely algebraic topological interpretation given by [18].

Generalizing the universal cover of $B\Gamma$, that corresponds to free actions, we now need to replace the role that hyperbolic space plays for discrete subgroups of $O(n,1)$. Let $\underline{E}\Gamma$ be the universal space for proper discontinuous Γ maps, i.e. the terminal equivariant homotopy type in the category of proper discontinuous Γ actions. It is characterized by having a contractible fixed sets for each finite subgroup of Γ and empty fixed set for each infinite subgroup of Γ. (For Γ torsion free, $\underline{E}\Gamma$ is merely $E\Gamma$.)

With this notation, then rigidity would boil down to an assembly map being an isomorphism:

$$H_n^{\Gamma}(\underline{E}\Gamma;\mathbf{L}) = H_n\left(\underline{E}\Gamma/\Gamma;\mathbf{L}(\Gamma_x)\right) \longrightarrow L_n(\Gamma).$$

A similar description can be given for the relative equivariant Whitehead group using the assembly map in Algebraic K-theory. In that last case it has long been known (through its connection to "the fundamental theorem of algebraic K-theory") that Nil groups obstruct this being an isomorphism.

Cappell defined a similar UNil group that arises in surgery theory. He showed that the above conjecture is false even for the extraordinarily simple group $\mathbf{Z}_2 * \mathbf{Z}_2$ (this group acts properly discontinuously on $\mathbf{R}$. Of course it is too low dimensional to apply surgery to, but ultimately these examples do give rise to counterexamples to equivariant rigidity for some crystallographic orbifolds.)

At the meeting, the talks by Davis, Quinn, Ranicki, and Reich all gave new information about the calculation of Nil and UNil groups by a mixture of algebraic and geometric methods. Indeed, this area is, in our view, one of the most exciting areas of development.

If one changes coefficients, to say replace $\mathbf{Z}$ by $\mathbf{Q}$ (or another field) in the assembly maps, then the Nil and UNil terms would vanish.

The Farrell–Jones conjecture (discussed below) is an extraordinary conjecture that applies even when the Nil and UNil groups are nonzero. It (conjecturally) reduces the study of these groups to groups that act properly discontinuously on $\mathbf{R}$ — the virtually cyclic groups. This conjecture grew out of their successful attempts to prove the Borel conjecture for compact non-positively curved manifolds and invisible but crucial role that geodesics played in that work.

In operator theory, there is no analogue of these Nil and UNil groups. In that case, one simply obtains a conjectural isomorphism (the Baum-Connes conjecture),

$$K_n^{\Gamma}(\underline{E}\Gamma) \longrightarrow K_n(C^*\Gamma).$$

Of course, there are different subtleties here, for instance that we use the reduced C^* algebra (i.e. the completion of $\mathbf{C}\Gamma$ with respect to the action of Γ on $L^2\Gamma$ by the regular representation). Using the completion with respect to arbitrary unitary representations runs afoul of Kazhdan's property T — a subject that makes several appearances in our story and will therefore be introduced in the next section.

The talks by Ji and Rosenthal gave many cases where it is possible to prove the integral split injectivity of these assembly maps. Higson's was devoted to his deep work with Kasparov on the Baum–Connes conjecture for groups that act isometrically on Hilbert space, and the injectivity of assembly maps (due to Skandalis, Tu, and Yu) for groups that embed in Hilbert space.

References for this section include: [3],[4], [6], [18], and [42].

Property T (Silberman)

Kazhdan's property T is a profound representation theoretic property of groups that has had an extraordinary range of application. It asserts that the trivial representation is isolated in the space of all unitary representations of the group. Equivalently, it can be viewed as a fixed point property: any affine isometric action of the group on Hilbert space must have a fixed point (or that a unitary representation with "almost invariant" unit vectors has a fixed vector).

A quick connection between property T and the problems we are considering here is that it immediately shows that a group like $SL_n(\mathbf{Z})$ for $n > 2$ (or any lattice therein) cannot act without fixed points on any tree — and therefore the Novikov conjecture cannot be proven using "splitting" methods (i.e. the inductive method that covers the case of the torus).

Other connections are via super-rigidity (see below). Note that the theorem of Higson and Kasparov asserting the Baum–Connes conjecture requires the exact opposite of property T — the existence of a proper discontinuous action of the group on a Hilbert space. This leads to another, more recent connection.

Margulis had long ago observed that property T can be used to construct families of graphs, called expanders that are of great interest to computer scientists. Gromov observed that a sequence of expanders cannot be embedded in any Hilbert space with any reasonable distortion. (Technically, they cannot be uniformly embedded in Hilbert space.) Thus a group containing expanders is a candidate for counterexamples to the Baum–Connes conjecture, and it was indeed verified by several authors that such a group provides counterexamples to variants of the Baum-Connes conjecture. However the original Baum-Connes Conjecture still stands.

The construction of these groups is done by random methods. These groups have very strong and unusual properties; Silberman in his talk explained some of these fixed point properties and described the use of "heat flow" techniques on them. This led, subsequently to the conference, to a joint work with Fisher wherein torsion free groups that do not smoothly and volume preservingly on any compact manifold are produced.

References for this section include: [5], [21], [24] and [29].

Techniques

There are, by now, many approaches to the Novikov conjecture. We first mention two that apply equally well in operator theoretic and topological contexts. One is the method of descent wherein coarse properties of the group with its word metric are then coupled with a family argument (or equivalently a homotopy fixed set argument) to give a result about the group as an algebraic object. In operator theory, this often goes via a "Dirac–dual Dirac" argument, and in topology/K-theory, one uses either controlled arguments growing out of the α-approximation theorem or more categorical alternatives to that.

Rosenthal's talk explained how to do topological descent for groups with torsion in purely algebraic settings, and also how to remove compactness hypothesis on $B\Gamma$. Ferry's talk was about the relevant properties of the group with its word metric, and alternatives to boundedness, with connections to de Rham cohomology.

The other general result (true in all the settings, by several different arguments) is that for groups of finite asymptotic dimension, all of these results are correct. Dransihnikov's talk was about the asymptotic dimension of Coxeter groups.

Bestvina's talk about the boundary of Teichmüller space and of outer space was designed to help determine whether these techniques could be applied to (outer) automorphisms of free groups.

Ji's talk surveyed many results that can approached by these methods—in particular, how one deals with many specific classes of linear groups. Prassidis's talk explained how to use a criterion of Lubotzky to show linearity of various groups.

General linear groups in the C^*-algebra setting have been disposed of by the embedding method of Skandalis–Yu–Tu, mentioned above. This does not imply the integral L-theory conjecture at the prime 2.

References for this section include: [2], [7], [18], [23], [10], [27], [37], [41], and [45].

Superrigidity (Melnick, Monod, Fisher)

Margulis has generalized Mostow's rigidity theorem in an extraordinary way: He showed that many homomorphisms of a lattice into another Lie group can (in the higher rank situation) be classified Lie theoretically.

This immediately suggests versions of Borel for embeddings, immersions, fiberings, and so on. Some of these are implicit in "twisted" versions of the conjecture—that turn out to be important in proving the original versions in certain cases.

Zimmer has extended Margulis's work to suggest a large scale research program involving lattice actions on manifolds. This involves intimately (through work of Benveniste, and Fisher–Margulis) versions of property T for other Banach spaces.

Monod described a cohomological approach to superrigidity for irreducible lattices in products that is very different from Margulis's original approach.

Fisher, on the other hand, described some strange quasi-isometric embeddings in a bare-handed tour de force, that violates the spirit of quasi-isometric super-rigidity.

Melnick's talk was also in this area: she combined dynamic, ergodic, and algebraic arguments to give rigidity results in the setting of Cartan geometries.

The reference included in this section is [46].

Farrell–Jones conjecture (Mineyev, Bartels, Reich, Davis)

We have already mentioned that for groups with torsion, all of these problems, even at the level of conjecture, are much more subtle. When we move to algebraic K-theory and L-theory, there are additional difficulties due to Nil and UNil groups.

In the course of their extraordinary work proving the Borel conjecture for torsion free lattices, Farrell and Jones were taken by the special role that closed geodesics on these spaces played in the proof. Indeed, this role is invisible if one considers just usual L-theory of K-theory of integral group rings, but it was readily apparent for pseudoisotopy and for analogues involving more general group rings.

Ultimately, they realized that one can reformulate everything in terms of isolating out subgroups that are virtually cyclic, and formulating an analogue to the statement for $\underline{E}\Gamma$ to another classifying space involving virtually cyclic isotropy.

The talks of Quinn and Reich devoted controlled ideas to the analysis of Nil. David discussed the Farrell–Jones conjecture for crystallographic groups.

Mineyev's talk was on the detailed structure of the boundary of hyperbolic groups, and how that can be applied geometrically. Indeed, it was so applied in Bartels's talk which sketched the proof of the Farrell–Jones conjecture for hyperbolic groups.

References in this section include: [13], [14], [32], and [33].

List of Participants

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Below we attach detailed program and abstracts of the talks given. Many of these have subsequently been written up for publication.

A highlight of the conference was a two evening five hour lively problem session. The participants asked and answered many of each others' questions. Given the interdisciplinary nature of the participation, this was very valuable in breaking down the barriers between fields.

Lior Silberman took excellent notes on the problem session, and it is anticipated that these will ultimately be polished and published.

Chapter 26

Canada-China Workshop on Industrial Mathematics (07w5072)

Aug 5 - Aug 10, 2007

Organizer(s): Arvind Gupta (MITACS), Huaxiong Huang (York University), Gong Qing Zhang (Peking University)

Overview

Industrial mathematics was identified as one of the main areas for further collaboration during the Canada-China Congress in Mathematics, held at the University of British Columbia in 2001. After the congress, the MITACS NCE in Canada and the Mathematics Center of the Ministry of Education (MCME) in China started a pilot program. About 10 young Chinese mathematicians were invited by MITACS, and selected by MCME from top Chinese universities to joint MITACS teams and work with Canadian researchers, for a period of six months. The purpose of this program was to jump-start the Chinese industrial mathematics program by training young and promising Chinese mathematicians in a collaborative environment such as the one found within a MITACS projects, where a team of mathematical scientists carry out applied researches relevant to industry. All the participants of the program found the experience rewarding and expressed strong desire to continue collaborations started by this pilot! program.

The Canada-China Workshop in Industrial Mathematics was organized after the successful conclusion of the MITACS-MCME pilot program. The objectives of the workshop are to provide a platform for the participants to 1. exchange ideas and insights on the development of industrial mathematics in both countries; 2. assess the success of existing collaboration between the two countries; 3. discuss future directions. In order to achieve these goals, we have invited prominent mathematical scientists as well as young researchers in both countries to show case their researches at the BIRS workshop. Round-table discussions were also organized for the participants to provide their insights and exchanges ideas on the development of industrial mathematics program in both countries.

Presentation Highlights

In the true spirit of industrial mathematics, presentations at the BIRS workshop cover a wide range of topics. Ian Friggard (UBC) gave an excellent talk on the relevance of mathematical modeling in the oil industry. He showed that simple models can go a long way in providing extremely useful insights in practical problems faced by the industry. In addition, he also showed that interesting mathematical problems arise naturally from the modeling exercises, which are nontrivial [4]. Yongji Tan (Fudan) presented an efficient method for solving the well-logging problem also used in the oil industry [1]. Ping Yan (Health Canada) gave a very informative lecture on the application of mathematical theory and stochastic models to the prediction of infectious diseases while Jianhong Wu (York) shared success stories of applying mathematical models during the SARS outbreak both in Canada and in China [7]. Guanghong Ding (Fudan) gave a very interesting overview of! the modeling efforts related to the traditional Chinese medicine, by incorporating biophysical mechanisms [5]. Christian Reidys (Nankai) reported the modeling work on bioinformatics done at the Center for Mathematical Biology in Nankai University. Nilima Nigam (McGill) present a case study on how mathematics and medicine can work together on predicting the bone remodeling [3]. Matt Davison (UWO) showcased the results from his MITACS project on modeling finance risk [8]. Shige Peng (Shandong) summarized his recent work on risk measure in finance and G-Brownian motion [2]. Evangelos Kranakis (Calton) presented modeling and computation on global communication in ad hoc networks. Sean Bohun (UOIT) summarized the MITACS funded work on crystals growth with a Canadian company Firebird [6].

Outcome of the Meeting

To build on the success of the pilot program, workshop participants have identified several directions for further developing the collaborations between Canada and China in the area of industrial mathematics. First of all, it was proposed that future collaborations should be focused on areas which are important to both countries, such as mathematical finance. Secondly, it was emphasized that the collaborations should be mutually beneficial. Finally, the collaborations should be built to take advantages of the strength of each country.

The workshop participants have also set milestones and developed strategies for followup activities. It was proposed that specific workshops in the areas of mathematical finance and biology to be held in the near future in Toronto [9]. It was also proposed that a industrial problem solving workshop in finance be jointly organized and held in China [10]. Both MITACS and MCME will develop joint programs and workshops in disease modeling and in other areas identified at and after the BIRS workshop.

List of Participants

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Chapter 27

Geometric Mechanics: Continuous and discrete, finite and infinite dimensional (07w5068)

Aug 12 - Aug 17, 2007

Organizer(s): Jerrold Marsden (California Institute of Technology), Juan-Pablo Ortega (CNRS, Universite de Franche-Comte), George Patrick (Department of Mathematics and Statistics), Mark Roberts (University of Surrey), Jedrzej Sniatycki (University of Calgary), Cristina Stoica (Wilfrid Laurier University)

Overview of the Field

Geometry, as learned by students of grade school, is the business of lines, angles, circles and triangles. It's useful because, locally, the concepts of geometry are similar to objects of our world. Using geometry, we can compute areas, heights, and angles. Almost all of us know some geometry. Many of us need it, from time to time.

It's mostly about lines. A line is the shortest distance between two points. On a manifold M, with Riemannian metric g, the length of a curve $m(t)$ from $t = a$ to $t = b$ is computed from

$$\text{length}^2 \equiv \int_a^b g\big(m'(t), m'(t)\big)\, dt.$$

To find the "straight line" between m_1 and m_2 is to minimize this length functional among curves $m(t)$ with $m(a) = m_1$ and $m(b) = m_2$.

What has that got to do with mechanics? With $F = ma$? To begin with, another notion of "straight line" is the path of a unforced particle. That is Newton's first law. At a far deeper level, Hamilton's principle [1, 43] of mechanics states that frictionless motion in a configuration space Q in general occurs on curves $q(t)$ which extremalize the functional

$$S \equiv \int_a^b L\big(q'(t)\big)\, dt, \tag{27.1}$$

where L (the Lagrangian) is typically the difference of the kinetic and potential energies. From L one can construct a two form ω on TQ, and an energy function E on TQ, such that the solutions of the variational principle (27.1) are exactly the integral curves of the Hamiltonian vector field X_E i.e. integral curves the unique vector field that satisfies

$$i_{X_E}\omega = dE. \tag{27.2}$$

Morphisms that preserve ω preserve these equations, which shows that conservative classical mechanics fundamentally occurs in the context of the category of symplectic manifolds.

Generally, we have a pair (M, ω, E), where ω is a closed nondegenerate two form on M, and E is a smooth function on M, and the evolution of the system is again via equations (27.2). If a Lie group acts of M in such a way that ω and E are invariant, then the solutions are preserved. These are the *Hamiltonian systems with symmetry*, and they are nongeneric in the set of all Hamiltonian systems. It is useful to separate the part of the dynamics that occurs due to symmetry from the generic part. By results collectively referred to as Noether's theorems, for every symmetry, there is a conserved quantity for the dynamics; and the conserved quantities in turn generate the symmetries. These conserved quantities, counting one for every independent symmetry, are typically packaged into a single *momentum map* from the phase space to the dual of the Lie algebra of the symmetry group. The quotients in the symplectic category are complicated because, besides reducing orbits to points, they also have to account for the momentum map, or else one does not get to a generic context. This is *reduction*, and the foundational work is [45]. Sequential partial reductions—reduction by stages—are an important part of the theory, because they allow for finer grained elimination of symmetry [39, 40]. And there are also ways to reduce from the basic Lagrangian variational principles [16, 17, 31, 42, 44]

Smooth quotients are only expected when the symmetry group acts freely, and singularities are expected where there is isotropy. In Hamiltonian systems, there are two sources for singularities in the quotients: from isotropy of the symmetry and from singularity in the momentum map. Because the symmetries and the momentum are so intimately related, these singularities occur at exactly the same places in phase space [2]. The singular quotients in symplectic geometry are Whitney stratified spaces, where the stratifications are by isotropy type and momentum [5, 54, 57, 67].

Equilibria are of course fundamental organizing solutions for dynamical systems. In the presence of symmetry one can seek equilibria in the quotients; these are the *relative equilibria* and they are solutions that are the actions of one parameter groups of the symmetry. Relative equilibria, such as occur in the restricted three body problem of celestial mechanics, are of great physical interest, and have been studied since the outset of mechanics. One of the important tools of Geometric Mechanics is special coordinates near relative equilibria, derived from symplectic geometry's various linear splittings together with the (equivariant) Darboux theorem [56, 58, 59, 65, 71]. The coordinates are helpful to determine the structure of the set of relative equilibria, as in [53, 58, 61, 64, 66, 74]. Extensions to relative periodic orbits occur in [55, 74, 75], and numerical continuation is considered in [76].

The stability of relative equilibria can be delicate, as the spectrum of the linearizations in Hamiltonian system cannot all be in the negative real half plane, and asymptotic stability is impossible. Using energy and momenta for the Lyapunov functions can provide conditions for robust stability. But generically there are also regions where such Lyapunov stability fails and the spectra is purely imaginary. This situation is a an important target of one of the principle advances in Science in the last century: the KAM theory, which arose from early questions about the stability of the solar system [4], and in many cases forms of stability may be recovered even here [24, 25, 52]. KAM stability, or, in higher dimensions, stability over exponentially long times is delicate. It can be destroyed through resonance [7], or by the introduction of arbitrarily small dissipation (dissipation induced instability) [9, 22, 33, 34].

There are other ways to write equations (27.2). Given functions f and g, the Poisson bracket of $\{f, g\}$ of f and g is

$$\{f, g\} \equiv \omega(X_f, X_g)$$

and the solutions of the variational principle (27.1) are equivalent to

$$\frac{df}{dt} = \{f, E\}.$$

This leads to Poisson geometry [43, 72]. There is little advantage between symplectic and Poisson geometry at this level. But in the quotients by the actions of groups the symplectic geometry occur as the leaves of the Poisson spaces; the latter coherently assembles the former into a whole. Both are important, and they are different viewpoints.

There are extensions. If a mechanical system is constrained in its velocities, such as a penny, ball, or egg, rolling on a table, then the geometry is altered. This is nonholonomic mechanics [13], and it necessitates generalizations, from symplectic to semi-symplectic, and Poisson to almost Poisson. Different categories mean different morphisms, and different basic properties. For example, energy is conserved, but not momentum, or you couldn't turn any rolling vehicle (local nonconservation of angular momentum). Since momentum and symmetry in holonomic mechanics are related, the quotients in these generalizations take a different form. The geometry of nonholonomic systems is complicated, and its development relatively recent [6, 11, 37, 38, 60, 68].

Other important extensions occur in field theories i.e. essentially, partial differential equations. A classic example is the realization of the motion of a perfect fluid as a geodesic flow on the group of volume preserving diffeomorphisms [3, 23]. Of course differential geometry and field theories are not strangers: the theories of general relativity, and gauge field theories [50, 51], are all heavily geometric, and no summary can do justice to the developments. Field theories may be generally cast geometrically, in a similar way as geometric mechanics, as is seen in the works [27, 28, 29], and the references therein.

All these ideal conservative systems are central. Most fundamentally they arise as above from variational principles, and give rise, as already mentioned, to the category of symplectic and Poisson manifolds, which themselves are of great current interest to Mathematics. For mechanical systems, such as rigid body systems, the phase spaces are finite dimensional, while for field theories they are infinite dimensional. While physical systems are seldom exactly symmetric, or exactly conservative, there are often physical regimes for which they are nearly so, and thus for which the dominant or organizing behaviors are those of the ideal systems. For example, small dissipation might be superposed on a conservative system, or a system might be obtained by breaking the symmetry of a perfectly symmetric system. The symmetric, conservative systems are an organizing center for the subject.

Discrete analogies occur, in 1956 [70], and then rediscovered in [18], as integration algorithms for the ordinary differential equations of holonomic mechanics. It was long known that the flows of mechanics are canonical, and so preserve the volume of phase space. In the geometric setting, the flows preserve the symplectic form, and so are morphisms of the symplectic manifolds. Integration algorithms that are iterations of a single symplectic map preserve the dynamical features of classical mechanical systems, over long times, better than generic algorithms. Thus began in 1990 a cascade of work of geometric integration algorithms, as discussed for example in the books [30, 36], and extended to nonholonomic systems [19, 48]. Significant is the development of discrete mechanics, which is towards discrete models that reflect physical reality so well that they have a status with continuous models. The continuous models are fundamentally variational. The variational foundations of discrete Lagrangian mechanics, where the continuous tangent bundle phase space is replaced by pairs of configurations, and the action integral is a sum over sequences of pairs, occur in [49], and as developed as such in [46, 63, 73]. This is extended to nonholonomic systems in [21, 20], and to first order variational field theories in [41].

"Ballistic" refers to the free flight of a projectile. But if the projectile is to arrive at a specific target, then controlled motion is far more effective. Automatic control is an important area in engineering; devices of all sorts have to be guided and stabilized to particular states or modes of motion. Control theory is one of the main application areas of geometric mechanics, as witnessed by the recent books of Bloch [8] and Bullo and Lewis [15].

In the most primitive form, one has a number of control inputs represented by coefficients of a sum of vector fields. The question is whether there is a curve in parameter space which causes a transit between any two given points of configuration or phase space. This is the issue of controllability, and it is an early application of differential geometry: the Lie bracket of two control vector fields may be generated by alternately running one, the other, and then the reverse, so the number of control vector fields required may be reduced if those vector fields do not commute [69]. Feedback control refers to the problem of designing a way of setting the control parameters, bases on the evolving state itself, so that a particular trajectory is achieved, and stable. Recent work of this in the geometric setting occurs for example in [10, 12]. Optimal control refers to the additional requirement that some quantity, such as fuel, energy, or time, be minimized. The constraints of optimal control lead to fundamentally different variational principles than the constraints of nonholonomic mechanics. Discrete mechanics is playing a key role in control as well, with the development of applications of discrete mechanics to optimal control [32]. And there is an interesting recent application to image restoration [47].

Differential geometry and mechanics are fundamentally related, as are differential geometry and physics. Geometric Mechanics enhances the traditional approach to mechanics by the inclusion of ideas from differential geometry, nicely balanced with analytical methods. While this idea has its roots going back to the founders of mechanics, such as Jacobi, there has been a resurgence of these ideas in the past few decades, with the infusion of many new ideas and links. For instance, it is a basic fact that the standard Hopf fibration of S^3 to S^2, usually thought of as belonging to pure topology (or bundle theory) already occurs in rigid body mechanics (going back to Euler and Lagrange). The geometric approach to mechanics flourishes today: it has its own internal beauty and research (such as stability theory and singular reduction theory), as well as substantial contributions to neighboring areas, such as molecular systems, classical field theories (fluids, solids, electromagnetism, gravity, etc.), to control theory, and to computational mechanics. In Geometric Mechanics today, we use concepts and compute properties that could not be easily discerned without the differential-geometric context, some of which did not exist even a decade ago.

Presentation Highlights

Special Lectures

We had three special lectures, of one hour duration, in the evenings.

Jerry Marsden (Caltech), spoke about variational integrators and optimal control, and Lagrangian coherent structures (LCS). While the free dynamics of conservative systems is always ideal and therefore of special application, the application of control is wide open. Jerry explained how the established work towards discrete analogies in mechanics can help solve to solve problems in optimal control. LCS can help understand mixing, transport and barriers in fluid flows (e.g., ocean and atmosphere) and other dynamical systems. It can also be used to decide drifter deployment, and understand pollution dispersion, oil spills.

Tudor Ratiu (Ecole Polytechnique Federale de Lausanne) considered the Lagrangian and Hamiltonian structures for an ideal gauge-charged fluids [26]. These are geometrically complex infinite dimensional examples. The discussion include a Kelvin-Noether theorem non-canonical Poisson bracket associated to these systems.

Jedrzej Sniatycki (Calgary) considered the commutativity of quantization and reduction. The important new aspect is the lack of assumptions on the group action, so that the Theorem below addresses possible singularities in the classical system:

Theorem 27.3 *Assumptions:*

1. *Let P be a Kahler manifold, ω be the Kahler form on P and F be given by antiholomorphic directions.*
2. *Suppose that an action of a connected Lie group G on (P, ω) has an Ad^*_G equivariant momentum map $J\colon P \to \mathfrak{g}^*$ and preserves F.*
3. *Let O be a quantizable co-adjoint orbit admitting a Kahler polarization F_O such that quantization of (O, ω_O) in terms of the polarization F gives rise to an irreducible unitary representation U_O of G.*
4. *Suppose that there exists a Lagrangian subspace of (P, ω) contained in $J^{-1}(O)$.*

Then the space of square integrable wave functions, obtained by quantization of algebraic reduction at O, defines on H a projection operator Π^O such that $\Pi^O(H)$ is the closed subspace of H on which the quantization representation U is equivalent to U^O. Here H denotes the space of square integrable holomorphic sections of $L \to P$.

Relative equilibria

James Montaldi (Manchester) presented on bifurcations of relative equilibria at zero momentum. A result of Roberts and Patrick [61], states that generically, $\mathrm{SO}(3)$ symmetric Hamiltonian systems have no equilibria. Montaldi answered why this is not the case for simple mechanical systems.

The problem of Riemann Ellipsoids has an long and important history, going back to Newton, MacLaurin, Dirichlet, Riemann and Poincare. It has its origins in the attempt to provide an explanation to the rotating figure of the Earth. Miguel Rodriguez-Olmos reviewed this problem from the point of view of Differential Geometry and discussed how the geometric perspective can give some insight into the nonlinear stability of some of its classical solutions.

Understanding the structure of the relative equilibria or periodic orbits of a system means following manifolds of them in phase space and understanding their bifurcations. It is generally impossible to do this exactly. Frank Schilder and Claudia Wulff discussed this problem, and presented the software system SYMPERCON for numerical bifurcation analysis of Hamiltonian relative periodic orbits. SYMPERCON accounts for, and takes advantage of, the nongenericity of Hamiltonian systems with symmetry in the class of all systems with symmetry.

Saari's conjecture is that every solution of the planar Newtonian N-body problem with constant moment of inertia is a relative equilibrium. Cristina Stoica (Wilfrid Laurier) presented recent work showing that, for generic rotationally-invariant vector fields in the plane, Saaris conjecture is true: the only constant-inertia solutions are the relative equilibria [66].

The stability of relative equilibria is sensitive to the topology of the orbit space of the coadjoint action of the symmetry group [62]. Claudia Wulff (Surrey) presented work proving that, in the Kirchhoff model for the motion of an axisymmetric underwater vehicle, relative equilibria that were thought to be robustly stabilized by spin are in fact only KAM stable. Furthermore, there is numerics that showing dissipation induced instability in this case.

Control

Future space missions like Terrestrial Planet Finder (NASA) and Darwin (ESA) will make use of a network of formation flying spacecraft. In these missions, the requirements on the accuracy on the relative positioning of the craft are extremely high. In addition, reconfigurations of the formation have to be performed at regular intervals with minimal energetic effort. Oliver Junge (Munich University of Technology) showed how the recently developed variational method DMOC (Discrete Mechanics and Optimal Control) for the numerical computation of optimal open-loop controls for mechanical control systems can be applied to this problem.

Suppose you have a network of sensor equipped vehicles and you want to coordinate and stabilize the motion of your fleet. Sujit Nair explained the you can couple the system to yield a multi-body Lagrangian system, and use controlled Lagrangians and matching conditions. The symmetry of these systems depends on the context: $\mathrm{SO}(3)$ for spacecraft and $\mathrm{SE}(3)$ for underwater vehicles. He also spoke about about coordinating hovercrafts for the purpose of surveillance, and showed a wonderful movie showing coordination of inverted pendula on connected rolling carts.

Andrew Lewis (Queens) spoke about energy shaping, which is a control strategy wherein one converts a given mechanical system (called the open-loop system) to another mechanical system (called the closed-loop system) with desired properties. He gave an affine differential geometric formulation of the energy shaping problem and gave a complete description of part of the problem using techniques from the formal theory of partial differential equations.

Discrete systems; numerics

Melvin Leok (Purdue University) discusses the synthesis of Lie group techniques and variational integrators to construct symplectic-momentum methods which automatically stay on Lie groups and homogeneous spaces without the need for constraints, local coordinates, or reprojection. These are integrators that simultaneously preserving the symplectic and Lie group properties.

George Patrick (Saskatchewan) presented a new development of variational discretizations, based on discrete analogues of tangent bundles, obtained by systematically extending tangent vectors to finite curve segments. He showed that existence and uniqueness of the discrete evolutions can be analyzed by blowing up the variational principles at zero time-step. These methods can automatically convert any one-step numerical method to a variational method of the same order.

Ari Stern (California Institute of Technology) presented applications of variational integrators to electromagnetism. This uses discrete versions of the exterior calculus of differential forms.

Other

In the wide group of participants, there are, inevitably, presentations that span areas or that do not seem to naturally classify with other presentations of this workshop.

Anthony Bloch (Michigan) considered Hill's equation with random forcing terms. Andreu Lazaro (Zaragoza), Nawaf Bou-Rabee (Caltech), and Stephane Chretien discussed aspects of stochastic systems, which is an emerging area which everyone anticipates will be important [14, 35]

Katlin Grubits (Hawaii) presented "Self-assembly of particles using isotropic potentials". Eva Kanso (University of Southern California) talked about low order models of swimming. Oleg Kirillov (Moscow M.V. Lomonosov State University) considered models of rotating bodies of revolution being in frictional contact. This has applications to well-known phenomena of acoustics of friction, such as the squealing disc brakes, and singing wine glass Rouslan Krechetnikov (Carleton) considered dissipation-induced instability phenomena in both finite-dimensional mechanical systems, and an infinite-dimensional two-layer quasi-geostrophic beta-plane model, which describes the fundamental baroclinic instability in atmospheric and ocean dynamics Antonio Hernandez-Garduno (Universidad Nacional Autonoma de Mexico) discussed the averaging of Lagrangian systems, illustrating it with the example of the forced inverted pendulum.

There were presentations on general theory. Lie groupoids are a unifying thrust of geometry. Manuel de Leon (Instituto de Matematicas y Fisica Fundamental) considers geometric Hamilton–Jacobi theory on almost Lie algebroids, with applications to nonholonomic mechanical systems. Rui Loja Fernandes (Instituto Superior Tecnico) asked what happens with respect to reduction for general (non-free) proper Poisson actions. This involved defining Poisson stratified spaces and results which establish that $\bigcup_{(H)} M_{(H)}/G$ is a Poisson stratification of M/G, where there is a Poisson action of G on M. Ivan Struchiner (UniCamp–Campinas–Brasil) began by asking, up to isometries, what are all constant curvature Riemannian metrics in a neighborhood of $0 \in \mathbb{R}^2$? Similar classification problems appear in many other settings, including geometric mechanics, pdes, variational problems, etc. He presented a general approach to such classification problems using Lie algebroids and Lie groupoids. Cotangent bundles are a principle example in geometric mechanics, and indeed every regular Lagrangian system and a cotangent bundle fomulation, wherein the symplectic form is canonical. Tanya Schmah (Macquarie) presented explicitly construction of symplectic tubes, particularly for T^*Q and a group G acting by cotangent lifts. Symplectic tubes are symmetry adapted coordinates near group orbits of G, and they are one of the more important applications of the symplectic geometry in mechanics. Dan Offin (Queen's) reviewed some translations of the Maslov index, and explained how to use them to predict stability and instability for global periodic solutions determined by variational principles.

Bifurcations, even in the absence of Hamiltonian structures, are highly geometric. Luciano Buono (University of Ontario Institute of Technology) considered steady-state bifurcations in reversible-equivariant vector fields. He showed that the analysis of these bifurcations can be reduced to the study of bifurcations of an equivalent equivariant vector field with no time-reversibility, sometimes also having parameter symmetry and for which a bifurcation theory already exists.

Outcomes of the Meeting

This was a meeting of leading experts in applications of differential geometry to mechanics. There were a great many informal discussions, the results of which cannot be catalogued. People who work in Geometric Mechanics are scattered worldwide and there was a large benefit of having many of them in one place. The scope of the discussion was terrific, and the expansive view of the area and the activity in it was of great value. This will have affected the thinking of many of the participants. The isolation of many was diminished, and this persisted in real ways after the workshop.

It could not have been done so well elsewhere, and perhaps not at all. The participants have a very high opinion of BIRS. It is regarded as a high profile opportunity for meeting by people who do not choose to attend every conference to which they are invited. In subsequent planning, BIRS is singled out as a place to which select opportunities might be directed. The meeting directly gave rise to three other BIRS proposals. The participants want to return to BIRS.

In modern day Science, we all should organize meetings. It might not so immediate to the minds of such as the BIRS directors and high level supporters, but it is true and it should be stated, that not everyone is naturally predisposed to this activity. An important effect of BIRS is to encourage faculty in its member Universities, and elsewhere in Canada, to participate in the organization of such high level meetings. They learn, in a supportive environment, that they really can participate in the organization of meetings, at the highest level. BIRS is great help for the visibility of faculty who might not otherwise organize such these activities. Such was the case for this workshop.

List of Participants

Bates, Larry (University of Calgary)
Bloch, Anthony (University of Michigan)
Bou-Rabee, Nawaf (New York University)
Buono, Luciano (University of Ontario Institute of Technology)
Chretien, Stephane (Universite de Franche Comte France)
Cushman, Richard (University of Calgary)
de León, Manuel (Instituto de Matemáticas y Física Fundamental)
Fernandes, Rui Loja (Instituto Superior Tecnico)
Gotay, Mark (University of Hawaii)
Grubits, Katalin (California Institute of Technology)
Helmuth, Tyler (University of Saskatchewan)
Hernández-Garduño, Antonio (Universidad Nacional Autonoma de Mexico)
Junge, Oliver (Munich University of Technology)
Kanso, Eva (University of Southern California)
Kirillov, Oleg (Moscow M.V. Lomonosov State University)
Krechetnikov, Rouslan (Carleton University)
Lamb, Jeroen (Imperial College London)
Lazaro, Andreu (University of Zaragoza)
Leok, Melvin (Purdue University)
Lewis, Andrew (Queens University)
Marsden, Jerrold (California Institute of Technology)
Montaldi, James (University of Manchester)
Nair, Sujit (Control and Dynamical Systems Caltech USA)
Offin, Daniel (Queen‘s University)
Ortega, Juan-Pablo (CNRS, Universite de Franche-Comte)
Patrick, George (Department of Mathematics and Statistics)
Plummer, Mike (University of Surrey)
Ratiu, Tudor (Ecole Polytechnique Federale de Lausanne)
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Roberts, Mark (University of Surrey)
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Chapter 28

Operator Spaces and Group Algebras (07w5013)

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Organizer(s): 5
Aug 19 - Aug 24, 2007
This workshop was organized by **Eberhard Kaniuth** (Univeristy of Paderborn), **Anthony To-Ming Lau** (University of Alberta), and **Zhong-Jin Ruan** (University of Illinois). The workshop had a total of 42 participants. Our main speaker was Professor Gilles Pisier (Paris VI/Texas A & M), who gave two 50-minute talks. The other 22 speakers each gave a 50-minute talk. The following are reports on their talks, problems arising, and impacts following the workshop.

The similarity problem

The two lectures of **G. Pisier** at the workshop concerned the similarity problem.

A locally compact group is *unitarizable* if any (continuous) uniformly bounded representation is unitarizable (i.e., if it is similar to a unitary representation). Dixmier asked already in 1950 whether ‘unitarizable’ implies ‘amenable’ (the converse was proved by him and Day independently). More precisely, a representation $\pi\colon\ G \to \mathcal{B}(H)$ is *unitarizable* if there is an invertible operator $\xi\colon\ H \to H$ such that $g \mapsto \xi\pi(g)\xi^{-1}$ is a unitary representation on G. In 1955, Ehrenpreis and Mautner showed that $SL_2(\mathbb{R})$ is not unitarizable, from which it follows formally that any discrete free group admitting it as a quotient is also non-unitarizable.

Motivated by this, Kadison [41] formulated the following conjecture: any bounded homomorphism $u\colon\ A \to \mathcal{B}(H)$ from a C^*-algebra into the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H is similar to a $*$-homomorphism, i.e., there is an invertible operator $\xi\colon\ H \to H$ such that $x \mapsto \xi u(x)\xi^{-1}$ satisfies $\xi u(x^*)\xi^{-1} = (\xi u(x)\xi^{-1})^*$ for all x in A. In this latter case, we say that ξ *unitarizes* u and that u is *unitarizable*. Without loss of generality, one may suppose that A has a unit. Then u is unitarizable if and only if its restriction to the unitary group of A is unitarizable as a group representation.

These conjectures remain unproved, although many partial results are known. In his series of two talks, Pisier surveyed those results, as well as more recent results on the closely related notion of *length* of an operator algebra that he introduced. In particular, he explained why ‘length equal to 2’ characterizes amenable groups or C^*-algebras. Moreover, he showed that, if one can always force the similarity ξ to be in the von Neumann algebra generated by the range, then the group (or the C^*-algebra) must be amenable. Here are some more precise definitions.

We denote by $\|u\|_{cb}$ the completely bounded (in short, c.b.) norm of a linear mapping between two operator spaces, i.e., two linear subspaces of the space $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. This plays a crucial role in similarity problems because of Haagerup's formula, valid for any (algebra) homomorphism $u\colon\ A \to \mathcal{B}(H)$ defined on a C^*-algebra A: $\|u\|_{cb} = \inf\{\|\xi\|\|\xi^{-1}\|\}$, where the infimum is over all invertibles ξ on H that 'unitarize' u.

The *similarity degree* of a unital operator algebra A is defined (see [58] and more references there) as the smallest $\alpha \geq 0$ for which there is a constant C such that any bounded morphism (= unital homomorphism) $u\colon\ A \to \mathcal{B}(H)$ satisfies $\|u\|_{cb} \leq C\|u\|^{\alpha}$.

On the other hand, an operator algebra $A \subset \mathcal{B}(H)$ is *of length* $\leq d$ if there is a constant K such that, for any n and any x in $\mathbb{M}_n(A)$, there are an integer $N = N(n,x)$ and scalar matrices $\alpha_0 \in \mathbb{M}_{n,N}(\mathbb{C})$, $\alpha_1 \in \mathbb{M}_N(\mathbb{C}), \ldots, \alpha_{d-1} \in \mathbb{M}_N(\mathbb{C})$, $\alpha_d \in \mathbb{M}_{N,n}(\mathbb{C})$ together with diagonal matrices $D_1, \ldots, D_d$ in $\mathbb{M}_N(A)$ satisfying

$$x = \alpha_0 D_1 \alpha_1 D_2 \ldots D_d \alpha_d \quad \text{with} \quad \prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leq K\|x\|\,.$$

We denote by $\ell(A)$ the smallest d for which this holds, and we call it the *length* of A (so that 'A has length $\leq d$' is indeed the same as '$\ell(A) \leq d$'). It is easy to see that, if $\ell(A) \leq d$, then, for any bounded homomorphism $u\colon\ A \to \mathcal{B}(H)$, we have $\|u\|_{cb} \leq K\|u\|^d$.

Now let G be a discrete group, and let $A = C^*(G)$. We wish to restrict the above factorization to the case when the entries of the diagonal matrices $D_1, \ldots, D_d$ sit in G itself (viewed as a subset of $C^*(G)$ in the usual way). We denote by $\ell(G)$ the smallest d as above, but for these restricted factorizations. Analogously, we denote by $d(G)$ the smallest α such that, for some C, all uniformly bounded representations π satisfy $\|u_\pi\|_{cb} \leq C \sup_{g\in G} \|\pi(g)\|^{\alpha}$, where $u_\pi\colon [G] \to \mathcal{B}(H)$ is the homomorphism linearly extending π.

To summarize the basic known results, we state the following from [55, 57]; the last assertion is due to Erik Christensen [12].

Theorem 28.1 (i) *For any discrete unitarizable group G, we have $d(G) = \ell(G)$.*

(ii) *Any G containing a non-abelian free group has infinite length* (*i.e. is not unitarizable*).

(iii) *The Dixmier question whether unitarizable groups are amenable is equivalent to the assertion that there are no groups G with $2 < \ell(G) < \infty$.*

(iv) *For an infinite discrete group, G is amenable if and only if $\ell(G) = 2$.*

Recall that a C^*-algebra A is *nuclear* (equivalently *amenable* in B. E. Johnson's sense) if, for all C^*-algebra B, there is a unique C^*-norm on $A \otimes B$.

Theorem 28.2 (i) *For any unital operator algebra A, we have $d(A) = \ell(A)$.*

(ii) *The Kadison conjecture that all C^*-algebras are unitarizable is equivalent to the assertion that there is a fixed d such that any C^*-algebra A has length $\ell(A) \leq d$.*

(iii) *For an infinite-dimensional C^*-algebra A, we have $d(A) = 2 \Leftrightarrow \ell(A) = 2 \Leftrightarrow A$* is nuclear.

(iv) *There are examples* (*e.g.*, $A = \mathcal{B}(\ell_2)$) *for which $\ell(A) = 3$.*

The Banff meeting was followed in October 07 by a conference at the American Institute of Mathematics (AIM) in Palo Alto on the dichotomy between amenable and non-amenable groups. The Dixmier problem was one of the main problems discussed in that workshop, but the participants were from a different background from those at Banff, with many from geometric (infinite) group theory or random walks on groups. Pisier again gave a series of two talks, concentrating on the group case rather than on operator algebras, at that meeting.

Operator algebras

A number of talks during the week circled around the interplay between groups and operator algebras, e.g., von Neumann II_1-factors arising either from ergodic group actions on probability spaces (through the Murray–von Neumann group-measure space construction) or as the II_1-factor $L(G)$ associated with the regular representation of a discrete group with infinite conjugacy classes. It must be observed that, if a countable group G acts ergodically on a probability space (X, μ), then the group-measure space factor $M = L^\infty(X, \mu) \rtimes G$ depends only on the equivalence relation induced by G on X; in particular, amenability properties of the equivalence relation translate into amenability properties of M.

At the workshop, **C. Anantharaman-Delaroche** gave a lucid survey talk on various possible definitions of amenability for equivalence relations, group actions, and more general groupoids. She put the emphasis on asymptotic properties of random walks on groupoids. A basic open question about II_1-factors M is: Up to conjugacy, how many Cartan subalgebras are there in M?

If M has no Cartan subalgebra, then M cannot come from the group-measure space construction; if M has at least two Cartan subalgebras, then M comes from at least two genuinely different actions (i.e., non orbit equivalent). The fact that the factor $L(\mathbb{F}_r)$ of the free group $\mathbb{F}_r$ has no Cartan subalgebra is a success of Voiculescu's free probability theory. The existence of a II_1-factor with two non-conjugate Cartan subalgebras is a joint result by two Fields medallists, A. Connes and V. F. R. Jones in 1982 [14].

N. Ozawa lectured about his joint work with Popa on II_1-factors with zero or one Cartan subalgebra. He sketched the proof [53] that, if M is a II_1-factor with the complete metric approximation property (CMAP), then $M \bar{\otimes} L(\mathbb{F}_r)$ has no Cartan subalgebra (a far-reaching generalization of Voiculescu's result), and that, if the probability space (X, μ) carries a profinite ergodic action of $\mathbb{F}_r$, then the group-measure factor $L^\infty(X, \mu) \rtimes \mathbb{F}_r$ has a unique Cartan subalgebra (namely $L^\infty(X, \mu)$).

A purely group-theoretical result of the Ozawa–Popa study is the fact that, if a wreath product $H \wr G$ has CMAP, then the acting group G is amenable. Since the workshop, Ozawa and Popa, generalizing further [54], have shown that, if Γ is a non-amenable group with a strong form of the Haagerup property (also called a-T-menability), which moreover has CMAP, then $L(\Gamma)$ has no Cartan subalgebra and $L^\infty(X, \mu) \rtimes \Gamma$ has a unique Cartan subalgebra when (X, μ) carries a profinite ergodic action of Γ. These results raise the questions of having more examples of groups satisfying CMAP and/or the Haagerup property.

During the workshop, two further talks addressed these questions. **E. Guentner** explained his result with Higson [35] about groups acting properly and isometrically on finite-dimensional $CAT(0)$. Also **A. Valette** explained his result with Y. de Cornulier and Y. Stalder [15] that the Haagerup property is preserved under wreath products. Together with the Ozawa–Popa result mentioned above, this disproves a conjecture by M. Cowling in 1996 that the class of CMAP groups coincides with the class of Haagerup groups; see [9]. Another by-product is that the finite-dimensionality assumption cannot be removed from the Guentner–Higson result. These three sets of results give rise to obvious questions about these classes. What are their permanence properties? In particular, which kinds of semi-direct products preserve these classes?

In a slightly different direction, **B. Bekka** explained his remarkable super-rigidity result [4] for $SL_n(\mathbb{Z})$, where $n \geq 3$, in a von Neumann-algebra setting. Let $f : SL_n(\mathbb{Z}) \to U(M)$ be a homomorphism to the unitary group of a finite factor M. Then either M is finite-dimensional (and f factors through a congruence subgroup), or there exists a finite index subgroup Γ of $SL_n(\mathbb{Z})$ such that f extends to a normal homomorphism $L(\Gamma) \to M$. A corollary is that the full C^*-algebra of $SL_4(\mathbb{Z})$ has no faithful trace, which answers negatively a question of Kirchberg [42] (a positive answer would have proved the famous Connes' embedding conjecture, also mentioned in **K. Dykema**'s talk during the week: Every countable group embeds into the unitary group of the ultra-power of the hyperfinite II_1-factor!) An interesting open question is: What happens if $SL_n(\mathbb{Z})$ is replaced by some other higher rank lattice?

Coming from theoretical physics, the Bessis–Moussa–Villani conjecture, going back to 1975, states that the function $t \mapsto \mathrm{Tr}(\exp(A + \mathrm{i}tB))$ is positive-definite on the real line for any two self-adjoint matrices A and B of the same size. Of course this conjecture can be generalized to any C^*-algebra with a faithful trace, and **M. Bozejko** explained the proof [6] of this generalized form when A and B are q-Gaussian random variables, with $q \in [-1, 1]$; the case where $q = 0$ corresponds to free probability. Interesting connections were made with completely bounded maps on Coxeter groups.

R. Archbold described some recent results [1, 2] that he had obtained with Kaniuth on the *stable rank* $sr(C^*(G))$ and *real rank* of group C^*-algebras $C^*(G)$ and of compact transformation group C^*-algebras $C_0(X) \rtimes G$. In the case of an almost connected, nilpotent, locally compact group, we have

$$sr(C^*(G)) = 1 + \left[\frac{1}{2}\mathrm{rank}(G/[G,G])\right],$$

which generalizesg the result of Sudo and Takai for simply connected, nilpotent Lie groups. On the other hand, an unresolved dichotomy was described for $RR(C^*(G))$ in the case where $\mathrm{rank}(G/[G,G])$ is even.

For a second countable transformation group (G, X) with G compact, detailed formulae were described for $sr(C_0(X) \rtimes G)$ and $RR(C_0(X) \rtimes G)$, subject to the proviso that the space X is locally of finite G-orbit type. The meeting stimulated further work on these problems, using some very recent results [7] of Brown on the real rank of CCR C^*-algebras. This has resulted in substantial progress by Archbold and Kanuith on the dichotomy for $RR(C^*(G))$, and also on the removal of the somewhat restrictive assumption of 'finite G-orbit type' in the case where G is a compact Lie group acting on a second countable space X.

A C^*-algebra A is *exact* if the minimal tensor product by A preserves short exact sequences of C^*-algebras. These algebras were the topic of the talk of **M. Dadarlat**.

A fundamental result of Kirchberg [43] asserts that the separable exact C^*-algebras are precisely those C^*-algebras which embed in the Cuntz algebra $\mathcal{O}_2$. A C^*-algebra which can be represented as an inductive limit of finite-dimensional C^*-algebras is called an *AF algebra*. A major open problem in the structure theory of C^*-algebras is to characterize the C^*-algebras which embed in separable AF-algebras. One has the following conjecture.

Conjecture: A separable C^*-algebra is AF-embeddable if and only if it is exact and quasidiagonal.

Recall that a separable C^*-algebra A is *quasidiagonal* if there is a sequence $\varphi_n : A \to \mathbb{M}_n(\mathbb{C})$ of completely positive contractions such that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b))\| \to 0$ as $n \to \infty$ for all $a, b \in A$. Voiculescu proved that quasidiagonality is a homotopy invariant in the class of separable C^*-algebras. While the above conjecture is very much open, there are some promising results towards its validity. N. Ozawa proved in [52] that AF-embeddability is a homotopy invariant in the class of separable exact C*-algebras. In particular the cone, and hence the suspension, of any separable exact C^*-algebra is AF-embeddable. Dadarlat [16] has verified the conjecture for the class of separable, residually finite-dimensional algebras that are equivalent in KK-theory to commutative algebras. Using a result of J. L. Tu, one then concludes that the C^*-algebra of a countable amenable residually finite group is AF-embeddable. This brings us to another very interesting open problem inspired by work of Vershik and Rosenberg.

Problem: Is the C^*-algebra of a countable amenable discrete group AF-embeddable or at least quasidiagonal?

Dadarlat has shown that this is the case for central extensions of amenable, residually finite groups by $\mathbb{Z}^n$. Nevertheless this problem seems very difficult even for the class of elementary amenable groups. It is expected that the methods required to solve this problem would inspire powerful generalizations of the Berg technique, and would lead to significant progress concerning the structure of C^*-algebras associated to amenable groups.

K. Dykema discussed Horn inequalities. Indeed, given a Hermitian $n \times n$ matrix A, let $\lambda_A(1) \geq \lambda_A(2) \geq \cdots \geq \lambda_A(n)$ be its eigenvalues listed according to multiplicity. The classical *Horn inequalities*, for Hermitian $n \times n$ matrices A, B, and C such that $A + B + C = 0$, are the inequalities

$$\sum_{i \in I} \lambda_A(i) + \sum_{j \in J} \lambda_B(j) + \sum_{k \in K} \lambda_C(k) \leq 0$$

for certain triples (I, J, K) of subsets of $\{1, \ldots, n\}$ with $|I| = |J| = |K|$, known as Horn triples.

It was proved about ten years ago, due to work of Klyachko [45] and Knutson and Tao [46] that the Horn inequalities, together with the trace equality

$$\sum_{i=1}^{n} \lambda_A(i) + \sum_{j=1}^{n} \lambda_B(j) + \sum_{k=1}^{n} \lambda_C(k) = 0,$$

exactly chacterize the set of possible eigenvalues of such A, B, and C.

A question asked in the talk of Dykema was whether the analogues of the Horn inequalities hold for self-adjoint elements in all II_1-factors. This is related to Connes' embedding problem, a deep question that has many equivalent formulations. The main new result that was given in the talk appeared about a year after the BIRS workshop [5]: it is that all Horn inequalities do hold in all II_1-factors. The proof of this result is actually a construction: given arbitrary flags $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ in a II_1-factor and a triple (I, J, K) whose corresponding Littlewood–Richardson coefficient is equal to 1, there is a projection in the intersection $S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)$ of the corresponding Schubert varieties. This construction seems to be new even in the case of matrices. A more intricate eigenvalue question, analogous to Horn's question, but with 'matrix coefficients'; by [13], this is equivalent to Connes' embedding problem.

R. Smith discussed masas in von Neumann algebras. For an inclusion $B \subseteq M$ of finite von Neumann algebras, a unitary operator $u \in M$ *normalizes* B if $uBu^* = B$. The group of normalizing unitaries is denoted by $\mathcal{N}(B)$, while $\mathcal{N}(B)''$ denotes the von Neumann algebra that it generates inside M. Interest in these operators goes back to Dixmier in the 1950's, who used $\mathcal{N}(B)''$ to classify various types of maximal abelian self-adjoint subalgebras (masas) in factors. It is always the case that $B \subseteq \mathcal{N}(B)''$, and B is *singular* if equality holds. A natural question is how singularity relates to tensor products and, as part of a larger study with Sinclair, White, and Wiggins [61], it was shown that the tensor product of singular masas is again singular. Subsequently this was generalized by Chifan [10] to the formula

$$\mathcal{N}(B_1 \mathbin{\overline{\otimes}} B_2)'' = \mathcal{N}(B_1)'' \mathbin{\overline{\otimes}} \mathcal{N}(B_2)''$$

for arbitrary masas in finite factors.

Since then Smith has investigated normalizers for irreducible inclusions of factors (with White and Wiggins [62]), where it is actually possible to determine them explicitly, unlike the masa case. This has led recently to the resolution of the case where the subalgebra satisfies $B' \cap M \subseteq B$, which includes the two special cases already mentioned. For tensor products one has to replace normalizers by a wider class of operators called *groupoid normalizers*, but then the analogous result holds true [25]. This has already had a slightly suprising application to the theory of maximal injective von Neumann subalgebras [8].

Various examples show that $B' \cap M \subseteq B$ is a natural boundary for such theorems, but all of this discussion holds true only for finite von Neumann algebras, and all such questions are open for the other type of von Neumann algebras. In the finite case one has available the basic construction algebra $\langle M, e_B \rangle$ of Jones. The main current difficulty is to find a suitable modification of the techniques to make progress on the other types of von Neumann algebras.

Fourier algebras

The Fourier algebra $A(G)$ of a locally compact group G has been a very fruitful object for interactions between operator algebras and harmonic analysis since the 1960s. Though the Banach algebra structure is commutative, the spatial structure is not, and the theory of operator spaces, which recognizes this noncommutativity, interacts surprisingly well with various aspects of $A(G)$. This has been known since the 1980s, with the pioneering work of Haagerup *et al.* on multipliers. In 1995, Ruan showed that questions of amenability, a Banach algebra property, can be resolved by tweaking that property in a way that takes operator space structures into account, which leads to the notion of *operator amenability*; indeed, he showed that $A(G)$ is operator amenable precisely when G is amenable [59]. Ruan's work led to more intense research into the operator space structure of $A(G)$, for example, see the work of Forrest, Kanuith, Lau and Spronk [27] on the complemented ideal problem, and work of Ilie and Spronk [39] on the structure of homomorphisms.

M. Neufang spoke on his impressive work with Junge and Ruan [40] on completely bounded multipliers of locally compact quantum groups. This extends theory pioneered by Haagerup in the $A(G)$ case, and Størmer and Ghahramani in the 'dual' group/measure algebra case. The present work unifies the aforementioned cases, and provides new insight into cases with less commutativity. This work has promoted further work between Hu, Neufang, and Ruan, who now have several papers (e.g., [38]) on various Banach algebras related to these multipliers, and on multipliers in a more classical setting. Following progress made at this meeting, Neufang was able to conduct work with his student Kalantar on defining and characterizing in various ways an intrinsic group for a locally compact quantum group, providing a beautiful analogue of the intrinsic group found in the theory of algebraic quantum groups. They also describe an invariant which is a certain subgroup of the torus, and in fact coincides with the latter in the case of Woronowicz's $SU_q(2)$ group. Promising insights into classification and structure of general locally compact quantum groups will stem from these efforts.

While the characterization of operator amenability of $A(G)$ was established by Ruan in 1995, the characterizations of other forms lagged by several years. Spronk in 2002 [63] showed that $A(G)$ is always *operator weak amenability*, and Forrest and Runde [26] characterized the amenability of $A(G)$ in 2004.

E. Samei presented on his work [28] with Forrest and Spronk, addressing such questions on certain Fourier algebras of symmetric spaces. Since this meeting there has been exciting progress. Critically using work presented at the meeting, Forrest, Samei, and Spronk [29] have characterized weak amenability of $A(G)$ for compact groups G. It remains an open question as to whether $A(G)$ is weakly amenable for various groups lacking a non-abelian connected compact subgroup; for example $G = SL_2(\mathbb{R})$, the $ax + b$ group, or any of the Heisenberg groups.

M. Monfared reported on his work [37] with Hu and Traynor on the *character amenability* of Banach algebras, extending work on a concept introduced and studied recently by Kanuith, Lau, and Pym and by Monfared. Based on consultations at the meeting, Runde and Spronk were able to improve one of their main results on the Fourier–Stieltjes algebra $B(G)$; the paper [60] considers the operator amenability of $B(G)$. However, it is an open question to classify the groups G for which $B(G)$ is operator amenable. When does $B(G)$ admit a point derivation?

The *Herz–Figá-Talamanca algebras* $A_p(G)$ can be defined for any $1 < p < \infty$. This class includes the Fourier algebras $A(G)$ at $p = 2$. Without the theory of von Neumann algebras associated to them, the algebras $A_p(G)$, in general, have a much more subtle theory than does $A(G)$. **V. Runde** reported on his work with Lambert and Neufang [47] establishing an operator space structure on $A_p(G)$ which allows a generalization of Ruan's amenability theorem. In new work, conducted in part at this meeting, Neufang and Runde have gained further insights into the fine structures of $A_p(G)$. Open questions remain as to whether the algebras $A_p(G)$, for amenable G, admit a nice homomorphism theorem, such as was proved for $A(G)$ by Cohen for abelian G and Ilie and Spronk for general amenable G.

N. Spronk reported on his generalization [64] of Feichtinger's Segal algebras from abelian groups to the general locally compact case, presented in a Fourier-algebra context. Through the meeting, he was invited to France to meet with Ludwig, and they found a minimality condition characterizing this algebra in the dual group algebra setting. This suggests that a definition for Feichtinger's Segal algebra for locally compact quantum groups may be possible.

Banach algebras and other topics

The lecture of **G. Dales** involved the topological centres of some Banach algebras.

Let A be a Banach algebra, and regard A as a closed subspace of its second dual A''. Then there are two natural products on A''; they are called the *first* and *second Arens products*, and are denoted by $\Box$ and $\Diamond$, respectively. We briefly recall the definitions. As usual, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\Phi \in A''$, define $\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle$ and $\langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle$ for $a \in A$, and, for $\Phi, \Psi \in A''$, define

$$\langle \Phi\Box\Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle\,, \quad \langle \Phi \Diamond \Psi,\, \lambda\rangle = \langle \Psi,\, \lambda \cdot \Phi\rangle \quad (\lambda \in A')\,.$$

Then $(A'', \Box)$ and $(A'', \Diamond)$ are both Banach algebras containing A as a closed subalgebra. The *left topological centres* of A'' is defined by

$$\mathfrak{Z}_t^{(\ell)}(A'') \quad = \quad \{\Phi \in A'' : \Phi\Box\Psi = \Phi \Diamond \Psi \ \ (\Psi \in A'')\}\,,$$

and similarly for the *right topological centre* $\mathfrak{Z}_t^{(r)}(A'')$. See [18, 20, 48, 49, 50, 51] for extensive discussions of these centres.

Let A be a Banach algebra. Then A is *Arens regular* if $\mathfrak{Z}_t^{(\ell)}(A'') = \mathfrak{Z}_t^{(r)}(A'') = A''$; *left strongly Arens irregular* if $\mathfrak{Z}_t^{(\ell)}(A'') = A$, *right strongly Arens irregular* if $\mathfrak{Z}_t^{(r)}(A'') = A$, and *strongly Arens irregular* if A is both left and right strongly Arens irregular. A subset V of A'' is *determining for the left topological centre* of A'' if $\Phi \in A$ whenever $\Phi \in A''$ and $\Phi \Box \Psi = \Phi \Diamond \Psi \ \ (\Psi \in V)$.

For example all C^*-algebras are Arens regular, but each group algebra $L^1(G)$ is strongly Arens irregular. There has been recent interest in improving the latter result by finding 'small' sets that are determining for the left topological centre of $L^1(G)''$.

Let S be a cancellative semigroup. Then it is shown in [20] that certain subsets of βS of cardinality 2 are determining for the left topological centre of $\ell^1(S)''$. The lecture, based on [17], discussed analogous results for various weighted convolution algebras of the form $\ell^1(S, \omega)$; see also [51]. There are several open questions in [20] and [17]; here is one from [17]. Is there a weight ω on $\mathbb{R}^+$ such that $\ell^1(\mathbb{R}^+, \omega)$ is Arens regular?

Current research is given in [21], where many related results are obtained. For example it is shown that, for each locally compact group G, the spectrum Φ of $L^\infty(G)$ is determining for the left topological centre of $L^1(G)''$. Here are two questions that are so far unresolved in [21]. (1) Is there a finite subset of Φ that is determining for the left topological centre of $L^1(G)''$? (2) Is the related measure algebra $M(G)$ strongly Arens regular for each locally compact group G? This is shown for non-compact groups G (of non-measurable cardinality) in [50], but it is open for the case where $G = \mathbb{T}$.

The lecture of **V. Paulsen** involved the projectivity and injectivity of C^*-algebras and G-maps.

There is a well-known contra-variant functor that connects compact spaces and abelian, unital C^*-algebras. Thus many results on the injectivity of C^*-algebras correspond to results about the projectivity of compact spaces. Gleason's classical theorem is central here. Paulsen gave an attractive, simple, and complete expositon of these notions, based on [34].

In the second part of the lecture, the above notions were generalized to a dynamical situation. Let G be a discrete group. An *action of G on a topological space X* is a homomorphism of G into the group of homeomorphisms of X that sends e_G to the identity map. Now X is a *G-space*. The notions of *G-cover* and *G-projective*, etc., are defined by analogy with the standard definitions. The definition of a *G-projective cover* requires care because the 'obvious' definition does not work.

The aim of the authors was to prove that every G-space has a G-projective cover, seeking to duplicate the theory of the first paragraph. This is an open question, and it leads to several interesting questions discussed in [34].

The authors do show that certain 'minimal' G-spaces have G-projective covers, and they derive various properties of G-projectivity that are related to topics in topological dynamics. For this, they use some results on the algebra of βG; these results were taken from the monograph [36], which was also refered to in some other talks at the meeting. Some of the questions raised in [34] concern amenable groups, another favourite topic of the meeting.

The lecture of **L. Turowska** was on operator multipliers.

The study of Schur multipliers has its origins in the work of Schur in the early $20^{\text{th.}}$ century. These objects have a simple definition: a bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is a *Schur multiplier* if, whenever a matrix $(a_{ij})_{i,j\in\mathbb{N}}$ gives rise to a (bounded) transformation of the space l_2, the matrix $(\varphi(i,j)a_{ij})_{i,j\in\mathbb{N}}$ does so as well. A characterization of Schur multipliers was given by Grothendieck in his *Résumé*: Schur multipliers are precisely the functions φ of the form $\varphi(i,j) = \sum_{k=1}^\infty a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ are such that $\sup_i \sum_{k=1}^\infty |a_k(i)|^2 < \infty$ and $\sup_j \sum_{k=1}^\infty |b_k(j)|^2 < \infty$. Schur multipliers have had many important applications in analysis, see e.g. [3], [23], and [56]. One of the forms of the celebrated Grothendieck inequality can be given in terms of these objects [56].

The lecture described generalization of an approach of Birman and Solomyak to the multi-dimensional setting, so extending many results known for classical Schur multipliers to ones about operator multipliers.

First Turowska introduced multi-dimensional Schur multipliers imposing some metric conditions as for the usual ('continuous') Schur multipliers and characterized them as elements of the *extended Haagerup tensor product*, generalizing results by Grothendieck and Peller. The result may be useful in connection with the theory of multi-dimensional operator integrals and their applications to the differentiation theory of operator functions and in the theory of perturbation.

Among other results she established a non-commutative and multi-dimensional version of the characterisation of Grothendieck and Peller which shows that the universal multipliers (i.e., multipliers with respect to any pair of representations) can be obtained as a certain weak limit of elements of the algebraic tensor product of the corresponding C^*-algebras with uniformly bounded Haagerup tensor norm. This was formulated as an open problem in [44], and is a generalization of the Grothendieck theorem to non-commutative multipliers.

A project discussed at the workshop concerns the study of compactness properties of operator multipliers. Schur multipliers φ whose associated linear operator $S_\varphi((a_{ij})) = (\varphi(i,j)a_{ij})$ is compact were studied by Hladnik. The notion of complete compactness is an operator space version of compactness which was defined and studied by Saar and Webster. A classification of completely compact universal operator was obtained.

The relations between completely compact and compact multipliers and between completely compact maps and compact maps are not fully understand so far. It was proved that the inclusion of completely compact multipliers in the set of compact ones is strict in general. To formulate necessary and sufficient conditions about automatic complete compactness of compact multipliers will be challenging. Some other interesting questions about multipliers which have to be investigated are: the connection between the space of the Fourier transforms of n-measures, completely bounded multipliers of the multidimensional Fourier algebra, and the space of multidimensional Schur multipliers; the property of closability of multipliers, important in connection with mathematical physics; factorisation of bounded multipliers and its connection with the study of means (geometric, algebraic, harmonic, and others) of Hilbert space operators.

The lecture of **Y. Zhang** concerned the approximate amenability of Banach algebras.

Let A be a Banach algebra A. Then: a continuous derivation $D : A \to X$ is *approximately inner* if there exists a net $(\xi_\nu) \subset X$ such that, for each $a \in A$, $D(a) = \lim_\nu (a \cdot \xi_\nu - \xi_\nu \cdot a)$; A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is *approximately inner*.

This definition and many variants were introduced by Ghahramani, Loy, and Zhang [30, 33]; see also [11, 7], for example. Many results about these properties are now known; for example, the Banach sequence algebras ℓ^p are not approximately amenable [22].

The lecture discussed Segal algebras. Let G be a locally compact group. A *Segal algebra* S on G is a dense left ideal of the group algebra $L^1(G)$ such that $(S, \|\cdot\|)$ is a Banach algebra for a norm $\|\cdot\|$, where $\|L_x f\| = \|f\| \geq \|f\|_1$ for $f \in S$ and $x \in G$. Of course, $L^1(G)$ itself is amenable, and hence approximately amenable. The conjecture is that any proper Segal algebra fails to be approximately amenable. This is proved in some cases in [32]; see also [19] for some further recent results, where the authors concentrate on the case where $G = \mathbb{T}$.

Related to this problem, it is shown in [11] that a Segal algebra on a SIN group is always approximately permanently weakly amenable.

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Chapter 29

Loss of compactness in nonlinear PDE: Recent trends (07w5087)

Aug 26 - Aug 31, 2007

Organizer(s): Pierpaolo Esposito (Universitaegli Studi Roma Tre), Frank Pacard (Université Paris 12-val de Marne), Gabriella Tarantello (Universita' di Roma Tor Vergata)

Overview of the Field

In the study of nonlinear elliptic PDEs, variational and topological methods are the essential tools to attack the existence question. They essentially rely on some compactness properties of the set of solutions or quasi-solutions. As we explain below, such properties fail in general for a large class of interesting problems which are relevant in physical, biological and theoretical models. This fact has opened in the sixties a very rich line of research which is far from being exhausted yet. In this context, there are several typical issues which we can roughly resume as follows:

- the basic description of the way a specific elliptic PDE exhibits a loss of compactness;
- the identification of the "good" situations where compactness is recovered and existence results can be established directly;
- a deeper description of the asymptotic behavior in the "bad" cases where usually both existence and non-existence can occurr;
- the construction of explicit solutions with a non-compact behavior through a combination of perturbative techniques and variational/topological devices in some specific "bad" situation.

The most famous and paradigmatic problem in this context is represented by the Yamabe equation in conformal geometry. On a compact $n-$dimensional manifold M, $n \geq 3$, with a background metric g_0, the problem of finding a conformal metric g to g_0 is equivalent to solve

$$-\frac{4(n-1)}{n-2}\Delta_{g_0} u + S_{g_0} u = c u^{\frac{n+2}{n-2}} \qquad \text{in } M, \tag{29.1}$$

for some $c = 0, \pm 1$ depending on the background metric g_0. Here, Δ_{g_0} is the Laplace-Beltrami operator with respect to g_0 and S_{g_0} is the scalar curvature of g_0, defined as a suitable trace of the Riemann tensor Riem_{g_0} of g_0.

Yamabe [22] proposed the following method to attack the existence. First, we replace the nonlinear term $u^{\frac{n+2}{n-2}}$ with $u^{\frac{n+2}{n-2}-\epsilon}$, $\epsilon > 0$. Compactness is recovered for every $\epsilon > 0$ yielding to a "ground-state" solution u_ϵ by standard variational methods. The solution of (29.1) is then obtained as the limit of u_ϵ as $\epsilon \to 0$. Yamabe in [22] claimed to be able to carry out this limiting procedure on u_ϵ.

Later, Trudinger [20] found a gap in Yamabe's proof, closely related to non-compactness phenomena for (29.1). In fact, not every solutions sequence u_ϵ has goood compactness properties, and the key point is to identify the "energy" levels where compactness might fail (the "energy" functional is here the associated Rayleigh quotient). In particular, the first "energy" threshold can be computed explicitly as $S^{\frac{n}{2}}$, where S is the Sobolev constant S of the embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. Indeed, whenever $\|u_\epsilon\|_\infty \to +\infty$ as $\epsilon \to 0$, around the maximum points x_ϵ of u_ϵ the sequence u_ϵ has an asymptotic profile given by a solution U of

$$\begin{cases} -\Delta U = cU^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \\ 0 < U \leq U(0) = 1 & \text{in } \mathbb{R}^n. \end{cases} \tag{29.2}$$

For $c = 0, -1$ problem (29.2) does not possess solutions. Trudinger [20] solved (29.1) for compact manifolds with non-positive Yamabe invariant ($c = 0, -1$) where the sign of c in (29.2) prevents non-compactenss phenomena.

In the difficult case of positive Yamabe invariant $c = 1$, the problem remained open for several years and became popular as the Yamabe conjecture. When $c = 1$ all the solutions of (29.2) are known, coincide with the extremals of the Sobolev inequality and have the same "energy" $S^{\frac{n}{2}}$. The limiting procedure on u_ϵ is still effective for $c = 1$ and produces a positive solution to (29.1) whenever the "energy" level of u_ϵ does not approach the first bad level $S^{\frac{n}{2}}$. Since the flat space $\mathbb{R}^n$ and the round sphere (S^n, h) are in correspondence through the stereographic projection, the key point is how to "measure" the difference between the geometry of a manifold (M, g) and that of (S^n, h). In this respect, the complete resolution has been given in two steps by Aubin [2] in $'76$ and then by Schoen [17] in $'84$.

In the Euclidean space $\mathbb{R}^n$, the Yamabe problem takes the simpler form (29.2) and the solution set is well understood. On a bounded domain $\Omega \subset \mathbb{R}^n$, the Yamabe problem can be supplemented by a Dirichlet boundary condition:

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{29.3}$$

The "ground state" energy of (29.3) is independent of Ω and coincides with $S^{\frac{n}{2}}$–the one on $\mathbb{R}^n$. No hope to produce solutions with the Yamabe procedure. The idea of Brezis and Nirenberg [8] was to introduce a perturbing term λu^q, $1 < q < \frac{n+2}{n-2}$, to see a similar effect as in the Yamabe problem. This line of research has been continued later by several authors along the last twenty years, but the results have not had a deep scientific impact due to the limited relevance of such class of PDEs.

We won't discuss here other very interesting equations in nonlinear PDEs, where similar non-compactness phenomena arise, such as the prescribed $H-$curvature problem for surfaces with given boundary [6] and the problem of harmonic maps [16].

More generally, the identification of a limiting problem and the knowledge of its solution set give insights on non-compact sequences of solutions. To the above list of typical issues we then add:

- identification of the limiting problem and the corresponding solutions set.

The Yamabe problem is the higher-dimensional version of the constant Gauss curvature problem for compact surfaces– referred to as the Uniformization Theorem– which can be written as the two-dimensional equation

$$-\Delta_{g_0} u + K_{g_0} = e^u \qquad \text{in } S,$$

where K_{g_0} denotes the Gauss curvature of g_0. The statistical mechanics of point vortices in the mean field limit leads to variants of it:

on a compact surface S

$$-\Delta_{g_0} u = \lambda \left(\frac{Ve^u}{\int_S Ve^u} - \frac{1}{\text{vol}\, S} \right) \qquad \text{in } S, \tag{29.4}$$

and on a bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = \lambda \frac{Ve^u}{\int_\Omega Ve^u} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{29.5}$$

where $\lambda > 0$. When $\inf_\Omega V > 0$ a first aymptotic description of non-compact sequences of solutions to (29.5) has been given by Brezis and Merle [7] without assuming any boundary conditions. Additional information (Dirichlet boundary conditions or compact surfaces without boundary) allow a complete picture of the good/bad situations in terms of λ (and not in terms of "energy" levels): solutions with λ away from the set $8\pi\mathbb{N}$ form a compact set in a strong norm, while for $\lambda \to 8\pi k$ non-compact sequences can generally be found. The limiting problem, which is responsible for such a quantization of critical situations, reads in this case as

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ U \leq U(0) = 0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u dx < +\infty. \end{cases} \tag{29.6}$$

Chen and Lin [11, 12] have derived very precise asymptotic estimates on non-compact sequences and the Leray-Schauder degree d_λ, $\lambda > 0$, of the associated nonlinear map has been explicitly computed. Fine existence results readily follow whenever $d_\lambda \neq 0$. A somehow related problem (for example, the limiting equation is the same) is the Euler-Lagrange equations associated to the Moser-Trudinger functional on $H_0^1(\Omega)$:

$$\begin{cases} -\Delta u = \lambda u e^{u^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{29.7}$$

where $\lambda > 0$. See [1, 14].

We conclude this overview with the mention of the huge class of singularly perturbed problems. The scalar case is of interest in the study of standing waves of Schrödinger operators in the semi-classical limit [15], in the study of phase transitions [13] (the so-called Allen-Cahn equation) and in the Gierer-Meinhardt model for the dynamics of biological populations. The complex case has been largely studied in connection with the classical Ginzburg-Landau theory in super-conductivity for type II super-conductors [4].

Recent Developments and Open Problems

Despite of the complete resolution of the Yamabe problem, there are still interesting questions to be addressed. Schoen [18] attempts to describe the set of metrics having constant curvature in a given conformal class, with an interest towards a-priori estimates and multiplicity results of the Yamabe equation. In particular, for locally conformally flat compact manifolds M which are not conformally equivalent to the round sphere he shows that the Yamabe solutions in a given conformal class with prescribed volume form a compact set in $C^{2,\alpha}(M)$. He leaves open the case of non locally conformally flat compact manifolds, in literature referred to as the Schoen conjecture. Later, several authors (Druet, Li, Marques, Zhu, Zhang) have established the validity of this conjecture in low dimension $n \leq 7$ and in every dimension under an additional assumption on g. Brendle has constructed counter-examples to this conjecture for dimensions $n \geq 52$, and it is still open to know the exact dimensions to have the validity of the Schoen conjecture.

The constant Gauss curvature problem extends to a fourth order equation on $4-$dimensional compact manifold, as well as the Yamabe problem to $n-$dimensional compact manifolds with $n \geq 5$. On $4-$manifolds, Paneitz in the first '80 discovered a fourth order operator P_g having the same transformation law of the Laplace-Beltrami operator Δ_g under conformal changes of the metric. The operator P_g is built with a principal part given by Δ_g^2. A notion of curvature –the so-called $Q-$curvature– was then introduced in terms of S_g

and the Ricci tensor Ric_g of g. The problem of prescribing a constant $Q-$curvature on a compact $4-$manifold leads to a fourth-order elliptic PDE with esponential nonlinearity:

$$P_{g_0}u + 8Q_{g_0} = 8ce^u \qquad \text{in } M, \tag{29.8}$$

where $c = 0, \pm 1$ depends on the sign of $\int_M Q_{g_0} dv(g_0)$. The interest in solving this problem is in the same spirit of the Uniformization Theorem and relies on a $4-$dimensional Gauss-Bonnet formula:

$$\int_M (Q_g + \frac{|W_g|^2}{8}) dv(g) = 4\pi^2 \chi(M),$$

where W_g is the Weyl tensor of (M, g) and $\chi(M)$ is the Euler characteristic of M. Existence results are available for compact manifolds with non-negative Yamabe invariant and

$$0 \le \int_M Q_{g_0} dv(g_0) < 8\pi^2.$$

It is completely open such a question when $\int_M Q_{g_0} dv(g_0) \ge 8\pi^2$.

In the flat case, equation (29.8) can be considered also on bouded domains $\Omega \subset \mathbb{R}^4$ in the mean field form

$$\Delta^2 u = \lambda \frac{Ve^u}{\int_\Omega Ve^u} \qquad \text{in } \Omega \tag{29.9}$$

and possibly supplemented by either Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

or Navier boundary conditions

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega.$$

Similarly as in two dimensions, we can consider the Euler-Lagrange equations associated to the Moser-Trudinger functional:

$$\Delta^2 u = \lambda u e^{u^2} \qquad \text{in } \Omega, \tag{29.10}$$

with either Dirichlet or Navier boundary conditions. Quantization issues for (29.9) and (29.10) are not known, and the “bad” situations have not been identified yet.

Another direction has been recently pursued in conformal geometry. Letting σ_k the $k-$th elementary symmetric polinomial, we have that S_g coincides with (a multiple of) $\sigma_1(\lambda_1(g), \dots, \lambda_n(g))$, where $\lambda_1(g), \dots, \lambda_n(g)$ are the ordered eigenvalues of the Schouten tensor of g –defined in terms of the Ricci tensor Ric_g of g and $S_g g$. Viaclowsky [21] has proposed the so-called $k-$Yamabe problem, which corresponds to finding a metric g in a given conformal class with $\sigma_k(\lambda_1(g), \dots, \lambda_n(g))$ being a constant. The case $k = 2$ has been considered by Chang, Gursky and Yang [10], and in its Euclidean counter-part by Caffarelli, Nirenberg and Spruck [9] for every $k = 1, \dots, n$. The geometric implications of solving the $k-$Yamabe problem are strong and make their study very interesting. Analytically, the problem is delicate and is still under investigation.

In several theories in super-conductivity (for example, in the Ginzburg-Landau and Chern-Simons models), there is a very special physical regime, referred to as the selfdual regime, where the Euler-Lagrange equations considerably simplify. The function $u = \ln |\phi|$ –where the Higgs complex function ϕ is an order parameter measuring the superconductivity state in the sample Ω– satisfies exactly equations like (29.4) or (29.5) with an additional singular source term supported at the zero set of ϕ. In all these models, such a set is always composed by finitely many points –the so-called vortices– with integer multiplicities.

Since the presence of a singular source term can be re-absorbed into a vanishing potential V, in many physical applications there is a definite interest in considering potentials V in (29.4) or (29.5) which vanish at finitely many points in Ω. From the physical point of view, the simplest situation of interest is a potential V in the form

$$V = |x-p|^\alpha K\,, \qquad \inf_\Omega K > 0$$

with $\alpha \in \mathbb{N}$. As when $\inf\limits_\Omega V > 0$, there is a quantization of the "bad" levels in the form

$$\lambda \in 8\pi\mathbb{N}$$

as shown by Tarantello [19]. In the easier case $\alpha \notin \mathbb{N}$ a similar quantization does hold according to [19] and sharp asymptotic estimates are available in [3]. When $\alpha \in \mathbb{N}$ these estimates are still missing, no spike solutions are known and a formula for the Leray-Schauder degree is not available yet.

In the context of Ginzburg-Landau theories, physicists have proposed several other models to take into account specific effects which have physical relevance. For example, an anisotropic Ginzburg-Landau model does not adequately describe the magnetic properties of a layered high-temperature superconductor. The interest in layered structures relies on the fact that many of high-temperature superconductors are obtained as highly anisotropic crystalline materials composed of stacks of copper oxide superconducting planes separated by insulating or weakly superconducting material. The Lawrence-Doniach model has been instead proposed to properly capture the effect of the layered structure, where stacks of parallel superconducting planes are coupled via the Josephson interaction. This effect is particularly noticable when an external "in-plane" field is applied to the sample, i.e. the magnetic field is parallel to the superconducting planes. As $\epsilon \to 0$ it is not clear the behavior of the Lawrence-Doniach model.

More generally, for singularly perturbed problems it is often possible to construct for small $\epsilon > 0$ solutions which are strongly localized around finitely many concentration points. They are obtained in a constructive way as small perturbations of a specific family of approximating solutions, built on as a suitable gluing of several local profiles of the associated limiting problem centered at such concentration points. The concentration points can't be chosen freely but are prescribed by the problem under consideration.

In the last years, in collaboration with several authors, Malchiodi has been able to construct solutions localized around higher dimensional concentration sets Γ. The limiting problem, which provides the good local profiles to build the approximating family, has to be considered in $\mathbb{R}^{n-\dim\Gamma}$, and Γ is prescribed by some geometric condition (when $\dim\Gamma = 1$, Γ is usually a geodesic with respect to some distance function). There are many analytical difficulties which have been overcome and clarified by Malchiodi's work. His ideas could be fruitfully applied in many problems (not only for the singularly perturbed ones) to construct solutions which exhibit a concentration and/or a blow-up behavior on a manifold Γ of positive dimension (in case the situation $\Gamma = \{p_1, \ldots, p_k\}$ is already well understood).

Scientific Progress Made

The speakers of our workshop have reported about the results they have recently established. They have given partial/complete answers to the open questions in our field as listed in the previous section. We will report in the sequel only about the most striking achievements and we won't attempt to give a complete account of the talks in our workshop.

In this respect, the first result to quote is the complete resolution of the Schoen conjecture as reported by Marques. In collaboration with Khuri and Schoen, they establish compactness of the set of metrics with volume one and constant scalar curvature in a given conformal class for all the non locally conformally flat compact manifolds M of dimension $n \leq 24$. In this context, the counter-examples of Brendle for $n \geq 52$ are extended by Marques and Brendle to all the remaining dimensions $25 \leq n \leq 51$. The picture is then complete in the compact case: the Schoen conjecture is valid on locally conformally flat manifolds (different from the round sphere) and on non locally conformally flat manifolds of dimension $n \leq 24$, and is false in general in the remaining situations.

In a joint work with Djadli, Malchiodi solves the constant $Q-$curvature problem on compact $4-$manifolds with non-negative Yamabe invariant and $\int_M Q_{g_0} dv(g_0) \geq 0$. For the associated "energy" functional J, by looking at the "energy" sublevels $J^a = \{J \leq a\}$ for a very negative, they see how to define a suitable min-max scheme to produce a critical "energy" level c. They also introduce in the equation a parameter λ so that the original problem corresponds to $\lambda = \int_M Q_{g_0} dv(g_0)$. To recover compactness, taking advantage of the Struwe monotonicity trick, they show the validity of the Palais-Smale condition for λ in a small dense subset aorund $\int_M Q_{g_0} dv(g_0)$ and find an associated solution. By a previous result of Malchiodi, compactness does hold when λ is far from $8\pi^2\mathbb{N}$ and, by a limiting procedure, an existence result is deduced for $\int_M Q_{g_0} dv(g_0) \in [0, +\infty) \setminus 8\pi^2\mathbb{N}$. Malchiodi has also shown that a Morse theoretical approach, similar in spirit to the one just described, works also for (29.4) when $\inf\limits_S V > 0$. It yields to existence results and simplifies the proof of a degree formula for d_λ obtained by Chen and Lin [12]. Let us also point out the contribution of the talk of Lucia on a general deformation lemma for a class of functionals that do not satisfy Palais-Smal condition, including the ones arising in the stuy of the mean field equation (29.4).

As far as (29.9), Robert in his talk has given a beautiful and complete description of the general state-of-art. When $\inf\limits_\Omega V > 0$, in dimension two a non-compact sequence u_n always satisfies a concentration property of the nonlinear term which is equivalent to have

$$u_n - \int_\Omega V e^{u_n} dx \to -\infty \qquad \text{in } C_{\text{loc}}(\Omega \setminus \{p_1, \ldots, p_k\}) \text{ as } n \to +\infty,$$

for a finite number of points $p_1, \ldots, p_k \in \Omega$. Surprisingly, it generally does not hold in dimension four and the situation can become quite weird. However, adding a L^1-bound on Δu_n or a Dirichlet/Navier boundary condition, the situation becomes similar to that of the two-dimensional case.

Wei has reported about a later work, in collaboration with Lin, Robert and Wang, concerning (29.9) on a domain $\Omega \subset \mathbb{R}^4$ with Dirichlet or Navier boundary conditions. They have established compactness for λ in compact sets of $[0, +\infty) \setminus 64\pi^2\mathbb{N}$, sharp asymptotic estimates for $\lambda \to 64\pi^2\mathbb{N}$ as well as a degree-counting formula for d_λ, in the same line of [11, 12]. Similarly, Struwe has reported on a quantization property for (29.10) with Navier boundary conditions: for positive solutions $u_k \rightharpoonup 0$ weakly in $H^2(\Omega)$ the concentration energy

$$\Lambda = \lim_{k \to +\infty} \int_\Omega |\Delta u_k|^2 dx$$

is quantized in integer multiples of $\Lambda_1 = 16\pi^2$. A discussion grows out on the possibility of obtaining for (29.10) sharp asymptotic estimates and a degree formula as for (29.9).

On the $k-$Yamabe problem, we can point out the talk of Ge on an analytic foundation for the fully non-linear equation

$$\frac{\sigma_2(\lambda_1(g), \ldots, \lambda_n(g))}{\sigma_1(\lambda_1(g), \ldots, \lambda_n(g))} = f$$

on compact manifolds M with positive Yamabe invariant. As an application, in a joint work with Wang and Lin, they prove that, if a compact $3-$dimensional manifold M admits a metric g with positive scalar curvature $S_g > 0$ and $\int_M \sigma_2(\lambda_1(g), \lambda_2(g), \lambda_3(g)) dv(g) > 0$, then it is topologically a quotient of the sphere. In the fully nonlinear context, Y.Y. Li has explained his contribution in terms of a Liouville-type result for entire solutions of general conformally invariant fully nonlinear elliptic equations of second order, motivated by the study of the limiting problem along non-compact sequences. The interest is strictly related to a-priori estimates for this class of problems on compact manifolds M.

Thanks to the quantization property in [19] and to the sharp asymptotic estimates in [3], Lin has been able to compute a degree formula for (29.4) with a potential V in the form

$$V = |x - p|^\alpha K, \qquad \inf_S K > 0$$

with $\alpha \notin \mathbb{N}$. In his talk, he gives the explicit expression for the degree formula when $\alpha \notin \mathbb{N}$ and describes how to get, with a limiting procedure, a similar degree formula for $\alpha \in \mathbb{N}$. In this way, for $\alpha \in \mathbb{N}$ it is possible to overcome the difficult identification of all the possible non-compact sequences (and their contribution to the changes of the degree when λ crosses the values in $8\pi\mathbb{N}$). A discussion grows out on how try to establish sharp asymptotic estimates for $\alpha \in \mathbb{N}$ and how to construct explicit non-compact sequences. In this respect, del Pino and Musso carry to the audience attention a recent partial result in collaboration with Esposito on non-compact sequences for simply connected domains. A general result should be in order via a suitable gluing argument.

Sandier in his talk presents the rigourous derivation of an anisotropic Ginzurg-Landau theory as the limit, in a certain regime, of the Lawrence-Doniach model. In particular, if the interlayer distance goes to zero no faster than ϵ, an extension of the order parameter between the layers converges weakly to the canonical harmonic map away from the vortices, and the magnetic potential converges to a vector field whose magnetic field satisfies the anisotropic London equation. In an ongoing joint work with Alama and Bronsard, they also find that anisotropic $3D$ models have interesting Gamma limits as $\epsilon \to 0$.

In collaboration with Bronsard and Millot, Alama is interested to describe the asymptotic behavior as $\epsilon \to 0$ of the energy minimizers of a two-dimensional superconductor under the effect of an external applied magnetic field. They are interested in determining the number and the distribution of the vortices, which are defects of the superconductive state and appear in the model as quantized singularities. For an external applied magnetic field near the "lower critical field", as $\epsilon \to 0$ these vortices concentrate along a curve determined by a classcial problem from potential theory. Here, the "lower critical field" represents the critical value of an external applied magnetic field for which vortices first appear in the superconductor.

Solutions of the Allen-Cahn equation with finite energy as $\epsilon \to 0$ admit a limiting profile given as an entire solution of the Allen-Cahn equation (with $\epsilon = 1$) having at most a growth R^{n-1} of the energy on $B_R(0)$, as $R \to +\infty$. Del Pino presents the results of some joint works with Kowalczyk, Pacard and Wei, where a solution of this type is constructed having a finite number of nearly parallel transition layers. The solution is constructed as a gluing of one-dimensional profiles with a single transition located very far appart one to each other. Similarly, for the stationary nonlinear Schrödinger equation mulitple bump lines are found, while Toda system is shown to rule out the asymptotic shape of these transition lines.

The last talk we would like to quote concerns solutions which concentrate and blow-up along a curve. In collaboration with del Pino and Pacard, Musso considers slightly sub-critical Yamabe equation (29.3) in a domain $\Omega \subset \mathbb{R}^n$. Since they are interested in boundary concentration and $\partial\Omega$ is a $(n-1)-$dimensional manifold, the right exponent in the Yamabe equation is not $\frac{n+2}{n-2}$ but the critical Sobolev exponent in dimension $n-1$, i.e. $\frac{n+1}{n-3}$. For this problem, they construct a family of solutions whose energy density concentrates as a Dirac line measure on Γ, where Γ is a closed geodesic in $\partial\Omega$ with negative curvature and satisfying some non-degeneracy condition.

Outcome of the Meeting

The meeting has given the opportunity of all the participants to exchange ideas and to communicate new results and research directions in this field. As planned in our proposal, we have had the big opportunity to gather junior and senior scientists, and let them the possibility of a fruitful exchange of experiences.

We hope to have stimulated new interactions in our mathematical community, whose revenues will certainly become manifest in next years.

List of Participants

Alama, Stanley (McMaster University)
Almaraz, Sergio (IMPA)
Davila, Juan (CMM & DIM University of Chile)
del Pino, Manuel (Universidad de Chile)
Druet, Olivier (Ecole Normale Supérieure de Lyon & CNRS)
Ge, Yuxin (University of Paris 12)
Ghoussoub, Nassif (BIRS)
Gladiali, Francesca (Universita' di Sassari)

Grossi, Massimo (Universita Roma 1)
Li, Yanyan (Rutgers University)
Lin, Chang-Shou (National Taiwan University)
Lucia, Marcello (Universitat zu Koln)
Malchiodi, Andrea (International School for Advanced Studies (SISSA))
Marques, Fernando Coda (IMPA)
Mazzieri, Lorenzo (Scula Normale Pisa)
Musso, Monica (Universidad Católica de Chile)
Pacard, Frank (Université Paris 12-val de Marne)
Pistoia, Angela (Universit La Sapienza di Roma, Dipartimento di Metodi e Modelli Matematici)
Robert, Frederic (Universite de Nice-Sophia Antipolis)
Sandier, Etienne (Universite Paris 12)
Shafrir, Itai (Technion-Israel Institute of Technology)
Stanczy, Robert (Uniwersytet Wroclawski)
Struwe, Michael (ETH Zentrum)
Tarantello, Gabriella (Universita' di Roma Tor Vergata)
Wei, Jun Cheng (Chinese University of Hong Kong)
Zhang, Lei (University of Florida)

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Chapter 30

Hochschild Cohomology of Algebras: Structure and Applications (07w5075)

Sep 02 - Sep 07, 2007

Organizer(s): Luchezar Avramov (University of Nebraska), Ragnar-Olaf Buchweitz (University of Toronto Scarborough), Karin Erdmann (University of Oxford), Jean-Louis Loday (Institut de Recherche Mathématique Avancée), Sarah Witherspoon (Texas A&M University)

Hochschild cohomology of associative algebras is important in many areas of mathematics, such as ring theory, commutative algebra, geometry, noncommutative geometry, representation theory, group theory, mathematical physics, homotopy theory and topology. Usually, associative algebras, occuring naturally, are not semisimple, and to understand their properties, homological methods are absolutely essential. Such algebras can be very different depending on the context, but fortunately, Hochschild cohomology is defined in complete generality, as it is constructed in terms of very basic linear algebra. In spite of this simplicity, it is a unifying concept for different areas. It has important invariance properties, for example it is invariant under derived, stable, and Morita equivalences.

The workshop concentrated on several algebraic aspects of Hochschild cohomology, reflecting connections among homological algebra, commutative algebra, representation theory, group theory and mathematical physics. It brought together experts from several of these fields, promoting an exchange of ideas and potentially new collaborations. Mathematicians using Hochschild cohomology presented recent results, techniques and open problems from their fields. This allowed us to take stock of progress achieved and led to discussions of possible solutions to open problems. A few mathematicians presented applications of Hochschild cohomology in fields outside of algebra proper, encouraging potentially useful interaction within a wider group of users of Hochschild cohomology, and broadening our understanding of the context to which the theory applies and what it all means.

During the week there were 21 talks. There were 41 participants in total, 5 of these Canadians.

Overview of the Field

Let A be an associative algebra over a field k, and M an A-bimodule (equivalently an $A \otimes_k A^{op}$-module where A^{op} is the algebra A with the opposite multiplication). The Hochschild cohomology of A, with coefficients in M, is the graded vector space $\mathrm{HH}^*(A, M) = \mathrm{Ext}^*_{A \otimes_k A^{op}}(A, M)$, where A is considered an $A \otimes_k A^{op}$-module under left and right multiplication. Equivalently, $\mathrm{HH}^*(A, M)$ is given in terms of the bar resolution of A by free $A \otimes_k A^{op}$-modules:

$$\cdots \xrightarrow{\delta_2} A \otimes_k A \otimes_k A \xrightarrow{\delta_1} A \otimes_k A \xrightarrow{\delta_0} A \to 0$$

where $\delta_i(a_0\otimes\cdots\otimes a_{i+1}) = \sum_{j=0}^{i}(-1)^j a_0\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_{i+1}$. Apply the functor $\mathrm{Hom}_{A\otimes_k A^{op}}(-, M)$ to the above sequence and let δ_i^* ($i \geq 0$) denote the induced map. Then $\mathrm{HH}^i(A, M) = \ker(\delta_i^*)/\mathrm{im}(\delta_{i-1}^*)$ and $\mathrm{HH}^*(A, M) = \oplus_{i\geq 0}\mathrm{HH}^i(A, M)$.

The cohomology space $\mathrm{HH}^*(A, A)$ has a cup product, under which it is an associative algebra, and a graded Lie bracket. Further, $\mathrm{HH}^*(A, M)$ is a module for $\mathrm{HH}^*(A, A)$. One wants to understand the structure of $\mathrm{HH}^*(A, A)$ as a graded vector space, as a ring, and as a graded Lie algebra, and similarly to understand its modules $\mathrm{HH}^*(A, M)$. In turn such information about cohomology sheds light on the structure of the algebra A itself and on its bimodules M.

Many of the deep results on the structure of Hochschild homology and cohomology involve methods and results particular to a given field. It is one of the constant challenges to understand the unifying principles or possibilities of transfer from one area to another. We mention as one example the Hodge decomposition of Hochschild (co)homology for commutative algebras in characteristic zero (Quillen, Gerstenhaber-Schack, Loday in the algebraic framework; Kontsevich, Swan in geometry), that has caused new insights in combinatorics, such as the study of Eulerian idempotents, and in Lie theory, where it helps to understand better the nature of the Poincaré-Birkhoff-Witt isomorphism (Reutenauer, Loday, Bergeron-Wolfgang). This decomposition has as well a counterpart for the Ext algebra of the residue field of a commutative ring in positive characteristic (Tate, Assmus, Gulliksen-Levin, Quillen, Gerstenhaber and Schack), the decomposition holds universally for complete intersections (Wolffhardt), and there are decomposition theorems whenever a group action is present. To what extent the decomposition theorem is compatible with the higher multiplicative structure on Hochschild cohomology is an important problem, where a lot of research is currently under way. There are many questions open here, of interest in a broad range of mathematics.

Recent Developments and Open Problems

The vanishing of Hochschild cohomology in high degrees has implications for the structure of algebras; for example for finitely generated commutative algebras it is equivalent to the geometrically motivated notion of smoothness. There is evidence that such implications may play an important role also for noncommutative algebras, but they are not well understood and so bear further investigation. Computations of the ring structure of Hochschild cohomology of various types of algebras have been completed in the past decade by researchers from different fields using specialized techniques. When the Hochschild cohomology ring is finitely generated modulo nilpotents, support varieties of modules can be defined, analogous to those from group cohomology, and this is currently an active area of research. At present it is not clear how to produce systematically algebras with this property, as proofs of finite generation typically depend on difficult techniques and deep results from some specific area. The workshop was an ideal venue for sharing ideas and potentially making further progress.

Vanishing of Hochschild (co)homology in high degrees

For projective commutative algebras of finite type, eventual vanishing of either series of invariants is equivalent to the geometric notion of smoothness, and to finite projective dimension over the enveloping algebra (Avramov, Iyengar, Rodicio, Vigué-Poirrier). However, vanishing of cohomology does not imply vanishing of homology (Han) nor finite global dimension (Buchweitz, Green, Madsen, Solberg). The discovery of very small self-injective algebras with finite-dimensional Hochschild cohomology ring was a great surprise, and clearly, this is a very interesting avenue for further research.

The condition of finite projective dimension has been proposed as a definition for smoothness of noncommutative algebras (van den Bergh), but its properties are largely unknown and its relation to other generalizations of smoothness are not yet completely understood. The joint expertise of the participants of the workshop provided a good setting for discussing smoothness and other geometrically motivated properties of commutative algebras, such as being locally complete intersection, in the context of more general associative algebras.

The ring and Lie algebra structures of Hochschild cohomology

Ideally one might aim for a presentation by generators and relations, though this appears to be a very difficult problem in general. There is recent progress, and various groups of researchers have developed completely different methods. For example, the ring structures of Hochschild cohomology of certain path algebras and related algebras were determined using quiver techniques (Erdmann, Holm, and Snashall). A computation of the Hochschild cohomology ring of a finite group algebra used group cohomology (Siegel and Witherspoon proved a conjecture originally due to Cibils). The Hochschild cohomology rings of invariant subalgebras of Weyl algebras and generalizations were found by adapting techniques from commutative homological algebra (Alev, Farinati, Lambre, Solotar, and Suarez-Alvarez). These are just a few examples of computations of the ring structure of Hochschild cohomology completed within the past ten years.

Varieties for modules

In group representation theory, the support variety of a module is a powerful invariant, for example its dimension is equal to the complexity of the module, and it encapsulates many of the module's homological properties. It has found many applications, and has subsequently been extended, first to restricted enveloping algebras and more recently to finite group schemes and to quantum groups. The very definition of support varieties for these algebras depends on the difficult results that their cohomology algebras are finitely generated.

One would like to have some analogue for more general associative algebras. Recently much work has been done, showing that Hochschild cohomology, which is graded commutative, can serve as a substitute for group cohomology. It was shown that at least for self-injective algebras, many properties of support varieties for group representations have analogues in this more general setting (work by Snashall, Solberg, joined by Erdmann, Holloway, Taillefer). This however needs suitable finite generation properties. The required condition that the quotient of Hochschild cohomology by a nilpotent ideal be finitely generated is conjectured to be true for artin algebras (Snashall and Solberg), and known to be true in some cases. It would be very interesting to establish this conjecture for classes of algebras, especially finite-dimensional Hecke algebras. For these, some more information is available via rank varieties, and some analogue of 'Quillen stratification' might be true. In the context of more general associative algebras, such properties are even less well understood.

For some algebras, all representations or modules have periodic projective resolutions; these include modular group algebras of finite groups with cyclic p-Sylow or quaternion Sylow subgroups on one end of the spectrum, but also, apart from self-injective algebras of finite type, all finite-dimensional preprojective algebras that are of interest for quiver varieties and quantizations of singularities. In all cases known, this phenomenon is explained by periodicity of the Hochschild complex—and then one has in particular finite generation of cohomology.

Hochschild cohomology in related areas

To conclude we mention that in addition to the above algebraic directions of research, many questions arise from the use of Hochschild cohomology in fields less related to algebra that accordingly were not central to this meeting.

Geometric applications of Hochschild homology and cohomology play a significant role in work of Barannikov, Kontsevich, Markarian and Tsygan. One instance would be the connection with graph-complexes à la Kontsevich that was reported on by Burgunder, see below.

There are as well attempts to define more generally Hochschild theory for abelian, exact, or triangulated categories, see for example Lowen's contribution below.

As a specific example of a geometric application, mathematicians studying orbifolds have looked at Hochschild cohomology of the bounded derived category of equivariant sheaves (Baranovsky, Căldăraru, Kaledin) or their quantizations (Etingof, Ginzburg, Kaledin); in case of C*-algebras, not touched upon here, this work centers on the celebrated Baum-Connes conjecture. In the case of an affine space, this Hochschild cohomology ring is simply the Hochschild cohomology of an extension of a function algebra by the group, and thus it may be studied algebraically.

Presentation Highlights

The first day of the workshop consisted of five talks by younger mathematicians who introduced and represented the broad topics into which the remaining talks were organized. We include the abstracts of the talks submitted by the presenters.

Hochschild cohomology and support varieties

Nicole Snashall introduced this topic on the first day, and other talks were given by Petter Andreas Bergh and Henning Krause.

Nicole Snashall (University of Leicester)
The Hochschild cohomology ring modulo nilpotence
For a finite-dimensional algebra Λ, it was conjectured by Snashall and Solberg that the Hochschild cohomology ring of Λ modulo nilpotence is itself a finitely generated algebra. This talk described the current position of this conjecture and its connection to support varieties of modules. Reference was also made as to whether or not it is known that the Hochschild cohomology ring itself is finitely generated in certain cases, as this plays an important role in support varieties for self-injective algebras.

In particular, Snashall gave an overview of what is known for finite-dimensional self-injective algebras, then discussed the cases where Λ is a monomial algebra or is in a class of special biserial algebras which arise from the representation theory of $U_q(\mathfrak{s}l_2)$ (Erdmann, Snashall, Taillefer).

Petter Andreas Bergh (NTNU Trondheim)
Hochschild (co)homology of quantum complete intersections
This is joint work with Karin Erdmann. We construct a minimal projective bimodule resolution for finite dimensional quantum complete intersections of codimension 2. Then we use this resolution to compute the Hochschild homology and cohomology for such an algebra. In particular, we show that when the commutator element is not a root of unity, then the cohomology vanishes in high degrees, while the homology is always nonzero. Thus these algebras provide further counterexamples to “Happel’s question”, a question for which the first counterexample was given by Buchweitz, Green, Madsen and Solberg. On the other hand, the homology of the quantum complete intersections behave in accordance with Han’s conjecture, i.e. the homology version of Happel’s question.

Henning Krause (Universität Paderborn)
Localising subcategories of the stable module category of a finite group
This is a report on recent joint work with Srikanth Iyengar and Dave Benson. We classify the localising subcategories of the stable module category for a finite group. This enables us to prove the telescope conjecture in this context, as well as give a new proof of the tensor product theorem for support varieties.

In my talk I explain the history of this classification problem as well as the strategy of the proof. The challenge is basically to make proper use of the group cohomology ring which acts as a ring of cohomological operators. Thus we reduce this classification to a problem from commutative algebra. The main tools are support varieties and local cohomology. Then we use a similar classification of localising subcategories for the derived category of a commutative noetherian ring, which Neeman obtained some 15 years ago.

Structure of Hochschild cohomology

Srikanth Iyengar introduced this topic, and other talks were given by Thorsten Holm, Maria Julia Redondo, Marco Farinati, Mariano Suarez-Alvarez, and James Zhang.

Srikanth Iyengar (University of Nebraska, Lincoln)
Gorenstein algebras and Hochschild cohomology
This is joint work with L. L. Avramov. A classical result of Hochschild, Kostant, and Rosenberg characterizes smoothness of commutative algebras essentially of finite type over a field in terms of its Hochschild cohomology. I will discuss a similar characterization of the Gorenstein property.

Thorsten Holm (Universität Magdeburg)
Bilinear forms, Hochschild (co-)homology and invariants of derived module categories
This is joint work with C. Bessenrodt and A. Zimmermann. Let A be a symmetric algebra over a perfect field k of positive characteristic p. For such algebras, B. Külshammer introduced, for any n, spaces $T_n(A)$ as those elements of A whose p^n-th power lies in the commutator subspace $K(A)$. He then considered the orthogonal spaces with respect to the symmetrizing bilinear form on the symmetric algebra A. These $T_n(A)^\perp$ are ideals of the center of A, i.e. of the degree 0 Hochschild cohomology of A.

It has been shown by A. Zimmermann that the sequence of these ideals is invariant under derived equivalences of the symmetric k-algebras.

In the talk we first briefly discuss Zimmermann's results and then explain how to extend this theory to arbitrary finite-dimensional, not necessarily symmetric, k-algebras. The way to achieve this is by passing to the trivial extension algebras. In this way we obtain new invariants of the derived module categories of finite-dimensional k-algebras. This can be seen as an extension of the well-known fact that the degree 0 Hochschild homology $A/K(A)$ is invariant under derived equivalence.

We also present recent applications of the above results, e.g. to blocks of group algebras.

María Julia Redondo (Universidad Nacional del Sur)
Hochschild cohomology via incidence algebras
Let A be an associative, finite dimensional algebra over an algebraically closed field k. It is well known that if A is basic then there exists a unique finite quiver Q and a surjective morphism of k-algebras $\nu : kQ \to A$, which is not unique in general, with $I_\nu = \operatorname{Ker} \nu$ admissible. The pair (Q, I_ν) is called a *presentation* of A. Given a presentation (Q, I) of A we associate an incidence algebra $A(\Sigma_\nu)$ and study the connection between their Hochschild cohomology groups. If the chosen presentation is homotopy coherent we define a morphism between the complexes computing these cohomology groups, which induces morphisms $\mathrm{HH}(\Phi^n)$: $\mathrm{HH}^n(A(\Sigma_\nu)) \to \mathrm{HH}^n(A)$. Finally we find conditions for these morphisms to be injective.

Marco Farinati (Universidad de Buenos Aires)
The cohomology of monogenic extensions in the noncommutative setting
We extend the notion of monogenic extension to the noncommutative setting, and we study the Hochschild cohomology ring of such an extension. As an application we complete the computation of the cohomology ring of the rank one Hopf algebras begun by S. M. Burciu and S. J. Witherspoon.

Mariano Suarez-Alvarez (Universidad de Buenos Aires)
Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology
We consider the change-of-rings spectral sequence as it applies to Hochschild cohomology, obtaining a description of the differentials on the first page which relates it to the multiplicative structure on cohomology. Using this information, we are able to completely describe the cohomology structure of monogenic algebras as well as some information on the structure of the cohomology in more general situations.

We also show how to use the spectral sequence to reprove and generalize results of M. Auslander et al. about homological epimorphisms. We derive from this a rather general version of the long exact sequence due to D. Happel for a one-point (co)-extension of a finite dimensional algebra and show how it can be put to use in concrete examples.

James Zhang (University of Washington)
Twisted Hochschild (co)homology for Hopf algebras
This is joint work with K.A. Brown. The Hochschild homology and cohomology groups with coefficients in a suitably twisted free bimodule are shown to be non-zero in the top dimension d, when A is an Artin-Schelter regular noetherian Hopf algebra of global dimension d. (Twisted) Poincaré duality holds in this setting, as is deduced from a theorem of Van den Bergh.

Relations with algebraic geometry

Andrei Căldăraru introduced this topic, and other talks were given by Hubert Flenner, Joseph Lipman, and Amnon Yekutieli.

Andrei Căldăraru (University of Wisconsin, Madison)
Non-flat base change and the orbifold HKR isomorphism
I shall discuss and prove a base change theorem for derived categories of smooth schemes, in the absence of flatness assumptions. As an application I shall present a Hochschild-Kostant-Rosenberg isomorphism for smooth, global quotient orbifolds.

Hubert Flenner (University of Bochum)
Hochschild cohomology of singular spaces
This is a report on joint work with R.-O. Buchweitz.

In analogy with the cotangent complex we introduce the so called (derived) Hochschild complex of a morphism of analytic spaces or schemes; the Hochschild cohomology and homology groups are then the Ext and Tor groups of that complex. We prove that these objects are well defined, extend the known cases, and have the expected functorial and homological properties such as graded commutativity of Hochschild cohomology and existence of the characteristic homomorphism from Hochschild cohomology to the (graded) centre of the derived category.

We further generalize the HKR-decomposition theorem to Hochschild (co-)homology of arbitrary morphisms between complex spaces or schemes over a field of characteristic zero. To be precise, we show that for each such morphism $X \to Y$, the Hochschild complex decomposes naturally in the derived category $D(X)$ into $\bigoplus_{p\geq 0} \mathbb{S}^p(\mathbb{L}_{X/Y}[1])$, the direct sum of the derived symmetric powers of the shifted cotangent complex, a result due to Quillen in the affine case. The proof shows that the decomposition is given explicitly and naturally by the *universal Atiyah–Chern character*, the exponential of the universal Atiyah class. We also give further applications, in particular to the semiregularity map.

Joseph Lipman (Purdue University)
Hochschild homology, the fundamental class of a scheme-morphism, and residues
Let S be a noetherian scheme, $g : X \to Y$ a flat finite-type separated S-morphism, $\delta : X \to X \times_S X$ and $\gamma : Y \to Y \times_S Y$ the diagonal maps. We define a natural functorial map $\mathbf{L}\delta^*\delta_*\mathbf{g}^* \to \mathbf{g}^!\mathbf{L}\gamma^*\gamma_*$, the "relative fundamental class of g" (where $g^!$ is the twisted inverse image functor from Grothendieck duality). This can be interpreted as an orientation in a bivariant theory involving Hochschild complexes. It globalizes interesting commutative-algebra maps, like residues (which can also be described via Hochschild homology), and traces of differential forms. The talk will be about this theory, which, though at least 25 years old, still needs to be properly exposed.

Amnon Yekutieli (Ben Gurion University)
Twisted Deformation Quantization of Algebraic Varieties
This is joint work with F. Leitner (BGU). Let X be a smooth algebraic variety over a field of characteristic 0, endowed with a Poisson bracket. A quantization of this Poisson bracket is a formal associative deformation of the structure sheaf O_X, which realizes the Poisson bracket as its first order commutator. More generally one can consider Poisson deformations of O_X and their quantizations.

I will explain what these deformations are. Then I'll state a theorem which says that under certain cohomological conditions on X, there is a canonical quantization map (up to gauge equivalence). This is an algebro-geometric analogue of the celebrated result of Kontsevich (which talks about differentiable manifolds).

It appears that in general, without these cohomological conditions, the quantization will not be a sheaf of algebras, but rather a stack of algebroids, otherwise called a twisted associative deformation of O_X.

In the second half of the talk I'll talk about twisted deformations and twisted quantization, finishing with a conjecture.

Generalized Hochschild cohomology and category theory

Wendy Lowen introduced this topic, and other talks were given by Amnon Neeman, Emily Burgunder, and Claude Cibils.

Wendy Lowen (Vrije Universiteit Brussel/Université Denis Diderot Paris 7)
Hochschild cohomology of abelian categories

In contrast to the well known connection between Hochschild cohomology and deformation theory of associative algebras, Hochschild cohomology of schemes is harder to interpret in terms of deformations. We explain how it describes the deformation theory of suitable abelian categories over the sheme, like the categories of (quasi-coherent) sheaves, and we generalize both Hochschild cohomology and deformation theory to arbitrary abelian categories. There is a characteristic morphism from the Hochschild cohomology of an abelian category into the graded centre of its derived category. This morphism encodes the obstructions to deforming single objects of the (abelian or derived) category, and describes which part of the (enhanced) derived category is deformable as a differential graded category.

Amnon Neeman (Australian National University)
Brown representability via Rosicky
For some years we have known that well generated triangulated categories satisfy Brown representability; there are three proofs in the literature. The corresponding statement for the dual has stumped us all; we had no idea how to proceed. Then came a remarkable result of Rosicky's.

Emily Burgunder (Université de Montpellier II)
Leibniz homology and Kontsevich's graph complexes
The homology of the Lie algebra of matrices $gl(A)$ over an associative algebra A can be computed thanks to the cyclic homology of A:

$$H(gl(A)) = S(HC(A))$$

This theorem is known as the Loday-Quillen-Tsygan theorem. Another well-known theorem in homology is due to Kontsevich which says that the homology of the symplectic Lie algebra $K[p_1, ...p_n, q_1, ...q_n]$ can be explicited thanks to the homology of a certain "graph complex" G:

$$H(K[p_1, ..., p_n, q_1, ...q_n]) = S(H(G))$$

These two theorems are, in fact, examples of a more general theorem in the operadic setting, that we will present. If we replace the Lie homology by the Leibniz homology, then the cyclic homology has to be replaced by the Hochschild homology (Cuvier-Loday theorem). We show that, in Kontsevich case, there exists a "nonsymmetric graph complex" which computes the Leibniz homology of the symplectic Lie algebra.

Claude Cibils (Université de Montpellier II)
The Intrinsic Fundamental Group of a Linear Category
Joint work with Maria Julia Redondo and Andrea Solotar.

The main purpose is to provide a positive answer to the question of the existence of an intrinsic and canonical fundamental group associated to a linear category. The fundamental group we introduce takes into account the linear structure of the category, it differs from the fundamental group of the underlying category obtained as the classifying space of its nerve (see for instance G. Segal [1968] or D. Quillen [1973]).

We provide an intrinsic definition of the fundamental group as the automorphism group of the fibre functor on Galois coverings. We prove that this group is isomorphic to the inverse limit of the Galois groups associated to Galois coverings. Moreover, the graduation deduced from a Galois covering enables us to describe in a conceptual way the canonical monomorphism from its automorphism group to the first Hochschild-Mitchell cohomology.

The fundamental group that we define is intrinsic in the sense that it does not depend on the presentation of the linear category by generators and relations. In case a universal covering exists, we obtain that the fundamental groups constructed by R. Martínez-Villa and J. A. de la Peña depending on a presentation of the category by a quiver and relations are in fact quotients of the intrinsic fundamental group that we introduce.

If a universal covering exists, the fundamental group that we define is isomorphic to its automorphism group. Otherwise we show that the fundamental group is isomorphic to the inverse limit of the automorphisms groups of the Galois coverings of B. In case each connected component of the category of the Galois coverings admits an initial object – in other words if "locally" universal coverings exist – the intrinsic group that we define is isomorphic to the direct product of the corresponding automorphism groups.

The methods we use are inspired from the topological case as presented for instance in R. Douady and A. Douady's book. They are closely related to the way the fundamental group is considered in algebraic geometry after A. Grothendieck and C. Chevalley.

This work is very much indebted to the pioneer work of P. Le Meur [2006].

Relations with representation theory

Travis Schedler introduced this topic, and other talks were given by Andrea Solotar, Silvia Montarani, and Ching-Hwa Eu.

Travis Schedler (University of Chicago)
Calabi-Yau Frobenius algebras, stable Hochschild cohomology, and preprojective algebras
We will define and study the notion of Calabi-Yau Frobenius algebras over arbitrary base commutative rings k (especially the integers). This includes preprojective algebras of ADE quivers and finite group algebras. Such algebras have Calabi-Yau stable module categories and have a duality between "stable" Hochschild cohomology and homology—concepts which we define for any Frobenius algebra and interpret using the (unbounded) derived category of k-modules. We show that the stable Hochschild cohomology of "periodic" CY Frobenius algebras has a BV Frobenius algebra structure, which is closely related to the BV string topology algebra for compact spherical manifolds. We show that this includes the preprojective algebras above and group algebras for finite groups that act freely on a sphere. As a consequence, we also give a new explanation why any symmetric algebra has a BV algebra structure on ordinary Hochschild cohomology. If time permits, we will also compare the Dynkin preprojective algebra results with the non-Dynkin case (which is usual Calabi-Yau, and infinite-dimensional).

Andrea Solotar (Universidad de Buenos Aires)
Representations of Yang-Mills algebras
Joint work with Estanislao Herscovich.

Given $n \in \mathbb{N}$ and a field k, let $\mathfrak{f}(n)$ be the k-free Lie algebra with n generators $x_1, ..., x_n$. Consider the k-Lie algebra

$$\mathfrak{ym}(n) := \mathfrak{f}(n)/\langle \sum_{i=1}^{n} [x_i, [x_i, x_j]] : 1 \leq j \leq n \rangle,$$

which has been called the Yang-Mills algebra with n generators. It is a $\mathbb{N}$-graded Lie algebra, locally finite dimensional.

We denote $\mathrm{YM}(n)$ its enveloping algebra $\mathcal{U}(\mathfrak{ym}(n))$. Also,

$$\mathrm{YM}(n) \simeq TV(n)/\langle \sum_{i=1}^{n} [x_i, [x_i, x_j]] : 1 \leq j \leq n \rangle,$$

where $V(n)$ is the k-vector space with basis $\{x_1, ..., x_n\}$. As a consequence $\mathrm{YM}(n)$ is an homogeneous cubic algebra. As it has been noticed by Connes and Dubois-Violette, $\mathrm{YM}(n)$ is 3-Koszul. They also have computed some of the Hochschild cohomology k-vector spaces of this algebra.

However, $\mathrm{YM}(n)$ is not easy to handle: while for $n = 2$, $\mathfrak{ym}(2)$ is isomorphic to the Heisenberg algebra with generators x, y, z subject to relations $[x, y] = z, [x, z] = [y, z] = 0$, and so $\mathrm{YM}(n)$ is isomorphic to the down-up algebra $A(2, -1, 0)$, for $n > 2$ the associative algebra $\mathrm{YM}(n)$ is non-noetherian. So, the representation theory of $\mathrm{YM}(n)$ for $n > 2$ is highly non-trivial.

Our main result is then, given $n > 2$, to find families of representations of $\mathrm{YM}(n)$ big enough to separate points of the algebra. We manage to do so by showing that some well-known algebras, such as all the Weyl algebras and enveloping algebras of nilpotent finite-dimensional Lie algebras are quotients of $\mathrm{YM}(n)$. A mix of both methods allows us to describe several families of representations of $\mathrm{YM}(n)$, including infinite dimensional ones—via the Weyl algebras—and also finite dimensional representations of $\mathrm{YM}(n)$. The tools we use include A_∞-algebras and results of Bavula and Bekkert on representations of generalized Weyl algebras.

The interest on the representations of this family of algebras is mainly motivated by its physical applications, related to classical field theory and also to the study of D-branes.

Silvia Montarani (Massachusetts Institute of Technology)
Finite dimensional representations of symplectic reflection algebras associated with wreath products
Symplectic reflection algebras were introduced by Etingof and Ginzburg. They arise from the action of a finite group G of automorphisms on a symplectic vector space V, and are a multi-parameter deformation of the algebra $S(V) \rtimes G$, smash product of G with the symmetric algebra of V. A series of interesting examples

is provided by the wreath product symplectic reflection algebras, when G is the semidirect product of the symmetric group S_n of rank n with G'^n, where G' is a finite subgroup of $SL(2, C)$.

In this talk we will explain how to produce finite dimensional representations of these algebras. Our method is deformation theoretic and uses some properties of the Hochschild cohomology for this kind of algebras.

Time permitting, we will illustrate how Wee Liang Gan was able to recover the same representations by defining “reflection functors” between the categories of modules over wreath product symplectic reflection algebras corresponding to different values of the deformation parameters.

Ching-Hwa Eu (Massachusetts Institute of Technology)
Hochschild cohomology of preprojective algebras of ADE quivers
Preprojective algebras of ADE quivers are Calabi-Yau and Frobenius. We use these properties to compute the structure of the Hochschild cohomology and its product.

List of Participants

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Benson, David (University of Aberdeen)
Bergh, Petter Andreas (NTNU)
Buchweitz, Ragnar-Olaf (University of Toronto Scarborough)
Burciu, Sebastian (Institute of Mathematics, “Simion Stoilow” of Romanian Academy)
Burgunder, Emily (Univerite de Montpellier II)
Butler, Michael C R (University of Liverpool)
Caldararu, Andrei (University of Wisconsin, Madison)
Cibils, Claude (Université Montpellier 2)
Erdmann, Karin (University of Oxford)
Eu, Ching-Hwa (Massachusetts Institute of Technology)
Farinati, Marco (Universidad de Buenos Aires)
Flenner, Hubert (University of Bochum)
Green, Ed (Virginia Tech)
Happel, Dieter (Technische Universität Chemnitz)
Holm, Thorsten (University of Magdeburg)
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Krause, Henning (Universität Paderborn)
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Chapter 31

Applications of Macdonald Polynomials (07w5048)

Sep 09 - Sep 14, 2007

Organizer(s): Jim Haglund (University of Pennsylvania), Francois Bergeron (Universit´ du Quebec a Montr´al), Jeff Remmel (University of California, San Diego)

Overview of the Field

In 1988 Macdonald [33], [34] introduced a new family of symmetric functions $P_\lambda(X;q,t)$ depending upon a partition λ, a set of variables $X = \{x_1,\ldots,x_n\}$, and two real parameters q,t. They were immediately hailed as a breakthrough in symmetric function theory as well as special functions, as they contained most of the previously studied families of symmetric functions as special cases, and yet satisfied many exciting properties, such as a multivariate orthogonality relation. Some of these properties were conjectural, like Macdonald's postivity conjecture for the coefficients $K_{\lambda,\mu}(q,t)$ in the expansion of P_λ into the "plethystic Schur" basis $s_\mu[X(1-t)]$, which became a famous problem. Garsia and Haiman [10] refined this conjecture, giving a representation theoretic interpretation for the coefficients in terms of Garsia-Haiman modules, an interpretation which was finally proved ten years later in 2000 by Haiman, who connected the problem to the study of the Hilbert scheme of n points in the plane from algebraic geometry [17]. Another famous problem, Macdonald's constant term conjectures, involved an extension of the P_λ to arbitrary affine root systems. (In this setting, λ is no longer a partition, but an element of a certain lattice associated to the root system.) Letting $<,>_{q,t}$ denote Macdonald's scalar product with resect to which the P_λ are orthogonal, which can be expressed as the constant term in a certain multivariate Laurent series, Macdonald introduced a specific value for $< P_\lambda, P_\lambda >_{q,t}$ which in type A reduced to the q-Dyson conjecture. After several special cases were proved by a varity of authors, Macdonald's constant term conjectures in full generality were finally solved in the mid 1990's by Cherednik [4], [5], who showed they have a natural interpretation in terms of the representation theory of fundamental objects he introduced now called double affine Hecke algebras, or Cherednik algebras.

In 1995 Macdonald introduced [35] polynomials $E_\alpha(X;q,t)$, where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a weak composition into n parts. He showed these polynomials also satisfy an orthogonality relation, and are a basis for the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n](q,t)$. The E_α can be thought of as a refinement of the P_λ, since the main properties of the P_λ can be easily derived from corresponding properties of the E's. Cherednik developed the theory further [6], showing how the E_α also have an interpretation in terms of the representation theory of the double affine Hecke algebra.

In conjunction with their study of the study of the $K_{\lambda,\mu}(q,t)$, Garsia and Haiman [16], [11] introduced a number of fascinating combinatorial problems involving the space of diagonal harmonics DH_n, an important

S_n module which contains the Garsia-Haiman modules as S_n sub-modules. (DH$_n$ is isomorphic to the quotient ring of diagonal coinvariants.) For example, Haiman conjectured the dimension of DH$_n$ is $(n+1)^{n-1}$, which equals the number of parking functions on n cars. He also conjectured the dimension of the subspace of diagonal harmonic alternants DH$_n^\epsilon$ is the nth Catalan number C_n. By taking into account the natural bigrading of these spaces we get q, t versions of the number of parking functions and also the nth Catalan number. Based on some ideas of Procesi, Garsia and Haiman were led to a conjectured formula for the character of DH$_n$, which involved a sum of rational functions, with symmetric functions $\tilde{H}_\mu(X; q, t)$, modified versions of the Macdonald polynomials, occurring in the numerators. The terms in this formula correspond to terms in the Atiyah-Bott-Leftschetz fixed point formula, which is connected to the problem via an underlying torus action on the Hilbert scheme. In 2000 Haglund [14] introduced a specific conjectured interpretation for the q, t-Catalan number, which was proved shortly after in joint work with Garsia [8], [9]. The proof used plethystic symmetric function identities involving Macdonald polynomials which were developed by F. Bergeron, Garsia, Haiman, and Tesler during the 1990's [2], [12]. In 2001 Haiman proved the rational function formula for the character of DH$_n$ using algebraic geometry[17]. Three years later Haglund, Haiman, Loehr, Remmel and Ulyanov [21] introduced a conjecture which is still open, the "shuffle conjecture" which gives a a combinatorial conjecture for this character in terms of statistics on parking functions. Haglund's book [15] includes a detailed discussion of the combinatorics of DH$_n$, while Bergeron's book [1] contains further information on DH$_n$ in the context of a more general discussion on coinvariant spaces. In another direction Iain Gordon [13] proved a conjecture of Haiman for the dimension of versions of DH$_n$ for other Weyl groups. A refinement of this result, involving an extra parameter q was later found by Cherednik [7].

Recent Developments and Open Problems

Although from the time of their introduction Macdonald polynomials enjoyed a very fruitful interaction between harmonic analysis and representation theory, until fairly recently little was known about the combinatorics. In 2004 Haglund introduced a conjectured combinatorial model for the expansion of the $\tilde{H}_\mu(X; q, t)$ into monomials, or into Gessel's fundamental quasisymmetric functions. The conjecture was proved shortly after by Haglund, Haiman, and Loehr, who also showed the formula led easily to an expansion of $\tilde{H}_\mu$ into LLT polynomials, symmetric functions, depending on a parameter q, introduced by Lascoux, Leclerc, and Thibon. These functions were conjectured to be Schur positive, and now Grownowski and Haiman have announced a proof of this conjecture, which gives a new proof (using the representation theory of Hecke algebras) of Macdonald's positivity conjecture that the $\tilde{H}_\mu$ are Schur positive. Moreover, Sami Assaf has developed an amazing combinatorial model which algorithmically constructs a graph from each LLT polynomial, which also proves Schur positivity of the LLT by giving an explicit recursive construction for the Schur coefficients. The details of Assaf's construction are rather complicated though, and a major open question is whether her construction can be simplified to obtain nice formulas for the $K_{\lambda,\mu}(q, t)$.

Another significant body of recent research with implications to Macdonald theory is work on the k-Schur function. This is a family of symmetric functions in a set of variables X which depends on a partition λ, a positive integer k, and a parameter q, and which reduces to the usual Schur function when k is large enough. Originally introduced by Lascoux, Lapointe, and Morse [25], the combintorial theory of the k -Schur was primarily developed by Lapointe and Morse during the period $2000-2007$ [26], [27], [28], at which time they discovered connections between the k-Schur and Gromov-Witten invariants [29]. This led a number of other researchers such as Lam, Shimozono, and Schilling to work on the subject. It is conjectured that when LLT polynomials of total "band width" k are expressed in terms of the k-Schur, the coefficients are nonnegative polynomials in q. This implies Macdonald's positivity conjecture. There are currently about seven different ways of defining the k-Schur, which are all conjectured to be equivalent, and there is also a growing body of conjectures surrounding the combinatorial properties of the k-Schur. Billey and Assaf have recently announced a solution to one of these conjectures, that one of the definitions of the k-Schur is Schur-positive, by utilizing Assaf's LLT-graph decomposition algorithm.

There have been a number of interesting recent developments in the study of the combinatorics of DH$_n$. First of all, Loehr and Warrington have introduced a very general conjecture of the following form: let ∇ be a linear operator defined on the $\tilde{H}_\mu(X;q,t)$ by

$$\nabla \tilde{H}_\mu(X;q,t) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu(X;q,t), \tag{31.1}$$

where $n(\mu) = \sum_i (i-1)\mu_i$. F. Bergeron first noticed that many important identities in Macdonald theory can be elegantly expressed using the∇ operator. Loehr and Warrington [32] give an explicit combinatorial expression for ∇ applied to any Schur function s_λ, as a sum over statistics on parking functions for "nested" lattice paths. If $\lambda = 1^n$, the Loehr-Warrington conjecture reduces to the shuffle conjecture. Another generalization of the shuflfle conjecture has been introduced by Haglund, Morse, and Zabrocki. By building on earlier work of N. Bergeron, Descouens, and Zabrock [3], they conjecture that ∇ applied to a Hall-Littlewood function can be expressed in terms of statistics on parking functions for lattice paths which hit the main diagonal in certain specified points. Garsia, Xin, and Zabrocki have announced a proof of the hook shape case of this conjecture, which generalizes Haglund's q,t-Schröder theorem. Both the Loehr-Warrington conjecture and the Haglund-Morse-Zabrocki conjecture give expressions for ∇ applied to a whole basis for the ring of symmetric functions, which hopefully will be easier to prove than looking at ∇s_{1^n} by itself.

In [19] Haglund, Haiman, and Loehr give a version of the combinatorial formula for the $\tilde{H}_\mu(X;q,t)$ for the $J_\lambda(X;q,t)$ (scalar multiples of the P_λ whose monomial coefficients are in $\mathbb{Z}[q,t]$), and in subsequent work [20] also a version involving the $\mathcal{E}_\alpha(X;q,t)$ (scalar multiples of the $E_\alpha(X;q,t)$ whose monomial coefficients also have no denominators). The formula for the $\mathcal{E}_\alpha(X;q,t)$ involves a sum over certain "nonattacking" fillings, of the diagram whose $n-i+1$st column has height α_i, with positive integers, with each such filling weighted by powers of q,t and also factors of the form $(1-q^a t^b)$ for certain powers a,b defined combinatorially. This model also contains a "basement", i.e. a row of squares below the diagram filled with the numbers $n, n-1, \ldots, 2, 1$. By changing the basement to $2n, 2n-1, \ldots, n+1$ and summing as before over nonattacking fillings we get a formula for J_μ, where μ is the rearrrangement of the parts of α into partition order. Thus there are actually a number of formulas for J_μ corresponding to the various ways to permute the parts of μ and shuffle with zeros to obtain a weak composition.

Ram and Yip [39] have introduced a general formula for the E_α for arbitrary affine root systems. Their formula is obtained by iterating recurrence relations which can be used to define the E_α (known as intertwiner relations), and is expressed in terms of "alcove walks" in a certain lattice associated to the root system. In type A their formula has many more terms than the formula in [19], but Lenart [31] has shown how to group together terms in the J_μ version of their formula to obtain exactly the formula from [20] for J_μ corresponding to the case where $\alpha = \mu$. An exciting question for future research is whether or not terms in the Ram-Yip formula for general affine root systems can be grouped together in a similar way to obtain a canonincal combinatorial formula for the E_α.

Since both Demazure characters (also known as key polynomials), and the standard bases (introduced by Lascoux and Schützenberger [30] in their study of Schubert varieties) are limiting or special cases of the E_α, new combinatorial formulas for these functions are a by-product of the new Macdonald combinatorics. (See [40] for more background on key polynomials.) In connection with her study of these identities Sarah Mason [37], [38] introduced a generalization of the RSK algorithm. Recently Haglund, Luoto, Mason, and van Willigenburg have introduced a new basis $\mathrm{QS}_\beta(X\}$ for the ring of quasisymmetric functions they call quasisymmetric Schur functions, and have used properties of Mason's RSK algorithm to prove the QS_β satisfy a generalzation of the Littlewood-Richardson rule [22], [23]. Lauve and Mason have announced they have been able to use this generalized Littlewood-Richardson rule to prove a conjecture of F. Bergeron and C. Reutenauer that gives an explicit basis for the quotient ring of quasisymmetric functions in n variables by the ring of symmetric functions in n variables.

One implication of the type A formula for J_μ in [19] is that the coefficient of a monomial symmetric function in

$$J_\mu(X;q,q^k)/(1-q)^n \tag{31.2}$$

is in $\mathbb{N}[q]$, for any positive integer k. Maple calculations led Haglund to conjecture the stronger relation that the coefficient of a Schur function in (31.2) is in $\mathbb{N}[q]$. Ram has suggested that a more general phenomena may hold, where you decompose $J_\mu(X;q,q^k)$ in terms of the basis $\{J_\lambda(X;q,q^{k-1})\}$, with some kind of positivity at each step ($J_\lambda(X;q,q)$ is a scalar multiple of the Schur function $s_\lambda(X)$).

Presentation Highlights

Many of the talks at the workshop, for example talks by N. Bergeron, I. Gordon, J. Haglund, N. Loehr, S. Mason, and J. Morse, involved topics discussed in the above two sections. Other talks were about other topics relevant to symmetric function theory and Macdonald polynomials of interest to researchers in this area. Below we include titles and abstracts for all the presentations.

Abstracts for Talks

Speaker: **Nick Loehr** (Virginia Tech, USA) (talk describes joint work with Jim Haglund and Mark Haiman)
Title: *Symmetric and Non-symmetric Macdonald Polynomials*
Abstract: Macdonald polynomials have played a central role in symmetric function theory ever since their introduction by Ian Macdonald in 1988. The original algebraic definitions of these polynomials are very non-explicit and difficult to work with. Haglund conjectured an explicit combinatorial formula for the Macdonald polynomials. This was later extended to a combinatorial formula for non-symmetric Macdonald polynomials in type A. This talk will discuss the algebraic and combinatorial definitions of both symmetric and non-symmetric Macdonald polynomials. We also sketch the main ideas in the proofs that the algebraic and combinatorial constructions are equal.

Speaker: **Jim Haglund** (Univ. of Pennsylvania, USA) will deliver a talk prepared by **Greg Warrington** (Wake Forest, USA) who had to cancel his trip
Title: *Combinatorical structures associated to the nabla operator*
Abstract: Over the past ten years, there has been a rich interplay among the modified Macdonald polynomials, the diagonal harmonics modules, the nabla operator, and the combinatorics of q,t-weighted lattice paths. In this talk, we review these connections, paying particular attention to the q,t-Catalan numbers. We finish with recent joint work of N. Loehr and G. Warrington regarding a nested-lattice-path interpretation for nabla applied to arbitrary Schur functions.

Speaker: **Sami Assaf** (Univ. of Pennsylvania, USA)
Title: *A combinatorial proof of Macdonald positivity*
Abstract:Taking Haglund's formula for the transformed Macdonald polynomials expressed in terms of monomials as the definition, we present a self-contained, combinatorial proof of symmetry and Schur positivity of Macdonald polynomials, and give a combinatorial interpretation of the Schur coefficients. The method of the proof uses the theory of dual equivalence graphs and a new generalization of them called D graphs.

Speaker: **Jennifer Morse** (Drexel Univ., USA)
Title: An update on the k-Schur approach to statistics problems
Abstract: We will review the k-Schur role in the theory of Macdonald polynomials and talk about some related open problems and new conjectures.

Speaker:**Thomas Lam** (Harvard Univ., USA)
Title:*k-Schur functions and the homology of the affine Grassmannian*
Abstract: I will explain the relationship between Lapointe, Lascoux and Morse's k-Schur functions and the Schubert basis of the homology $H_*(Gr)$ of the affine Grassmannian of $SL(n)$. I will state some general facts about $H_*(Gr)$ then describe Peterson's work on affine Schubert calculus. Peterson's work can be connected to k-Schur functions via the Fomin-Stanley subalgebra and the theory of Stanley symmetric functions.

Speaker: **John Stembridge** (Univ. of Michigan, USA)
Title: *Kostka-Foulkes polynomials of general type and their variations*
Abstract: In this talk we plan to discuss the general features of Kostka-Foulkes polynomials for finite root systems. We will pose several problems or conjectures aimed at developing a general framework for explaining the nonnegativity of their coefficients in a combinatorial way.

If there is time, we will also discuss some additional families of univariate polynomials that also occur in representation theoretic contexts and have the same combinatorial flavor– one related to the Blattner multiplicity formula, and another related to Demazure modules.

Speaker: **Iain Gordon** (University of Edinburgh, United Kingdom)
Title: *Rational Cherednik algebras, diagonal coinvariants, and other animals*
Abstract: I will explain how the representation theory of rational Cherednik algebras is used to get a handle on diagonal coinvariants for Weyl groups. This is quite well understood, but may only be part of a broad scheme. Beyond diagonal invariants there is a dream that the representation theory could shed new light on the n! theorem and its conjectural generalisations to wreath products.

Speaker: **Bogdon Ion** (Univ. of Pittsburgh, USA)
Title: *Nonsymmetric Macdonald polynomials and applications*
Abstract: I will give a quick survey of nonsymmetric Macdonald polynomials and their properties and I will also describe some of their applications (geometric formulas for weight multiplicities and random walks on buildings).

Speaker: **Sarah Mason** (Davidson College, USA)
Title: *A specialization of nonsymmetric Macdonald polynomials*
Abstract: The nonsymmetric Macdonald polynomials can be specialized to polynomials which decompose the Schur functions. We describe several combinatorial properties of these polynomials and their connections to Demazure characters. We discuss a related family of polynomials called ”key polynomials” and two new methods for constructing key polynomials.

Speaker: **Adriano Garsia** (Univ. California at San Diego, USA)
Title: *Constant terms and Kostka-Foulkes Polynomials*
Abstract: A problem that arose in Gauge Theory led us to the evaluation of a constant term with a variety of ramifications into several areas from Invariant Theory, Representation Theory, the Theory of Symmetric-Functions and Combinatorics. A significant by-product of our evaluation is the construction of a trigraded Cohen Macaulay basis for the Invariants under an action of SLn(C) on a space of 2n + n2 variables.

Speaker: **Mike Zabrocki** (York University, Canada)
Title: *Combinatorial aspects of generalized Hall-Littlewood symmetric functions*
Abstract: We overview a number of open problems involving the combinatorics of generalized Hall-Littlewood polynomials.

Speaker: **Nantel Bergeron** (York University, Canada)
Title: $\nabla\, k\, \Lambda$
Abstract: We present a series of problems related to ∇ applied to symmetric functions. We show that the analogue results are true for non-commutative symmetric functions.

Speaker: **Tom Koornwinder** (University of Amsterdam, Netherlands)
Title: *The relationship between Zhedanov's algebra AW(3) and DAHA for Askey-Wilson*
Abstract: Zhedanov's algebra $AW(3)$ will be considered with explicit structure constants such that, in the basic representation, the first generator becomes the second order q-difference operator for the Askey-Wilson polynomials. This representation is faithful for a certain quotient of $AW(3)$ such that the Casimir operator is equal to a special constant. A central extension of this quotient of $AW(3)$ can be embedded in the double affine Hecke algebra (DAHA) by means of the faithful basic representations of both algebras. Next I will discuss the relationship between $AW(3)$ and the spherical subalgebra of the DAHA for Askey-Wilson. This one-variable exercise should be a stepping stone for exploring analogues of $AW(3)$ in higher rank.

Abstracts for Posters

Presenter(s): **J. Haglund** (Univ. of Pennsylvania) and **L. Stevens** (UC San Diego)
Title: An extension of the Foata map to standard Young tableaux
Abstract: We define an inversion statistic on standard Young tableaux. We prove that this statistic has the same distribution over SY T () as the major index statistic by exhibiting a bijection on SY T () in the spirit of the Foata map on permutations.

Presenter(s): **L. Tevlin** (Yeshiva Univ.)
Title: Noncommutative Hall-Littlewood Polynomials and q-Cauchy Identity
iAbstract: This poster will contain a proposal for a noncommutative version of Hall-Littlewood (H-L) polynomials. These seem to be natural analogs of classical objects as ribbon H-L polynomials interpolate between ribbon Schur functions and noncommutative monomial symmetric functions, while fundamental H-L polynomials interpolate between noncommutative fundamental and monomial symmetric functions.

Presenter(s): **Alex Woo** (UC Davis)
Title: Garnir modules, Springer fibers, and Ellingsrud-Stromme cells on the Hilbert scheme
Abstract: We calculate defining ideals for certain Sn invariant subspace arrangements of the braid arrangement and relate them (in part using duality) to the cohomology rings of Springer fibers as studied by Garsia and Procesi. This allows us to calculate their graded characters to be particular sums of Hall-Littlewood polynomials. We also relate these subspace arrangements to closed unions of cells on the Hilbert scheme. This is joint work with Mark Haiman.

Scientific Progress Made

In the spring of 2007 there was a workshop at the Center de Rescherches Mathématiques (CRM) in Montréal on Combinatorial Hopf algebras and Macdonald polynomials. Many of the speakers at this workshop were also here at the BIRS workshop, and there were benefits from being able to meet again a few months later. It turns out there was quite a bit of significant progress made in the study of Macdonald polynomial combinatorics in the intervening time. For example, Sami Assaf announced that she had successfully completed her ambitious program of trying to prove Schur positivity of LLT polynomials by a combinatorial construction, and she gave a nice presentation of her result at BIRS. This gives the first combinatorial proof of Macdonald's positivity conjecture, and further analysis of her algorithm will undoubtedly lead to exciting new identities for the q, t-Kostka coefficients. In addition, since the conjecture in [21] for the character of DH_n can be expressed as apositive sum of LLT polynomials, Assaf's work also gives an interpretation for the Schur coefficients in this character. Another significant development made in the interim was the discovery by Loehr and Warrington of a conjectured expression for the monomial expansion of the ∇ operator applied to any Schur function, which also made an exciting presentation.

The various researchers at the BIRS workshop all had different points of view on Macdonald polynomials, and it was a joy to hear about all the varioius avenues of research where important applications of Macdonald polynomials arise. The general attitude of the participants seemd to be quite positive about the experience and everyone was happy they attended the workshop. A number of collaborations were begun or enhanced during this time, for example J. Haglund, S. Mason, S. van Willigenburg continued to build on a collaboration started during the CRM workshop. This eventually led to a conjectured generalization of the Littlewood-Richardson rule connected to the study of a new basis for the riung of quasisymmetric functions, a conjecture which was proved by Haglund, Mason, van Willigenburg, and K. Luoto during a recent week long stay at BIRS as part of the Focused Research Group program.

List of Participants

Allen, Ed (Wake Forest University)
Assaf, Sami (MIT)
Bandlow, Jason (University of California, San Diego)
Bergeron, Francois (Universit´ du Quebec a Montr´al)
Bergeron, Nantel (York University)
Biagioli, Riccardo (Universite Claude Bernard Lyon I)
Can, Mahir (University of Western Ontario)
Descouens, Francois (Universite de Marne-la-Vallee)
Fishel, Susanna (Arizona State University)
Garsia, Adriano (University of California, San Diego)
Gordon, Iain (University of Edinburgh)

Haglund, Jim (University of Pennsylvania)
Hivert, Florent (University of Rouen)
Ion, Bogdan (University of Pittsburgh)
Jing, Naihuan (North Carolina State University)
Kasatani, Masahiro (Kyoto University)
Koornwinder, Tom (KdV Institute for Mathematics, University of Amsterdam)
Lam, Thomas (Harvard University)
Lapointe, Luc (Universidad de Talca)
Li, Huilan (Fields Institute)
Loehr, Nick (College of William and Mary)
Mason, Sarah (Davidson College)
Morse, Jennifer (University of Miami)
Remmel, Jeff (University of California, San Diego)
Schilling, Anne (University of California, Davis)
Schlosser, Michael (University of Vienna)
Shimozono, Mark (Virginia Tech)
Stembridge, John (University of Michigan)
Stevens, Laura (University of California, San Diego)
Stump, Christian (University of Vienna)
Suzuki, Takeshi (Okayama University)
Tevlin, Lenny (Yeshiva University)
Thiery, Nicolas M. (Univ Paris-Sud)
van Willigenburg, Stephanie (University of British Columbia)
Vazirani, Monica (University of California, Davis)
Woo, Alexander (University of California Davis)
Yoo, Meesue (University of Pennsylvania)
Zabrocki, Mike (York University)

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Chapter 32

Group Embeddings: Geometry and Representations (07w5034)

Sep 16 - Sep 21, 2007

Organizer(s): Michel Brion (Universite de Grenoble), Stephen Doty (Loyola University Chicago), Lex Renner (University of Western Ontario), Ernest Vinberg (Moscow State University)

Overview of the Field

The theory of algebraic group embeddings has developed dramatically over the last twenty-five years, based on work of Brion, DeConcini, Knop, Luna, Procesi, Putcha, Renner, Vust and others. It incorporates torus embeddings and reductive monoids, and it provides us with a large and important class of spherical varieties. The interest in these, and related, topics has led to vigorous and celebrated mathematical activity in Europe, Asia and North America. This BIRS workshop provided a timely opportunity to enhance the visibility of this highly interdisciplinary industry, by bringing together some of the principal players in group embeddings and representation theory.

Certain aspects of representation theory are well connected to the geometry of group embeddings, especially through the examples of linear algebraic monoids. The study of representations of reductive monoids has links with quasihereditary algebras and highest weight categories, important topics in the theory of finite dimensional algebras. Important work in this area includes that of Cline, Parshall, and Scott, Donkin, Erdmann, Green, Ringel and others. There are also significant interactions with the representation theory of finite groups, Hecke algebras, and quantum groups, and combinatorics.

The widespread interest in group embeddings results from the inherent richness and depth of the results; combining techniques from commutative algebra, algebraic geometry, representation theory, convex geometry, linear algebra, spherical embeddings, semigroup theory, and combinatorics. This subject is mature yet still growing, and there are many interesting open questions.

The ”wonderful compactification” stands out among all others as the one embedding of a semisimple group that serves as the role model for further development. DeConcini and Procesi [13] originally calculated the Betti numbers and cell decomposition using the method of Bialinycki-Birula. Since then, important work has been done by Brion [10] and Springer [46] on the geometry of orbit closures. Kato [29] and Tchoudjem [48] have found an analogue of the Borel-Weil-Bott theorem, and Kato [30] described all the equivariant vector bundles. The wonderful compactification can also be described in terms of certain reductive monoids [43]. Several authors have discovered explicit cell decompositions of the wonderful compactification [10], [29], [43], [46]. Luna has recently identified a more general class of wonderful embeddings.

There will be generalized Schur algebras (in the sense of Donkin [17]) implicit in the coordinate bialgebra of any reductive monoid, and the representation theory of the monoid breaks up into a direct sum of the representation theories of the various generalized Schur algebras, which are finite dimensional quasihereditary algebras. All of this extends the classic motivating example of polynomial representation theory of general linear groups. Although this goes back to Schur's dissertation, at the very inception of representation theory as a mathematical discipline, it remains in positive characteristic a vigorous and intensively studied aspect of modern representation theory, with important work in the past twenty-five years by Green, James, Donkin, Erdmann, Cline, Parshall, and Scott, Dlab, Ringel, and others.

Solomon has studied certain finite monoids (the Renner monoids), analogues of the Weyl group of a reductive group. The simplest examples are the *rook monoids* which are natural generalizations of the symmetric groups. There is recent interesting work of Halverson and Ram (see [24], [25]) on a q-analogue of the rook monoid. This is just the tip of a large unchartered iceberg; many intriguing structures exist in abundance, and little is known in general. Recent results of Steinberg [47] have brought some much-needed clarity to the representation theory of inverse semigroups.

Recent Developments and Open Problems

Recent Developments

There are a great many of these. We give a sketch of the some of the important ones.

1. Complexity of group actions [5], [50].

2. Stable reductive varieties [2], [3].

3. Analogues of the Bruhat decomposition for spherical varieties [11], [41], [49].

4. Equivariant compactifications of spherical homogeneous spaces [8], [32], [13].

5. Cartier and Weil divisors on spherical varieties [9], [51].

6. Universal G_m torsors on certain moduli spaces [12], [27].

7. Embeddings of G_a^n; an important beginning with many interesting examples [28].

8. Compactification of Jacobian varieties [1].

9. Among group embeddings and spherical varieties there should be many opportunities to identify and study examples of ”Cox rings” [12, 5].

10. Moduli spaces of group compactifications; generalizations of the toric Hilbert scheme [2, 3, 4, 26, 37].

11. More general wonderful embeddings [34]. See also recent work of P. Bravi and G. Pezzini, and P. Bravi and S. Cupit-Foutou.

12. Counterexample to Renner's conjecture regarding blocks of algebraic monoids [15]

13. The issue of *normality* for symplectic and orthogonal monoids now settled in [22].

14. Putcha's penetrating assessment of conjugacy classes on a reductive monoid [39]

15. Steinberg's recent results [47] on the representation theory of inverse semigroups, using Moebius inversion and Munn-Ponizovski to obtain information about multiplicities and character formulas.

Open Problems

1. Describe the cohomology rings of smooth, complete group embedddings, by generators and relations. For toric varieties, the answer is given by a result of Jurkiewicz and Danilov. For ”regular” group embeddings, the equivariant cohomology ring has been described by Bifet, De Concini and Procesi as a subring of a larger ring. But no generators of the cohomology ring are known in general.

2. Construct $B \times B$-equivariant desingularizations of the closures of $B \times B$-orbits in G-embeddings (G a connected reductive group, B a Borel subgroup). Does there exist such a desingularization with only finitely many fixed points of the maximal torus $T \times T$? This reduces easily to constructing a $B \times B$-equivariant desingularization of the closure of B; for regular embeddings, this closure is almost always singular. An affirmative answer to this question implies that the intersection cohomology of $B \times B$-orbit closures vanishes in all odd degrees. This was proved by Springer for the wonderful compactification, by combinatorial arguments.

3. Study the topology of hypersurfaces in smooth complete group embeddings: determine their numerical invariants, by generalizing the known results for toric varieties.

4. (Related to problem 1). In the case of a smooth, projective group embedding X, find an explicit cell decomposition of X, similar to what was done by Brion, Springer and Renner for the wonderful compactification. How is each cell made up of $B \times B$-orbits? Can one find an explicit Bialinycki-Birula 1-parameter subgroup? Is there a monoid-theoretic way to do it in some cases?

5. Compute the characters of simple modules (in the describing characteristic) for reductive normal algebraic monoids. This remains the most fundamental problem in representation theory.

6. Describe the blocks for reductive normal algebraic monoids. Donkin has a combinatorial description of the blocks of the monoid of all $n \times n$ matrices, but there are no results in any other case thus far. DeVisscher’s recent counterexample in Type C shows that the answer is not easily predicted from the (known) description of the blocks of reductive algebraic groups.

7. Establish characteristic-free, double-centralizer results for classical groups and other related situations. In a sense, this goes back to early work of DeConcini and Procesi [14].

8. Describe the structure and representation theory of various centralizer algebras arising from double centralizer situations. Such algebras tend to be ”diagram” algebras with a strong combinatorial flavour, and the modular representation theory of such algebras is wide open.

9. Obtain a class of embeddings for nonreductive groups that is well-behaved geometrically. Such embeddings could help lead to numerical/cohomological information about representations of nonreductive groups.

Abstracts of Talks Given

Speaker: **Henning Haahr Andersen** (Aarhus)
Title: *Combinatorial categories and Kazhdan-Lusztig theory*
Abstract: In joint work with Jantzen and Soergel [1] in the early 1990’s we constructed a combinatorial category $\mathcal{K}$. We used it to compare representations of (small) quantum groups to modular representations of the corresponding (infinitesimal) semisimple algebraic group. In recent work Peter Fiebig [2] considers another combinatorial category $\mathcal{B}$ and he gives a functor $\mathcal{B} \to \mathcal{K}$. This he then applies to the related Kazhdan-Lusztig theories.

We shall discuss the constructions of the two categories, the functor between them, and the consequences in representation theory.

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[2] P. Fiebig, *Sheaves on affine Grassmannians, Projective Representations and Lusztig's Conjectures*, Preprint (Universität Freiburg 2007).

Speaker: **Ivan V. Arzhantsev** (Moscow State University)
Title: *Geometric invariant theory via Cox rings*
Abstract: (Joint work with Jürgen Hausen.) The passage to a quotient by an algebraic group action is often an essential step in classical moduli space constructions of Algebraic Geometry, and it is the task of Geometric Invariant Theory (GIT) to provide such quotients. Starting with Mumford's approach of constructing quotients for actions of reductive groups on projective varieties via linearized line bundles and their sets of semistable points [7], the notion of a "good quotient" became a central concept in GIT, compare [10] and [3].

A good quotient for an action of a reductive group G on a variety X is an affine morphism $\pi : X \to Y$ of varieties such that Y carries the sheaf of invariants $\pi_*(\mathcal{O}_X)^G$ as its structure sheaf. In general, a G-variety X need not admit a good quotient, but there may be many (different) invariant open $U \subseteq X$ with a good quotient; we will call them the good G-sets. In this talk, we present a combinatorial construction of good G-sets $U \subseteq X$, which are maximal with respect to the properties either that the quotient space $U//G$ is quasiprojective or, more generally, that it comes with the A2-property, i.e., any two of its points admit a common affine neighbourhood.

Our first step is to consider actions of G on factorial affine varieties X. The basic data for the construction of good G-sets of X are *orbit cones*. They live in the rational character space $\mathbb{X}_{\mathbb{Q}}(G)$, and for any $x \in X$ its orbit cone $\omega(x)$ is the convex cone generated by all $\chi \in \mathbb{X}(G)$ admitting a semiinvariant f with weight χ such that $f(x) \neq 0$ holds. It turns out that there are only finitely many orbit cones and all of them are polyhedral.

To any character $\chi \in \mathbb{X}(G)$ we associate its GIT-*cone*, namely

$$\lambda(\chi) := \bigcap_{\chi \in \omega(x)} \omega(x) \subseteq \mathbb{X}_{\mathbb{Q}}(G).$$

We say that a collection Φ of orbit cones is *2-maximal*, if for any two members their relative interiors overlap and Φ is maximal with respect to this property.

Theorem. *Let a connected reductive group G act on a factorial affine variety X.*

(i) The GIT-cones form a fan in $\mathbb{X}_{\mathbb{Q}}(G)$, and this fan is in a canonical order reversing bijection with the collection of sets of semistable points of X.

(ii) There is a canonical bijection from the set of 2-maximal collections of orbit cones onto the collection of A2-maximal good G-sets of X.

For the case of a torus G this result was known before. The first statement is given in [2]. Moreover, a result similar to the second statement was obtained in [4] for linear torus actions on vector spaces, and for torus actions on any affine factorial X, statement (ii) is given in [1].

To obtain the general statement, we reduce to the case of a torus action as follows. Consider the quotient $Y := X//G^s$ by the semisimple part $G^s \subseteq G$. It comes with an induced action of the torus $T := G/G^s$, and the key observation is that the good T-sets in Y are in a canonical bijection with the good G-sets in X. This approach turns out to be as well helpful for computing GIT-fans, because Classical Invariant Theory in many cases provides enough information on the algebra $\mathbb{K}[X]^{G^s}$ of invariants.

Our next aim is to study quotients of certain non-affine G-varieties X, e.g., the classical case of X being a product of projective spaces. More precisely, we consider normal varieties X with a finitely generated Cox ring

$$\mathcal{R}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

where the divisor class group $\mathrm{Cl}(X)$ is assumed to be free and finitely generated. The "total coordinate space" $\overline{X}$ of X is the spectrum of the Cox ring $\mathcal{R}(X)$. This $\overline{X}$ is a factorial affine variety [2] acted on by the Neron-Severi torus H having the divisor class group $\mathrm{Cl}(X)$ as its character lattice. Moreover, X can be reconstructed from $\overline{X}$ as a good quotient $q : \widehat{X} \to X$ by H for an open subset $\widehat{X} \subseteq \overline{X}$.

After replacing G with a simply connected convering group, its action on X can be lifted to the total coordinate space $\overline{X}$. The actions of H and G on $\overline{X}$ commute, and thus define an action of the direct product

$\overline{G} := H \times G$. Given a good $\overline{G}$-set $W \subseteq \overline{X}$, we introduce a "saturated intersection" $W \sqcap_G \widehat{X}$. The main feature of this construction is the following.

Theorem. *The canonical assignment $W \mapsto q(W \sqcap_G \widehat{X})$ defines a surjection from the collection of good $\overline{G}$-sets in $\overline{X}$ to the collection of good G-sets in X.*

So this result reduces the construction of good G-sets on X to the construction of good $\overline{G}$-sets in $\overline{X}$, and the latter problem, as noted before, is reduced to the case of a torus action. Again, this allows explicit computations. Note that our way to reduce the construction of quotients to the case of a torus action has nothing in common with the various approaches based on the Hilbert-Mumford Criterion, see [3], [5], [7], [9] and [11].

As a first application of this result, we give an explicit description of the ample GIT-fan, i.e., the chamber structure of the linearized ample cone, for a given normal projective G-variety X with finitely generated Cox ring. Recall that existence of the ample GIT-fan for any normal projective G-variety was proven in [5] and [11]. As an example, we compute the ample GIT-fan for the diagonal action of $\mathrm{Sp}(2n)$ on a product of projective spaces $\mathbb{P}^{2n-1}$.

A second application of the above result are Gelfand-MacPherson type correspondences. Classically [6], this correspondence relates orbits of the diagonal action of the special linear group G on a product of projective spaces to the orbits of an action of a torus T on a Grassmannian. Kapranov [8] extended this correspondence to isomorphisms of certain GIT-quotients and used it in his study of the moduli space of point configurations on the projective line. Similarly, Thaddeus [12] proceded with complete collineations. We put these correspondences into a general framework, relating GIT-quotients and also their inverse limits. As examples, we retrieve a result of [12] and also an isomorphism of GIT-limits in the setting of [8].

Finally, we use our approach to study the geometry of quotient spaces of a connected reductive group G on a normal variety X with finitely generated Cox ring. The basic observation is that in many cases our quotient construction provides the Cox ring of the quotient spaces. This allows to apply the language of bunched rings developed in [2], which encodes information on the geometry of a variety in terms of combinatorial data living in the divisor class group.

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Speaker: **Stephen Donkin** (York)
Title: *Calculating the cohomology of line bundles on flag varieties in characteristic p*
Abstract: Let G be a connected reductive group over an algebraically closed field of characteristic p and let B be a Borel subgroup. The character of the cohomology of a the line bundle on the flag variety G/B is not well understood (by contrast with the situation in characteristic zero where this is given by Weyl's character

formula, via the Borel-Weil-Bott Theorem). We describe some general methods of calculation and a complete solution for the case $G = SL_3(k)$.

Speaker: **Jürgen Hausen** (Tübingen)
Title: *Cox rings and combinatorics*
Abstract: (Joint work with I.V. Arzhantsev and F. Berchtold.) Suppose that X is normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and free, finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup $K \subset \mathrm{WDiv}(X)$ of the group of Weil divisors mapping isomorphically onto $\mathrm{Cl}(X)$. The *Cox ring* $\mathcal{R}(X)$ is the algebra of global sections of a sheaf of K-graded algebras:

$$\mathcal{R}(X) \;:=\; \Gamma(X, \mathcal{R}), \qquad \text{where} \quad \mathcal{R} \;:=\; \bigoplus_{D \in K} \mathcal{O}(D).$$

Note that multiplication in the Cox ring is just multiplication of rational functions on X. Up to isomorphy, the Cox ring does not depend on the choice of K. A basic observation is that Cox rings are unique factorization domains.

The sheaf $\mathcal{R}$ defines moreover a generalized *universal torsor* $'X \to X$. Suppose that $\mathcal{R}$ is locally of finite type; this holds for example, if X is locally factorial or if $\mathcal{R}(X)$ is finitely generated. Then we may consider the relative spectrum $'X := \mathrm{Spec}_X\, \mathcal{R}$, which turns out to be a quasiaffine variety. The K-grading of R defines an action of the torus $H := \mathrm{Spec}\; \mathbb{K}[K]$ on $'X$, and the canonical morphism $p\colon {'X} \to X$ is a good quotient, i.e., it is an H-invariant affine morphism satisfying $\mathcal{O}_X = (p_* \mathcal{O}_{'X})^H$.

If X has a finitely generated Cox ring $\mathcal{R}(X)$, then $'X$ is an invariant open subvariety of the *total coordinate space* $\overline{X} := \mathrm{Spec}\; \mathcal{R}(X)$. Thus, varieties with finitely generated Cox ring are obtained as good quotient spaces of certain affine torus actions on factorial affine varieties. Such quotients in turn admit a description by combinatorial data, which we call "bunches of cones". We describe basic geometric properties of X in terms of its defining bunch of cones, for example, we discuss singularities, the ample cone, Fano criteria, and modifications. Moreover, we give some applications to almost homogeneous spaces.

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[2] F. Berchtold, J. Hausen: Cox rings and combinatorics. Transactions of the AMS 359, No. 3, 1205–1252 (2007), `math.AG/0311105`

Speaker: **Xuhua He** (SUNY - Stony Brook)
Title: *G-stable-piece decomposition of a wonderful compactification*
Abstract: Let G be a connected semisimple algebraic group of adjoint type over an algebraically closed field. Let us consider the diagonal G-action on the wonderful compactification X of G. The classification of the G-orbits were obtained by Lusztig in terms of G-stable pieces. He also used the G-stable-piece decomposition to construct certain simple perverse sheaves on X (which are called character sheaves on X). In this talk, I will discuss some geometric properties of the G-stable pieces. First, we will talk about some relation between the G-stable pieces and the $B \times B$-orbits in X, where B is a Borel subgroup of G. We will then use this relation to study the closure relation of the G-stable pieces and some algebro- geometric properties of the closures of G-stable pieces. Although the closures are not normal in general, they do have "nice" singularities (for example, they admit a Frobenius splitting and as a consequence, they are all weakly normal). If time allows, we will also discuss some generalization to complete symmetric varieties.

Speaker: **Lizhen Ji** (Michigan)
Title: *Borel-Serre compactification of locally symmetric spaces and applications*
Abstract: Let $\mathbf{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. Let $X = G/K$ be the symmetric space of noncompact type associated with the real locus $G = \mathbf{G}(\mathbb{R})$. Assume that the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, or equivalently, the locally symmetric space $\Gamma\backslash X$ is noncompact. In studying both Γ and $\Gamma\backslash X$, an important role is played by the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$, which is the quotient by Γ of a partial compactification $\overline{X}^{BS}$ of X. For example, together with the Solomon-Tits Theorem for Tits building of $\mathbf{G}$, $\overline{X}^{BS}$ can be shown that Γ is a virtual duality group, but not a virtual Poincare duality group.

In this lecture, I will explain this and other applications, together with the following topics:

1. $\overline{X}^{BS}$ is a Γ-cofinite universal space for proper actions of Γ.
2. A uniform Borel-Serre method to construct compactifications of both symmetric and locally symmetric spaces, in particular, the reductive Borel-Serre compactification.
3. Analogues for Teichmuller spaces and mapping class groups and applications.

Speaker: **Kiumars Kaveh** (Toronto)
Title: *Newton polytopes for flag and spherical varieties*
Abstract: The goal of the talk is to give a natural geometric description of the string polytopes for flag varieties and spherical varieties analogous to the definition of the Newton polytopes for toric varieties. This will be a generalization of a result of Okounkov for Gelfand-Cetlin polytopes of $SP(2n, \mathbb{C})$.

The classical construction of Gelfand and Cetlin associates a convex polytope to each irreducible representation of $GL(n, \mathbb{C})$ in such a way that the integral points in the polytope parameterize the elements of a natural basis for the representation. Equivalently one can think of them as polytopes associated to the ample line bundles on the flag variety. A main feature of the G-C polytopes is that the self-intersection number of a generic section of the line bundle is given by the volume of the corresponding polytope. This can be viewed as the flag variety analogue of the well-known Kushnirenko theorem in toric geometry. Since then G-C polytopes have been generalized to all reductive groups, called “string polytopes”, by the works of Littelmann, Bernstein, Zelevinsky and others. Even further, it has been generalized to spherical varieties by Okounkov (for classical groups) and by Alexeev-Brion for all reductive groups.

After an introduction to G-C and string polytopes, I will discuss the main result of the talk. Namely, we see that the integral points in the string polytope of a dominant weight λ can be interpreted as the highest terms of the elements of the corresponding irreducible representation V_λ (regarded as polynomials on the big cell) with respect to a natural valuation or term order.

In the second part of the talk, I generalize this construction to any algebraic variety equipped with an ample line bundle (even without a group action!). That is, we associate a convex set (which in most cases turns out to be a polytope) to an ample line bundle, hence arriving at a far reaching generalization of Kushnirenko theorem in toric geometry. In particular the Okounkov-Brion-Alexeev polytope of a spherical variety can be obtained in this way. Part of this work is joint with A. G. Khovanskii.

Speaker: **Valentina Kiritchenko** (Jacobs University Breman)
Title: *The Euler characteristic of complete intersections in reductive groups*
Abstract: Consider the following class of hypersurfaces in a complex reductive group: for each representation of the group take all generic hyperplane sections corresponding to this representation. I will present an explicit combinatorial formula for the Euler characteristic of complete intersections of such hypersurfaces. The Euler characteristic is expressed in terms of the weight polytopes of the corresponding representations. In particular, this formula extends the formulas of Bernstein and Khovanskii (all complete intersections in a complex torus) and of Brion and Kazarnovskii (zero-dimensional complete intersections in an arbitrary reductive group).

The main ingredients of my formula are Chern classes of a reductive group. These classes are related to the usual Chern classes of regular compactifications of the group. An adjunction formula involving these Chern classes allows to express the Euler characteristic via the intersection indices of the Chern classes with hyperplane sections. The latter are then computed using the De Concini-Procesi algorithm, which was originally devised for the intersection indices of divisors in wonderful compactifications of symmetric spaces. I will show how to refine this algorithm so that it produces explicit formulas for the intersection indices.

Speaker: **Jon Kujawa** (University of Georgia)
Title: *Cohomology and Support Varieties for Lie Superalgebras*
Abstract: (Joint work with Brian D. Boe and Daniel K. Nakano.) Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a simple classical Lie superalgebra over the complex numbers as classified by Kac [3]. The classical Lie superalgebras are the simple Lie superalgebras whose $\mathfrak{g}_{\overline{0}}$-component is a reductive Lie algebra. Let $G_{\overline{0}}$ be the reductive algebraic group such that $\operatorname{Lie} G_{\overline{0}} = \mathfrak{g}_{\overline{0}}$.

This project entails developing a support variety theory for Lie superalgebras much like the theory for representations in prime characteristic. We first construct detecting subalgebras of $\mathfrak{g}$ and show that these subalgebras arise naturally by using results from invariant theory of reductive groups by Luna and Richardson [5]. In particular, if $R = H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is the relative cohomology for the Lie superalgebra $\mathfrak{g}$ relative to $\mathfrak{g}_{\bar{0}}$ then there exists a Lie subsuperalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ such that

$$R \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{e}_{\bar{1}}^*)^W \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W,$$

where W is a finite pseudoreflection group. By using the finite generation of R we develop a theory of support varieties for modules over the Lie superalgebra (cf. [2]). This description allows us to conclude that the representation theory for the superalgebra over $\mathbb{C}$ has similar features to looking at modular representations of finite groups over fields of characteristic two.

One of our main objectives was to uncover deeper results about combinatorics of the blocks for finite dimensional representations of the Lie superalgebra $\mathfrak{g}$. The "defect" of a Lie superalgebra and the "atypicality" of a simple module (due to Kac-Wakimoto and Serganova) are combinatorial invariants used to give a rough measure of the complications involved in the block structure. We can now provide cohomological and geometric interpretations of the defect of a Lie superalgebra. In particular, this suggests that one could give a more general and functorial definition of defect.

A focus of recent work is the calculation of support varieties in specific cases. We calculate the support varieties for the finite dimensional universal highest weight supermodules (ie. Kac supermodules) for several infinite families of classical Lie superalgebras. When $\mathfrak{g} = \mathfrak{gl}(m|n)$ we are able to use powerful results of Serganova [7] to calculate the support varieties of the simple supermodules. In particular, this allows us to confirm our "atypicality conjecture" discussed in the previous paragraph in the case of $\mathfrak{gl}(m|n)$. These calculations also show that there are striking differences between this theory and the classical theory of support varieties for finite groups.

Let us also mention recent joint work of Irfan Bagci, Jonathan Kujawa, and Daniel K. Nakano on the type W simple Lie superalgebra which suggests that the theory extends to the Lie superalgebras of Cartan type.

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Speaker: **D. Luna** (Fourier Institute - Grenoble)
Title: *Examples of wonderful varieties*
Abstract: Wonderful varieties of rank bigger than 2, under a semi-simple group G, are difficult to describe explicitely. So by "examples" I mean couples (H, S), where H is a subgroup of G such that G/H has a wonderful completion, and where S is the "spherical system" of this completion (i.e. its main combinatorial invariant). I will concentrate on examples for groups of type D_4 and F_4.

Speaker: **Volodymyr Mazorchuk** (Uppsala)
Title: *Schur-Weyl dualities for symmetric inverse semigroups*
Abstract: In this talk I would like to present new Schur-Weyl type dualities which connects the classical symmetric inverse semigroup on $\{1, 2, \ldots, n\}$ (the rook monoid) and the relatively young dual symmetric inverse semigroup on $\{1, 2, \ldots, n\}$. This generalizes both the classical Schur-Weyl duality, the Schur-Weyl type duality between the symmetric group and the partition algebra, and the Schur-Weyl type dualities for the rook monoid discovered by Solomon. An interesting point here is the fact that the dual symmetric inverse semigroup, which was originally defined via a dual categorical construction, now appears as the dual object for the symmetric inverse semigroup from the representation theoretical point of view.

Speaker: **Claus Mokler** (Wuppertal)
Title: *The face monoid associated to a Kac-Moody group*

Abstract: The face monoid and its coordinate ring are obtained from the category of integrable modules of the category O of a symmetrizable Kac-Moody algebra by a Tannaka reconstruction. The face monoid contains the Kac-Moody group as open dense unit group. Its idempotents are related to the faces of the Tits cone. It has similar structural properties as a reductive algebraic monoid. In my talk I will give an overview (on slides) of the algebraic and algebraic geometric results obtained for this monoid as well as for the complex-valued points of its coordinate ring.

Speaker: **Brian Parshall** (University of Virginia)
Title: *Some new highest weight categories with applications to filtrations*
Abstract: Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of positive characteristic $p > h$ (the Coxeter number of G). Let $\mathcal{C}$ be the category of rational G-modules. Assume that for each restricted, dominant weight, the Lusztig character formula holds for the character of the irreducible G-module $L(\lambda)$. In this talk, we present two new highest weight categories $\mathcal{C}^{\rm reg}_{\rm even}$ and $\mathcal{C}^{\rm reg}_{\rm odd}$, which might be called the "even" and the "odd" categories of rational G-modules. These categories are (perhaps remarkably) full subcategories of $\mathcal{C}$. This fact depends on the use of the realization of the standard modules $\Delta^{\rm red}(\lambda)$ and costandard modules $\nabla^{\rm red}(\lambda)$ using quantum groups. We indicate some applications of the result; for example, we mention how it is related to a filtration conjecture.

This talk is based on:

[1] E. Cline, B. Parshall, and L. Scott, "Reduced standard modules and cohomology," Trans. Amer. Math. Soc., in press (2007).

[2] B. Parshall and L. Scott, "Some new highest weight categories," to appear in conference proceedings for ICRT-IV Tibet (2007).

Paper [1] initiates a cohomological study of the modules $\Delta^{\rm red}(\lambda)$, $\nabla^{\rm red}(\lambda)$ and applies this to the conjecture of Gurnalnick on 1-cohomology of finite groups. Paper [2] proves the main result mentioned above.

Speaker: **Mohan Putcha** (North Carolina State University)
Title: *Decompositions of reductive monoids*
Abstract: A reductive monoid M is the Zariski closure of a reductive group G. We will discuss three basic decompositions of of M, each leading to a finite poset via Zariski closure inclusion:

1. The decomposition of M into $G \times G$-orbits. The associated poset is the cross-section lattice Λ. This is a generalization of the face lattice of a polytope. While in general the structure of Λ is quite complicated, it is possible to compute the Möbius function on it.

2. The decomposition of M into $B \times B$-orbits, where B is the Borel subgroup of G. The associated poset R is the Renner monoid of M whose unit group W is the Weyl group of G. For $M_n(k)$, combinatorists know R as the *rook monoid* and semigroup theorists know R as the *symmetric inverse semigroup*. We will discuss the rich algebraic and combinatorial structure of R.

3. There is a decomposition of M related to conjugacy classes that is in betweeen the above two decompositions. The underlying finite *conjugacy poset* $\mathcal{C}$ is yet to be fully understood, but promises to have a very rich combinatorial structure. For the matrix monoid $M_n(k)$, $\mathcal{C}$ consists of partitions of m, $m \leqslant n$, ordered by a generalization of the dominance order on partitions of n. As an application of this decomposition we derive a description of the irreducible components of the nilpotent variety M_{nil} of M.

Speaker: **Daniel K. Nakano** (University of Georgia)
Title: *Cohomology for algebraic groups and Frobenius kernels*
Abstract: (joint work with Christopher P. Bendel, Cornelius Pillen.) Let G be a connected reductive algebraic group scheme, B be a Borel sub- group of G, and U be the unipotent radical of B. One of the outstanding open problems is to generalize the Bott-Borel-Weil theorem to understand the structure of the line bundle cohomology groups $H^\bullet(\lambda) := \mathcal{H}^\bullet(G/B, \mathcal{L}(\lambda))$ over fields of positive characteristic. A related question and significant part of this problem involves computing the rational B-cohomology groups $H^\bullet(B, \lambda)$ where λ is a one-dimensional character.

Let $F : G \to G$ be the Frobenius map and G_r (resp. B_r , U_r) be the r-th Frobenius kernels of G (resp. B, U). In this talk I will discuss recent progress in computing cohomology groups for algebraic groups and Frobenius kernels. My objectives for the talk are as follows:

1) Outline how the cohomology calculations for $H^\bullet(B, \lambda)$, $H^\bullet(G_r, H^0(\lambda))$, $H^\bullet(B_r, \lambda)$, $H^\bullet(U_r, k)$, and $H^\bullet(u, k)$ (ordinary Lie algebra cohomology for $u = \mathrm{Lie}\, U$) are interrelated.

2) Briefly discuss connections with B-cohomology and computing cohomology for Specht modules for symmetric groups due to Hemmer-Nakano [HN]. This topic falls under Section 3 of the Conference Objectives.

3) Discuss two conjectures related to these cohomological calculations:

a) Donkin's Conjecture [D]: This conjecture has a counterexample which was discovered by van der Kallen [vdK]. However, a modified version will be explained and formulated.

b) Induction Conjecture: This conjecture connects the B_r-cohomology with G_r-cohomology.

4) Exhibit explicit cohomological calculations (via slides) for H^1 and H^2 even for small primes [BNP1, BNP2, W]. Generic behavior will be discussed.

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Speaker: **Zinovy Reichstein** (Univ. of British Columbia)
Title: *Essential dimension and group compactifications*
Abstract: The essential dimension of an algebraic object (e.g., of a finite-dimensional algebra, a polynomial, an algebraic variety or a group action) is the minimal possible number of independent parameters required to define the underlying structure. In recent years this notion has been studied by a number of algebraic, geometric and cohomological techniques. In the first part of this talk I will give an overview of this topic. In the second part, based on recent joint work with Ph. Gille, I will discuss a particular lower bound on the essential dimension, conjectured by J.-P. Serre. Our proof of this bound relies on the existence and properties of regular group compactifications.

Speaker: **Nicolas Ressayre** (Monpellier)
Title: *Geometric invariant theory and eigenvalue problem*
Abstract: Let A be an Hermitian matrix: it is diagonalizable with real eigenvalues. Let $\lambda(A)$ denote its increasing spectrum. Set

$$\Delta(l) = \{(\lambda(A_1), \cdots, \lambda(A_l)) \mid A_i \text{ Hermitian with } A_1 + \cdots + A_l = 0\}.$$

The set $\Delta(l)$ is actually a convex polyedral cone. The cone $\Delta(l)$ may also be described in terms of the tensor products of representations of $\mathrm{SL}(n)$ and there are has generalizations for all simple groups.

The question to determine explicitely the inequalities fullfilled by the points of $\Delta(l)$ began with H. Weyl in 1912. Recently, Belkale and Kummar have proposed a list of inequalities which characterize the cone $\Delta(l)$ (and its genralisations for the others simple groups) paramitrized by a condition expressed in terms of a new product on the cohomology group of the flag varieties. Here, we assert that the list of Belkale and Kummar is minimal. The proof is made by using GIT.

Speaker: **Alvaro Rittatore** (Universidad de la Republica)
Title: *The structure of algebriac monoids: the affine case*
Abstract: In the 80's, L. Renner asked the following questions: "Is it true that if an algebraic monoid M is such that its unit group is affine, then M is affine?"; "is it possible to extend Chevalley's Theorem on the structure of algebraic groups to the case of algebraic monoids?". Recently (M. Brion '07), such a structure theorem has been proved. In this talk we concentrate on the first step of this study, namely we show that the first question has a positive answer.

Speaker: **Leonard L. Scott** (Univesity of Virgina)
Title: *Semistandard filtrations in highest weight categories*
Abstract: A definition of semistandard filtration of an object in a highest weight category is given, assuming finiteness of the indexing weight set and of all composition series in the latter category. These filtrations are studied especially in maximal submodules of standard modules, and their behavior under exact functors, such as translation to a wall in an algebraic groups setting, is examined. Some applications are given to extension groups for irreducible modules.

Speaker: **Benjamin Steinberg** (Carleton University)
Title: *Möbius functions and semigroup representation theory*
Abstract: Using Rota's theory of Möbius inversion, we are able to make very explicit the work of Munn and Ponizovskii on representations of inverse semigroups. In particular, one can obtain a formula for multiplicities of representations using only knowledge of the characters of maximal subgroups and the Mobius function of the idempotent semilattice. Since most important inverse monoids, such as Renner monoids of algebraic monoids, have Eulerian semilattices, this leads to relatively simple formulas.

The results for inverse semigroups can be made to work for other classes of semigroups including semigroups of upper triangular matrices over a field. This leads to applications in computing spectra of random walks on such semigroups.

Speaker: **V. Uma** (Madras)
Title: *Equivariant K-theory of compactifications of algebraic groups*
Abstract: In this talk we shall describe the $G \times G$-equivariant K-ring of X, where X is a regular compactification of a connected complex reductive algebraic group G. Furthermore, in the case when G is a semisimple group of adjoint type, and X its wonderful compactification, we shall describe its ordinary K-ring $K(X)$. More precisely, we prove that $K(X)$ is a free module over $K(G/B)$ of rank the cardinality of the Weyl group. We further give an explicit basis of $K(X)$ over $K(G/B)$, and also determine the structure constants with respect to this basis.

The above results have recently appeared in my paper titled "Equivariant K-theory of compactifications of algebraic groups" in Transformation Groups, Vol. 12, No.2, 2007, pp. 371–406.

Some Comments on the Outcome of the Meeting

It is clear that "embedding theory", as portrayed in the activities of this conference, constitute a deep and important part of mathematics. Embedding theory has its roots in the 19th century work of Cayley, Klein, Schubert, Cartan and Hilbert. Since then it has been infused with the 20th century developments of Chevalley, Weil, Nagata, Borel, Tits and Mumford. Some of the main general questions are

1. What "is" symmetry?

2. How is it compactified?

3. What does it "look like" at infinity?

4. How is it measured?

5. How do singularities play a role?

Each of these fundamental questions involves some important statements from the geometry of embeddings, combined with some important statements from representation theory.

The first important outcome of this exciting meeting was to "reaquaint" some of the main players in representation theory with some of the main players in embedding geometry. Many important, developing themes in algebra stem from the interaction of the following general themes.

1. Schur algebras, Highest weight categories and character formulas,

2. embedding theory of reductive groups,

3. algebro-geometric methods in representation theory,

4. geometric-topological methods in representation theory, and

5. intersection homology.

The second major outcome of the meeting was to aquaint some of the established researchers with some of the developing young people. This was particularly successful. We had ten young and vigorous researchers participating (Can, Cupit-Foutou, He, Kiritchenko, Kaveh, Maffei, Parker, Tchoudjem, Therkelsen, Uma). Many of these young scholars have already made significant contributions to embedding theory. Unfortunately there was not sufficient time for all of them to give a presentation.

List of Participants

Alexeev, Valery (University of Georgia)
Andersen, Henning Haahr (Aarhus University)
Arzhantsev, Ivan (Moscow State University)
Brion, Michel (Universite de Grenoble)
Can, Mahir (University of Western Ontario)
Carrell, James (University of British Columbia)
Cupit-Foutou, Stephanie (Universitaet zu Koeln)
De Concini, Corrado (University of Rome)
Dlab, Vlastimil (Carleton University)
Donkin, Stephen (University of York)
Doty, Stephen (Loyola University Chicago)
Giaquinto, Anthony (Loyola University Chicago)
Godelle, Eddy (Universite de Caen)
Hausen, Juergen (Universitaet Tuebingen)
He, Xuhua (SUNY - Stony Brook)
Ji, Lizhen (University of Michigan)
Kato, Syu (Univeristy of Tokyo)
Kaveh, Kiumars (University of Toronto)
Kiritchenko, Valentina (Jacobs University Bremen (Germany))
Knop, Friedrich (Rutgers University)
Kujawa, Jonathan (University of Oklahoma)
Luna, Dominique (Fourier Institute - Grenoble)
Maffei, Andrea (Universita di Roma 1)
Mazorchuk, Volodymyr (University of Uppsala)

Mokler, Claus (Wuppertal University (BUGH))
Nakano, Daniel K. (University of Georgia)
Parker, Alison (University of Leeds)
Parshall, Brian (University of Virginia)
Putcha, Mohan (North Carolina State University)
Reichstein, Zinovy (University of British Columbia)
Renner, Lex (University of Western Ontario)
Ressayre, Nicolas (Universite Montpellier 2- I3M- Monpellier(France))
Rittatore, Alvaro (Universidad de la Republica)
Scott, Leonard (University of Virginia)
Steinberg, Benjamin (Carleton University)
Tchoudjem, Alexis (Universit Lyon I)
Therkelsen, Ryan (North Carolina State University)
Timashev, Dmitri (Moscow State University)
Uma, V. (Indian Institute of Technology, Madras)

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Chapter 33

Entropy of Hidden Markov Processes and Connections to Dynamical Systems (07w5103)

Sep 30 - Oct 05, 2007

Organizer(s): Brian Marcus (University of British Columbia), Karl Petersen (University of North Carolina), Tsachy Weissman (Stanford University)

Workshop Overview

The focus of this workshop was entropy rate of Hidden Markov Processes (HMP)'s, related information-theoretic quantities and other connections with related subjects. The workshop brought together thirty mathematicians, computer scientists and electrical engineers from institutions in Canada, US, Europe, Latin America and Asia. While some participants were from industrial research organizations, such as Hewlitt-Packard Labs and Philips Research, most were from universities, representing all academic ranks, including three postdocs and four graduate students.

The participants came from a wide variety of academic disciplines, including information theory (source, channel and constrained coding), dynamical systems (symbolic dynamics, iterated function systems and Lyapunov exponents), probability theory, and statistical mechanics.

There were eighteen 50-minute lectures from Monday through Friday (see section 33 and the Appendices). This left plenty of time for spirited, yet informal, interaction at breaks, over meals, at a Sunday evening reception and a Wednesday afternoon hike. In addition, there were two problem sessions, one on Tuesday evening to collect and formulate open problems and one on Thursday evening to discuss approaches and possible solutions.

The workshop introduced and improved contacts among groups of experts previously working apart, with different languages and approaches. New joint work is now in progress.

Overview of the Subject

An HMP is a stochastic process obtained as the noisy observation sequence of a Markov chain. A simple example is a binary Markov chain observed in binary symmetric noise, i.e., each symbol (0 or 1) in a binary state sequence generated by a 2-state Markov chain is flipped with some small probability, independently in time.

Since any stochastic process can be approximated by a Markov process of some order, and since Markov processes are easy to describe, it is not surprising that HMP's are encountered in an enormous variety of applications involving phenomena observed in the presence of noise. These range from speech and optical character recognition, through target tracking, to biomolecular sequence analysis. HMP's are important objects of study, in their own right, in many areas of pure and applied mathematics including information theory, probability theory, and dynamical systems. An excellent survey on HMP's can be found in [10].

A central problem is computation of the Shannon entropy rate of an HMP. The entropy rate of a stationary stochastic process is, by the Shannon-McMillan-Breiman theorem, the asymptotic exponential growth rate of the effective number of different sequences that can be generated by the process. The entropy rate is a measure of randomness of a stochastic process and is the most fundamental notion in information theory. It is closely related to important quantities in statistical mechanics [28].

There is a very simple closed-form formula for the entropy rate of a finite-state Markov chain. But there is no such formula for entropy rate of HMP's, except in very special cases.

The problem of computing entropy rate for HMP's was addressed fifty years ago by Blackwell [6], who discovered an expression for the entropy rate of an HMP as the integral of a very simple integrand with respect to a typically very complicated (and usually singular) measure on a simplex. Shortly afterwards, Birch [5] discovered excellent general upper and lower bounds on the entropy rate. However, until recently, there had been very little progress, with only a few papers on the subject.

Closely related is the problem of computing the capacity of channels with memory. The capacity is defined as the maximum mutual information rate of input and output processes, over all possible input processes. For some channels, this amounts to maximizing the entropy rate of the output process, which would be an HMP if the input process were Markov.

This problem has received a good deal of attention from people working in many different areas, primarily information theory, dynamical systems, statistical mechanics, and probability theory. And in symbolic dynamics, there is the related problem of characterizing which mappings between symbolic dynamical systems always map Markov chains to Markov chains or always lift Markov chains to Markov chains [7]. [34].

Recent Developments

Recently there has been a rebirth of interest in information-theoretic aspects of HMP's and, in particular, in the computation of entropy rate:

- Application of ideas from filtering, de-noising and state estimation problems to estimation of entropy rate [23].
- New approaches to computing asymptotics of entropy rate as process parameters tend to extreme values [17], [23], [12] [36].
- Application of ideas from statistical mechanics [36], [8].
- Novel variations on earlier techinques [5] giving more efficient upper and lower approximations to entropy rate and bounds on the rates of convergence of the approximations [9], [13].
- Smoothness and analyticity of entropy rate of HMP's [12].
- Connections between Lyapunov exponents for random matrix products and entropy rate of HMP's [16], [17] (earlier, closely related papers on Lyapunov exponents are [3], [4], [25], [26], [29], [30]).
- Monte-Carlo estimation of entropy rate [1], [27]
- Optimization of mutual information rate and application to capacity of noisy channels with memory [16], [19], [33].
- New results on computation of capacity of channels with feedback [35].
- Connections with iterated function systems, a growing area of dynamical systems [2], [31] [32].

Overview of Lectures

For more content on the lectures, see the list of abstracts in Appendix 2. Most of the lecturers have provided links to their talks, which can be found at:

http://www.math.unc.edu/Faculty/petersen/

- Monday:

  The first day focused on *tutorials*, aimed to bridge the language and viewpoint barriers among the diverse group of participants. Also included was a short pictorial history, honouring the "masters" who have made fundamental contributions: D. Blackwell, R. Bowen, H. Furstenburg, A. Markov, A. Lyapunov, C. Shannon, and B. Weiss.

  Weissman, an information theorist, begin with an overview of entropy rate of HMP's and connections to source and channel coding, filtering and de-noising. He prepared so much interesting material, that, by popular demand, he continued his lecture on Monday evening. Boyle, a symbolic dynamicist, followed with a survey of results and problems on mappings from one symbolic dynamical systems to another for which there exists a fully supported Markov chain on the range whose inverse image is fully supported Markov. Verbitsky, who works on the interface of statistical mechanics and symbolic dynamics, surveyed connections between hidden Markov chains and statistical mechanics, including the role of Gibbs states, in one dimension and higher dimensions. Finally, Juang, an electrical engineer, gave a survey of applications of HMP's to speech recognition.

- Tuesday:

  The first three talks focused on work related to *statistical mechanics*. Ugalde presented specific results relating symbolic dynamics, HMP's and Gibbs states in one dimension; in particular, he gave sufficient conditions for an HMP to be a Gibbs state. Zuk followed by describing relations between HMP's and the Ising model, as well as various methods in statistical mechanics that can be used to approximate entropy rate of HMP's, e.g. how to compute higher-order derivatives, with respect to the underlying Markov chain parameters of entropy rate. Then Montanari presented many interesting results on entropy quantities in statistical mechanics.

  The last two talks focused on *computing and approximating entropy rates*. Ordentlich described deterministic approaches, as opposed to Monte-Carlo simulation, for computing entropy. He focused on the precision-complexity trade-off for each approach. Cuff followed with results relating a rather specific HMP to a classical channel (Blackwell's trapdoor channel), showing how results on the channel capacity shed light on entropy of the HMP.

- Wednesday:

  Guo presented a framework for computing entropy rate of HMP's using dynamic programming.

  Slomczynski followed with a survey on connections among HMP's, iterated function systems and quantum systems.

- Thursday:

  The first two talks focused on *analyticity* issues. The first, by Peres, was set in the more general context of Lyapunov exponents of random matrix products. Peres also described applications to computing Hausdorf dimension of fractal sets which are invariant for certain smooth dynamical systems. The second, by Han, focused on analyticity of entropy rate of HMP's and explicit computations of Taylor series coefficients, generalizing techniques introduced in Zuk's talk.

  The last three talks focused on *computation of noisy channel capacity*, rather than entropy rate. Pfister led off with approach to computing derivatives of entropy rate of HMP's and then made the transition to computing derivatives of channel capacity (more precisely derivatives of optimized mutual information rate over inputs of a fixed Markov order) for channels with memory. Vontobel continued with results and questions regarding optimization of the output entropy rate, or mutual information rate, over all Markov chain input processes to a finite state channel. Finally, Jaquette presented results on entropy rate and noisy channel capacity for channels with hard constrained inputs.

- Friday:

  Kavic discussed the general problem of channel capacity, for channels that allow *feedback*, highlighting a recent result showing that Markov chain inputs can achieve channel capacity (in contrast to the situation without feedback).

  Finally, Pollicott presented intriguing theoretical results that yield *numerical schemes*, with very fast convergence, for computation of Lyapunov exponents.

Highlights of Problem Sessions

Most of the lectures included statements of open problems. In the first problem session (Tuesday evening), the open problems stated earlier in the lectures were organized and expanded. Discussion on these problems continued throughout the workshop, with special attention given in the second problem session (Thursday evening). Some partial results and even complete results were given in the second session.

- Calculation of aymptotics of entropy rate of HMP's.

  In Zuk's talk, the problem was posed of computing the asymptotics in the "rate transitions" regime, i.e., the case of a binary symmetric Markov chain, in binary symmetric noise, with transition probability p from one state to the other as p tends to zero. At the second problem session, Quas presented a solution, with subsequent refinements by Peres; a paper on this result is in preparation.

- Estimates on decay rate of correlations for HMP's.

  The decay rate of correlations (as conditioning length increases) is exponential. Estimates on the decay rate should have impact on the domain of analyticity of entropy rate. Verbitsky's lecture contained several estimates and a conjectured improved estimate, which was proven by Peres later in the workshop.

  This is intimately connection to computation of best approximations to contraction factor for HMP's,

- Behaviour of measures under continuous factor maps ϕ from one irreducible SFT to another.

  Characterize those ϕ that are Markovian, i.e., is the inverse image of some fully supported Markov chain is again fully supported Markov? There is a characterization in terms of functions, known as compensation functions, but it is not clear how to make this effective.

  Characterize those ϕ such that every fully supported Markov measure on the domain SFT maps to a Gibbs measure on the range. Chazottes and Ugalde have partial positive results. At the second problem session, Boyle gave a complete characterization in the special case that ϕ is finite-to-one.

- Constructing special Markov chain covers of HMP's:

  Any HMP can be realized from many different Markov chains. There does not appear to be a canonical Markov chain for a given HMP. It was proven long ago [21] that there need not be a covering Markov chain of the same entropy as the HMP. But in the second problem session, Peres gave a procedure for constructing covering Markov chains with entropy arbitrarily close to that of the given HMP. This procedure is conceptually simple, but the covering Markov chain is somewhat complicated and impractical. This raises the question of devising less complex alternative schemes.

- Complexity-Precision trade-offs for computing entropy rate of an HMP:

  This topic was the major focus of Ordentlich's talk. The main problems are to determine the fundamentally optimum precision-complexity tradeoff as well as effective bounds on the approximation error as a function of the number of operations needed to construct the approximation.

  Ordentlich described various approaches for deterministic approximations of the entropy rate. One approach is based on the classical Birch approximations and their more recent variable length improvements due to Egner et. al. [9]. For the former, bounds on the exponential rate of convergence of the approximations are known, but the exact rate is not known. For the latter, only empirical evidence was known.

Another approach is to approximate, via quantization, the stationary distribution of a simplex-valued Markov process which is closely related to Blackwell's measure and, hence, is closely linked to the entropy rate. This approach leads to an approximation procedure, described by Ordentlich, yielding the best known precision-complexity tradeoff.

- Biological models:

  As described above, an HMP is the result of a Markov chain observed at the receiving end of a noisy channel. In this case, the channel is described by conditional probabilities of changing one symbol into another. In a biological setting, these changes could be viewed as "mutations." In addition to mutations, one could also allow insertions and deletions. The analysis of this type of model is important in biology, where, for instance, one might want to decide if two given genomes were derived from a common ancestor and, if so, what was the sequence of changes (mutations (M), insertions (I) and deletions (D)) that led to the two outcomes. More precisely, given two strings Y and Y' one could ask to find the most likely string of M's, I's and D's, which would transform a common ancestor of Y and Y' into Y and Y'. It was observed that even noisy channels which allow only deletions is difficult to analyze.

- Characterization of those stationary stochastic processes that have only countably many conditional measures.

  Kitchens and Tuncel [20] characterized those stationary stochastic processes that have only finitely many conditional measures. The analogous countable case is of interest since if the process were an HMP, with countable support of the Blackwell measure, entropy rate can be expressed as a series expansion.

- Capacity of finite-state channels

  Continuing on the thread of Pfister's and Vontobel's talks, the longstanding problem of computing the capacity of a noisy finite-state channel was discussed. One approach would, to some extent, depend on the question of whether a particular information rate function is concave as a function of the input distribution [33]. Compuatation of capacity for channels which allow feedback is better understood [35], but there are still open problems in that area.

- Capacity of noisy-constrained channels

  Low-order asymptotics have been developed for capacity of binary symmetric channels whose inputs are constrained by a finite-type constraint [18], [14]. There was discussion of how to extend this work to higher order asymptotics and more complex channels.

- Ising models on trees and graphs: There was considerable discussion on the problem of extending results on entropy quantities to these kinds of models.

Appendix 2: Abstracts

MONDAY:

- *T. Weissman (EE, Stanford):*

  Title: Overview of entropy rate of HMP's.

  Abstract: I will present an overview on the entropy rate of hidden Markov processes (HMP's), information and coding-theoretic motivation for its study, and some of its connections to dynamical systems, to non-linear filtering, and to statistical physics. Particular attention will be given to:

  – Alternative representations: via the Blackwell measure, as a Lyapunov exponent, and as a partition function in statistical physics.

  – Bounds and approximations (stochastic and deterministic), and their complexity-precision tradeoffs.

  – Asymptotic regimes and analyticity.

- *M. Boyle (Math, Maryland):*

  Title: Overview of Markovian maps

  Abstract: A topological Markov shift is the support of a Markov chain (measure); that is, it is the set of infinite sequences all of whose finite subwords have strictly positive probability (measure). A topological Markov shift can support many different Markov chains, including higher-order chains (on which the past and future become independent after conditioning on finitely many steps in the past).

  Now let f be a sliding block code from a topological Markov shift S onto another topological Markov shift T. We assume S is irreducible (it is the support of an irreducible/ergodic Markov chain).

  Then there is a dichotomy: either every Markov measure on T lifts (via f) to a Markov measure on S, or every Markov measure on T does not lift to a Markov measure on S. In the former case, the map f is called Markovian. The Markovian condition is a thermodynamic phenomenon and is the first of a range of conditions on the regularity of the map f. I will try to explain this condition, the related conditions, and related work due to myself, Petersen, Quas, Shin, Tuncel, Walters and others.

- *E. Verbitsky (Philips-Eindhoven):*

  Title: Thermodynamics of Hidden Processes

  Abstract: Hidden Markov processes have a number of very strong properties. I will argue that some of these properties can be explained using the language of Statistical Mechanics and Thermodynamic Formalism. For similarly defined hidden Markov random fields ($d > 1$) the picture is much more complex. I will illustrate it with a number of examples and open questions.

- *B.H. Juang (ECE, Georgia Institute of Technology):*

  Title: Hidden Markov Model and its Application in Speech Recognition – A Tutorial.

  Abstract: Speech signals are produced everyday and are considered one physical-behavioral phenomenon that is most unique and intriguing. A speech signal carries a code that can be understood (decoded) by the listener but the same code may be realized as acoustic signals with vastly different physical properties. Variations across pitch, power level, prosodic manifests, and talker (including gender, the articulatory apparatus, etc.) are observed to be wide and broad. The hidden Markov model/process has been successfully developed as an effective modeling tool for this rather complex signal in several applications, most notably automatic speech recognition. In this talk, we present justifications for use of hidden Markov process for speech modeling, elaborate the mathematical development of such a tool over the past two decades, and discuss applications of this mathematical formalism in practical systems that are in use in our daily life.

TUESDAY:

- *E. Ugalde (Math, Universida Autonama de San Luis Potosi):*

  Title: On Gibbs measures and lumped Markov chains

  Abstract: Gibbs measures are fully determined by continuous functions or potentials, and admit a nice thermodynamic characterization. In the symbolic case, for finite type or sofic subshifts, they may arise as measures induced from Markovian ones, after amalgamation of symbols in the original alphabet. We have found sufficient conditions for the induced measure to be Gibbsian, and under those conditions we are able to determine the nomalized potential of the measure. Even though in general the measure induced by amalgamation of symbols is not Gibbsian, an induced potential can be defined almost everywhere. The induced measure admits therefore a thermodynamic characterization (joint work with J.-R. Chazottes)

- *O. Zuk (Physics of Complex Systems, Weizmann Institute):*

  Title: HMP's Entropy Rate - Statistical Mechanics and Taylor Series Expansions

  Abstract: Hidden Markov Models are very similar to models encountered in statistical physics - specifically the Ising model in a random field. In this talk I will discuss the similarities and differences between the two models.

  I will also describe our asymptotic results for the entropy rate and other quantities of interest in both models in various regimes, including an algorithm for calculating the Taylor series coefficients of the entropy rate.

  I will end with numerical results and conjectures for the radius of convergence of the Taylor expansion. (based on joint work with E. Domany, I. Kanter and M. Aizenman).

- *A. Montanari (EE, Stanford):*

  Title: The rank of random band diagonal matrices in the Kac limit

  Abstract: Consider a stream of iid Bernoulli(1/2) x_t. At time t, you observe the mod 2 sum

  $$y_t = x_{i_1}(t) + ... + x_{i_k}(t)$$

  where $i_1, .., i_k$ are uniformly random in $[i - R, i + R]$, through an erasure channel. Further you know $i_1, ...i_k$. I provide several estimates of $H(X^t|Y^t)/t$ in the large R limit. In physics this is known as the Kac limit after the seminal work of Marc Kac.

- *E. Ordentlich (HP-Labs, Palo Alto):*

  Title: Deterministic algorithms for computing/approximating the HMP entropy rate.

  Abstract: We survey known deterministic algorithms for approximating the entropy rate of hidden Markov models. We will consider the well known approach based on truncated conditional entropies as well as less studied approaches based on quantized likelihood processes. The various approaches will be compared on a complexity versus accuracy basis, to the extent that this tradeoff is known.

- *P. Cuff (EE, Stanford):*

  Title: Entropy Rates of Hidden Markov Processes emerge from Blackwell's Trapdoor Channel

  Abstract: Blackwell's trapdoor channel is a simple channel with memory that has been widely investigated during the past four decades. The non-feedback channel capacity has not been solved analytically, but we find the feedback capacity to be the logarithm of the golden ratio.

  During the investigation of the trapdoor channel we find that, with the assistance of feedback, the channel can be transformed into an equivalent memoryless channel with a constrained input. A Markov source as input satisfies the constraints, but calculating the resulting mutual information requires finding the entropy rate of a Hidden Markov process.

  In hindsight, after finding the capacity of the trapdoor channel with feedback, we recognize that we can express the entropy rate of a particular class of Hidden Markov processes in closed form. This is the class of Hidden Markov processes for which the conditional entropy rate of the states given the observations is zero, so the entropy rate is that of the joint states and observations, which is a Markov process. We will comment on relations between the HMP transition probabilities and satisfiability of said conditional entropy rate condition.

WEDNESDAY:

- *D. Guo (EECS, Northwestern):*

  Title: On The Entropy and Filtering of Hidden Markov Processes Observed Via Arbitrary Channels

  Abstract: We study the entropy and filtering of hidden Markov processes (HMPs) which are discrete-time binary homogeneous Markov chains observed through an arbitrary memoryless channel. A fixed-point functional equation is derived for the stationary distribution of an input symbol conditioned on all past observations. While the existence of a solution to this equation is guaranteed by martingale theory, its uniqueness follows from contraction mapping property. In order to compute this distribution, the fixed-point functional equation is firstly converted to a linear system through quantization and then solved numerically using quadratic optimization. The entropy or differential entropy rate of the HMP can be computed in two ways: one by exploiting the average entropy of each input symbol conditioned on past observations, and the other by applying a relationship between the input-output mutual information and the stationary distribution obtained via filtering.

- *W. Slomczynski (Jagiellonian University):*

  Title: Entropy integral formula: from hidden Markov processes to quantum systems.

  Abstract: We investigate the notion of dynamical entropy (or entropy rate) in the context of the statistical (or operational) approach to dynamical systems. In this approach we can distinguish the kinematical and dynamical parts. In kinematics we define states of the system and observables, and in dynamics, the evolution operators describe changes of the state. Moreover, we describe mathematical objects depicting measurement procedures, called measurement instruments. This approach makes it possible to describe both classical and quantum phenomena by a single mathematical formalism. States are defined as the positive elements of the unit sphere in a certain ordered vector space and evolution operators as Markov (stochastic) operators in this space. By an equilibrium we mean a fixed point of a given Markov operator.

  We present a method of computing entropy rate based on an integral formula. This method enables us to generalise some old formulae for dynamical entropy and to prove new ones, to work out numerical methods for computing entropy, and to investigate the basic properties of dynamical systems.

  The reasoning leading to the proof of the integral formula is based on: attributing an iterated function system to each dynamical system and measurement instrument, investigating the properties of the iterated function system guaranteeing the existence and the uniqueness of an invariant measure, and justifying the integral formula using the properties of the iterated function system.

  Integral formulae for entropy rate were previously shown in particular cases, where the state space was finite-dimensional, by David Blackwell for functions of a Markov chain and by Mark Fannes, Bruno Nachtergaele and Leo Slegers for the so-called algebraic measures. Here we present a unified approach to the problem and show general results utilising two techniques: the first uses the compactness of subsets of the state space in certain weak topologies, the second is based on employing the projective metric in the state space. Applying these methods, we obtain results concerning iterated function systems on the state space and dynamical entropy for many concrete state space types.

  Applications of the integral formula include hidden Markov processes, kernel operators, Frobenius-Perron operators, and quantum systems (Srinivas-Pechukas-Beck-Graudenz entropy, quantum jumps, coherent states entropy).

THURSDAY:

- *Y. Peres (Microsoft):*

  Title: Analyticity of Lyapunov exponents

  Abstract: I will describe the relevance of the entropy of HMM and of Lyapunov exponents to determining dimension of slices and projections of fractals, then survey the analyticity of Lyapunov exponents via the polynomial approximation approach, the significance of obtaining explicit domains of analyticity and the Hilbert metric. Finally, I will work out a simple but illuminating example of integration with respect to coin tossing measures and determine a domain of analyticity there.

- *G. Han (Math, Hong Kong U.):*

  Title: Analyticity and Derivatives of entropy rate for HMP's

  Abstract: We prove that under mild positivity assumptions the entropy rate of a hidden Markov chain varies analytically as a function of the underlying Markov chain parameters. A general principle to determine the domain of analyticity is stated. We also show that under the positivity assumptions the hidden Markov chain itself varies analytically, in a strong sense, as a function of the underlying Markov chain parameters. For a natural class of hidden Markov chains called "Black Hole", we show that one can exactly compute any derivatives of entropy rate.

- *H. Pfister (ECE, Texas A& M):*

  Title: The Derivatives of Entropy Rate and Capacity for Finite-State Channels

  Abstract: This talk discusses a number of topics related to the entropy rate and capacity of finite-state channels. A simple formula is given for the derivative of the entropy rate and it is used to compute closed-form expansions for the channel capacity in the high noise regime. The relationship between this formula and previous results is discussed.

  The derivative formula is then extended to the Lyapunov exponent of a sequence of random matrices. In particular, we discuss i.i.d., Markov, and hidden Markov matrix sequences. The last case is closely related to the derivative of the divergence between two hidden Markov processes.

  The talk concludes with a short discussion of ergodic properties and mixing conditions of the forward Baum-Welch (a.k.a. BCJR) algorithm.

- *P. Vontobel (HP-Labs, Palo Alto):*

  Title: Optimizing Information Rate Bounds for Channels with Memory

  Abstract: We consider the problem of optimizing information rate upper and lower bounds for communication channels with (possibly large) memory. A recently proposed auxiliary-channel-based technique allows one to efficiently compute upper and lower bounds on the information rate of such channels.

  Towards tightening these bounds, we propose iterative expectation-maximization (EM) type algorithms to optimize the parameters of the auxiliary finite-state machine channel (FSMC). We provide explicit solutions for optimizing the upper bound and the difference between the upper and the lower bound and a method for the optimization of the lower bound for data-controllable channels with memory. We discuss examples of channels with memory, for which application of the developed theory results in noticeably tighter information rate bounds.

  Interestingly, from a channel coding perspective, optimizing the lower bound is related to increasing the achievable mismatched information rate, i.e. the information rate of a communication system where the maximum-likelihood decoder at the receiver is matched to the auxiliary channel and not to the true channel.

  (This talk is based on joint work with Parastoo Sadeghi (ANU) and Ramtin Shams (ANU).)

- *P. Jacquet (INRIA):*

  Title: Entropy of HMP and asymptotics of noisy input-constrained channel capacity

  Abstract: In this talk, we consider the classical problem of noisy constrained capacity in the case of the binary symmetric channel (BSC), namely, the capacity of a BSC whose input is a sequence from a constrained set. We derive an asymptotic formula (when the noise parameter is small) for the entropy rate of a hidden Markov chain, observed when a Markov chain passes through a binary symmetric channel. Using this result we establish an asymptotic formula for the capacity of a binary symmetric channel with input process supported on an irreducible finite type constraint, as the noise parameter ϵ tends to zero. For the (d, k)-Run Length Limited (RLL) constraint, we show that when $k \leq 2d$, the difference between the noisy capacity and noiseless capacity is $O(\epsilon)$ and when k¿2d, it is $O(\epsilon \log \epsilon)$ with explicitly computable constants (joint work with G. Han, B. Marcus, G. Seroussi, and W. Szpankowski).

FRIDAY:

- *A. Kavcic (ECE, Hawaii):*

  Title: Markov and hidden Markov Processes in communication channels used with feedback

  In this talk, we consider finite memory communications channels (finite-state channels, or state-space representable channels). Such channels are reasonably good models for magnetic and optical data storage, wireless communications in multipath environments, and communications through band-limited media. The channel capacity is typically obtained by optimizing the channel input process to maximize the entropy of the channel output. If the channel input is a Markov process, then the channel output is a hidden Markov process, and the problem is equivalently stated as the maximization of the entropy of a hidden Markov process. It is well known that even if the channel has finite memory, the channel capacity is generally not attained by a finite-memory channel input process, so generally, finite-memory Markov processes do not achieve the capacities of finite-memory channels. However, if feedback from the receiver to the transmitter is utilized, then a certain class of finite-memory conditionally Markov sources do achieve the feedback capacities of finite-memory channels. We establish some basic results for this case: 1) Finite-memory conditionally Markov sources achieve the capacities of finite-memory channels, 2) The optimal processor of the feedback is the forward recursion of the sum-product algorithm (i.e, the forward recursion of the Baum-Welch algorithm, or the Kalman-Bucy filter, depending on the application), 3) This generalizes Shannon's well-known result that memoryless sources achieve the (feedback) capacities of memoryless channels, i.e., we now have that finite-memory conditionally Markov sources achieve the capacities of finite-memory channels. An interesting consequence is that decoders for codes that achieve feedback capacities need not utilize long buffer memories, but rather the decoders can be implemented using extremely simple detection/estimation techniques already available in the statistical signal processing literature. We give several examples of how this applies to some well-known single-input-single-output channels. Further, we consider the open problem of establishing the capacity (or capacity bounds) for the relay channel, and show that similar results apply for relay channels with either deterministic or randomly fading finite intersymbol interference memory.

- *M. Pollicott (Math, Warwick):*

  Title: Computing integrals, Lyapunov exponents and entropy using cycle expansions

  Abstract: I will describe an approach which is based upon the study of certain analytic functions (called dynamical determinants) and studied by Ruelle. In certain cases, some of the above quantities can be "read off" from these functions. Using some classical ideas on determinants (originating with Grothendeick in the 1950s) one can rapidly approximate these analytic functions by polynomials. "Cycle expansions" refers to the explicit method, used by Cvitanovic et al, for computing these polynomials (and thus computing numerically the associated quantities).

List of Participants

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Bailey Frick, Sarah (Ohio State University)
Boyle, Mike (University of Maryland)

Cuff, Paul (Stanford)
Glynn, Peter (Stanford University)
Guo, Donging (Northwestern University)
Han, Guangyue (University of Hong Kong)
Jacquet, Philippe (Inst. Natl.Recherche en Inform. et Automat.)
Juang, Fred (Georgia Tech)
Kavcic, Alek (University of Hawaii)
Luo, Jun (Northwestern University)
Marcus, Brian (University of British Columbia)
Montanari, Andrea (Stanford University)
Moon, Taesup (Stanford)
Ordentlich, Erik (HP Labs)
Pavlov, Ronnie (University of British Columbia)
Peres, Yuval (Microsoft Research)
Petersen, Karl (University of North Carolina)
Pfister, Henry (Texas A&M University)
Pollicott, Mark (University of Warwick)
Quas, Anthony (University of Victoria)
Shin, Sujin (Korea Advanced Institute of Science and Technology)
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Weissman, Tsachy (Stanford University)
Williams, Susan (University of South Alabama)
Yayama, Yuki (University of North Carolina, Chapel Hill)
Zuk, Or (Weizmann Institute)

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Chapter 34

Recent progress on nonlinear elliptic and parabolic problems and related abstract methods (07w5004)

Oct 07 - Oct 12, 2007

Organizer(s): E. Norman Dancer (University of Sydney), Yihong Du (University of New England), Konstantin Mischaikow (Rutgers University), Peter Polacik (University of Minnesota), Xiaoqiang Zhao (Memorial University of Newfoundland)

Overview

In recent years, significant progress has been made on the analysis of a number of important features of nonlinear partial differential equations of elliptic and parabolic type. These equations arise from mathematical biology, chemical reaction theory, material science, water waves and many other areas of science. For applications as well as for completeness of the mathematical theory, new methods have been established for the study of spatial and temporal patterns, sharp layers and spikes, blow-up, travelling waves, to mention but a few. These progress depend, to a large extent, on the interaction and further development of an extensive range of techniques, including topological and variational methods, bifurcation theory, singular perturbation, infinite dimensional dynamical systems, elliptic and parabolic estimates.

The workshop brought together some of the most distinguished mathematicians in the world and many of the top young researchers in the field. It provided in a very timely fashion a platform to facilitate the dissemination of the most recent research ideas and techniques in this active area of research. The talks during the workshop stimulated extensive discussions and collaborations, and we have every reason to expect more fruitful collaborations after the workshop.

Apart from the scientific programs, which are detailed below, Professor Jean Mawhin and Professor Henri Berestycki gave two very interesting informal presentations at an evening session to celebrate Professor Norman Dancer's 60th birthday.

Presentation Highlights

Walter Allegretto (Univ of Alberta), *On some parabolic equations motivated by biological problems*: We consider nonlinear parabolic equations that model some physical problems for which boundary effects are extremely important in determining the response of the system. Both theoretical properties and numerical simulations are presented. Examples from mathematical biology are used as illustrations.

Thomas Bartsch (Univ. Giessen) *On a parabolic semiflow with small diffusion*: We consider the singularly perturbed semilinear parabolic problem $u_t - d^2\Delta u + u = f(u)$ with homogeneous Neumann boundary conditions on a smoothly bounded domain $\Omega \subset R^N$. Here f is superlinear at 0 and ∞ and has subcritical growth. For small $d > 0$ we construct a compact invariant set X_d in the boundary of the domain of attraction of the asymptotically stable equilibrium 0. The main features of X_d are that it consists of nonnegative functions that are pairwise non-comparable, and that its topology is at least as rich as the topology of $\partial\Omega$ in a certain sense. This implies the existence of connecting orbits within X_d.

Peter Bates (Michigan State Univ) *Invariant Manifolds of Spikes*: Many singularly perturbed nonlinear elliptic equations have spike-like stationary solutions. These can be found through various methods, including Lyapunov-Schmidt schemes that, in the neighborhood of a proposed spike solution, decompose the operator equation into one that is restricted to a ”normal subspace” and one in a ”tangential subspace”. Here, these subspaces correspond to eigenstates of the operator, linearized at an approximate spike solution, and where ”tangential” means ”corresponding to eigenvalues near zero”, and ”normal” means ”complementary”. In this talk We will describe a more global decomposition in which the ”tangential subspace” is replaced by a finite-dimensional manifold of spike-like states and this manifold is invariant with respect to the corresponding nonlinear parabolic equation and also normally hyperbolic. The stationary spike-like states lie on this manifold.

Henri Berestycki (EHESS) *Generalized travelling fronts passing an obstacle*

Daniel Daners (Univ of Sydney) *The Faber-Krahn inequality for Robin Problems*: We present a Faber-Krahn inequality for the first eigenvalue of the Laplacian with Robin boundary conditions, asserting that amongst all Lipschitz domains of fixed volume, the ball has the smallest first eigenvalue. We discuss the idea of the proof as well as uniqueness of the minimizing domain. The method avoids the use of symmetrisation arguments. This is partly joint work with James Kennedy.

Marek Fila (Comenius Univ.) *Large time behaviour of solutions of a semilinear heat equation with a supercritical nonlinearity*: We consider global positive solutions of the Cauchy problem for a parabolic equation which has an ordered family of regular steady states and a singular steady state. All of them are stable in a suitable sense. We shall discuss the rate of convergence to these steady states which turns out to depend explicitly on the spatial decay rate of the initial function.

Nassif Ghoussoub (University of British Columbia) *Bessel potentials and optimal Hardy and Hardy-Rellich inequalities*: We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball Ω of radius R in R^n, $n \geq 2$, so that the following inequalities hold for all $u \in C_0^\infty(\Omega)$:

$$\int_\Omega V(x)|\nabla u|^2 dx \geq \int_\Omega W(x) u^2 dx$$

and

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \left(\frac{V(x)}{|x|^2} - \frac{v'(|x|)}{|x|}\right)|\nabla u|^2 dx.$$

We then identify a large number of such couples (V, W) – that we call Bessel pairs – and the best constants in the corresponding inequalities. This will allow us to complete, improve, extend, and unify most related results –old and new– about Hardy and Hardy-Rellich type inequalities which were obtained by Caffarelli-Kohn-Nirenberg, Brezis-Vázquez, Adimurthi-Chaudhuri-Ramaswamy, Filippas-Tertikas, Adimurthi-Grossi-Santra, as well as some very recent work by Tertikas-Zographopoulos, Liskevich-Lyachova-Moroz, and Blanchet-Bonforte-Dolbeault-Grillo-Vasquez, among others.

This is joint work with Amir Moradifam.

Massimo Grossi (Univ. of Rome I) *Existence results in the supercritical case*: We show some existence results for supercritical nonlinearities in the unit ball of R^n, for $n > 2$. The basic result concerns the existence of a positive solutions but also nodal solutions and related problems will be considered.

Changfeng Gui (Univ of Connecticut) *A Hamiltonian identity for PDEs and it's application*: In this talk We will present a new hamiltonian identity for PDEs and systems of PDEs. We will also show some interesting applications of the identity to problems related to entire solutions. In particular, we show the Young's law in triple junction configuration for a vector-valued Allen Cahn model in phase transition, and derive a necessary condition for the existence of saddle solutions for Allen-Cahn equation with asymmetric double well potential.

Francois Hamel (Universite Aix-Marseille III) *Uniqueness and further qualitative properties of monostable pulsating fronts*: The talk is concerned with qualitative properties of pulsating travelling fronts in a general periodic framework, for monostable reaction-diffusion equations. First, the uniqueness of pulsating fronts for a given speed is established for Kolmogorov-Petrovsky-Piskunov nonlinearities. These results provide in particular a complete classification of KPP pulsating fronts. To do so, the main tool is to prove the exponential behavior of the front when it approches its unstable limiting state. In the general monostable case, the logarithmic equivalent of the front is proved. For a noncritical speed, the decay rate is the same as in the KPP case. This talk is based on joint works with L. Roques.

Danielle Hilhorst (Univ. Paris-Sud) *Peak solutions of a chemotaxis-growth system*: We consider an elliptic reaction-diffusion system which both contains a nonlinear reaction term and a gradient one. In the case that the gradient term has a dispersal effect, we prove the existence of a boundary peak solution, and in both the cases that the gradient term has a dispersal or an aggregation effect we discuss the existence of an interior peak solution. This is joint work with N. Dancer and Shusen Yan.

Thomas Hillen (Univ of Alberta) *A classification of spikes and plateaus*: In this talk We will propose a simple classification of spike versus plateau local maxima (and minima). The classification uses non-local gradients and the fourth order derivative. It confirms the classical notion of plateaus for Cahn-Hilliard and spikes for Gierer-Meinhardt. Further, We will show that this classification can be used to study stability of spatial patterns.

Vera Mikyoung Hur (MIT) *Steady free-surface water waves with vorticity*: The mathematical problem for free-surface water waves embodies the equation of hydrodynamics, the concept of wave propagation, and the critically important role of boundary dynamics. We will give a precise account of its formulation and discuss its distinct features. Particular emphasis is given to the effects of vorticity. Existence theories of traveling waves will be presented with proofs, at least their ideas. Steady periodic waves on a deep water are traditionally referred to as Stokes waves, whose existence theory is established for a general class of vorticity distributions as an application of generalized degree theory of Healey-Simpson and Rabinowitz's global bifurcation theory. Solitary waves near a KdV soliton are constructed for an arbitrary vorticity profile via a Nash-Moser implicit function theorem. Other mathematical aspects of steady water waves, such as the Cauchy problem, symmetries, and stability/instability will be discussed, if time permits.

Meiyue Jiang (Beijing Univ.) *Semilinear Elliptic Equations with Indefinite Nonlinearities*: Let $\Omega \subset R^n$ be bounded domain with smooth boundary, we consider the Dirichlet problem:

$$\begin{aligned} -\triangle u &= \lambda u + a_+(x)|u|^{q-1}u - a_-(x)|u|^{p-1}u + h(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $a_\pm : \overline{\Omega} \to \mathbf{R}$ are continuous functions and $h : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a C^1 function which is sublinear in u at infinity, and λ is a real parameter, $1 < q < \frac{n+2}{n-2}$ and $p > 1$.

In this talk, based on computations of critical groups and Morse theory, we will present various existence and multiplicity results of solutions.

This is a joint work with K. C. Chang.

Congming Li (Univ of Colorado) *Classification of solutions to some integral systems*: We will present the recent work (joint with W. Chen, C. Jin, J. Lim, and B. Ou) on systems of integral equations related to the Hardy-Littlewood-Sobolev (HLS) inequality. The focus is on the Euler-Lagrange equations of the HLS. The main object is to classify all the nonnegative solutions through the study of symmetry, monotonicity, regularity and the asymptotic of the solutions. The integral form of the method of moving planes is the main tool.

Yuan Lou (Ohio State Univ) *Principal Eigenvalue and Eigenfunction of Elliptic Operator with Large Advection and its Application*: The asymptotic behavior, as the coefficient of the advection term approaches infinity, of the principal eigenvalue of an elliptic operator is determined. As an application a Lotka-Volterra reaction-diffusion-advection model for two competing species in a heterogeneous environment is investigated. The two species are assumed to be identical except their dispersal strategies: one disperses by random diffusion only, and the other by both random diffusion and advection along environmental gradient. When the advection is strong relative to random dispersal, both species can coexist. In some situations, it is further shown that the density of the species with large advection in the direction of resources is concentrated at the spatial location with maximum resources.

Hiroshi Matano (Univ of Tokyo) *Convergence and sharp thresholds for propagation in nonlinear diffusion problems*: We study the Cauchy problem for the equation

$$u_t = u_{xx} + f(u)$$

on R^1, where $f(u)$ is a locally Lipschitz continuous function satisfying $f(0) = 0$. We show that any nonnegative bounded solution with compactly supported initial data converges to a stationary solution as t tends to infinity. Moreover the limit is either a constant or a symmetrically decreasing stationary solution. Note that we assume no condition on the symmetry or monotonicity for the initial data.

We next consider the special case where $f(u)$ is either a bistable nonlinearity or a combustion nonlinearity, and prove the sharpness of transition between propagation and extinction, thus extending the earlier result of Zlatos for any compatcly supported nonnegative initial data.

Finally we discuss how one can extend the above results to the case where f has some sort of discontinuity.

This is a joint work with Yihong Du.

Jean Mawhin (Universite Catholique de Louvain) *Maximum and antimaximum principles around an eigenvalue with constant eigenfunction*: We consider the existence of a maximum and/or an antimaximum principle for the solutions of some abstract linear equations, around an eigenvalue with constant eigenfunction. Applications are given to various boundary value problems for ordinary and partial differential equations.

This is a joint work with J. Campos and R. Ortega.

Pavol Quittner (Comenius Univ.) *Very weak solutions of elliptic equations with nonlinear boundary conditions*: We study very weak solutions of linear (and nonlinear) elliptic equations in bounded domains with nonlinear Neumann boundary conditions. We find a sharp critical exponent with the following property: If the growth of the nonlinearity is subcritical then any very weak solution is bounded. We also find sufficient conditions for the existence of uniform a priori bounds of positive very weak solutions. In the supercritical case we prove the existence of multiple positive unbounded very weak solutions blowing up at a prescribed point of the boundary. This is a joint work with Wolfgang Reichel.

Paul Rabinowitz (Univ of Wisconsin-Madison) *On A class of infinite transition solutions for an Allen-Cahn model equation*: For an Allen-Cahn model equation, we discuss some recent results on the existence of a class of solutions which undergo an infinite number of spatial transitions.

Juan Luis Vazquez (Universidad Autonoma de Madrid) *Nonlinear elliptic and parabolic equations with "incompatible" measures as data*: We consider semilinear ellitpic equations like

$$-\Delta u + e^u = \mu$$

posed in R^2 with right-hand side a measure that contains Dirac deltas. If the intensity of one of these deltas is larger than 4π, then the problem cannot be solved. Parabolic equations shed light on this mystery by producing a natural generalized measure solution.

Wenxian Shen (Univ of Auburn) *Variational Principle for Spatial Spread and Propagation Speeds in Time almost and Space Periodic KPP Models*: The present talk is concerned with spatial spread and propagation speeds in time almost periodic and space periodic KPP models. A notion of spatial spread (propagation) speed interval in any given direction for a time almost periodic and space periodic KPP model is first introduced, which extends the concept of the spatial spread (propagation) speed for time independent or periodic KPP models and can be applied to more general time dependent KPP models. Then a variational principle for the spatial spread (propagation) speed is established. To prove the variational principle, we apply the recently developed principal spectrum theory for general time dependent linear parabolic equations. In addition, based on the variational principle, the influence of time and space variation on the spread speed is discussed. It is shown that the time and space variation speeds up the spatial spread.

Hal Smith (Arizona State Univ) *Applications of monotone systems theory to parabolic systems of partial differential equations*: I would summarize joint work with Moe Hirsch and with German Enciso.

Charles Stuart (EPFL) *A stable branch of solutions of a nonlinear Schrdinger equation*: In joint work with François Genoud, we consider the problem

$$\Delta u(x) + V(x)\,|u(x)|^{p-1}\,u(x) - \lambda u(x) = 0 \text{ with } u \in H^1(R^N)\backslash\{0\} \tag{34.1}$$

where $N \geq 3$ and $V \in C^1(R^N\backslash\{0\})$ is such that

$$\begin{aligned} &(i) \quad \text{for some } b \in (0,2), \quad |x|^b\, V(x) \text{ is bounded as } x \to 0 \\ &\qquad\qquad \text{and} \qquad\qquad |x|^b\, V(x) \to 1 \text{ as } |x| \to \infty, \\ &(ii) \quad 1 < p < 1 + \tfrac{2(2-b)}{N-2}. \end{aligned}$$

We prove that there exist $\lambda_0 > 0$ and $U \in C^1((0,\lambda_0), H^1(R^N))$ such that $(\lambda, U(\lambda))$ satisfies (1) for all $\lambda \in (0,\lambda_0)$. Furthermore, if $1 < p < 1 + \frac{2(2-b)}{N}$, the standing wave $e^{i\lambda t}U(\lambda)(x)$ is an orbitally stable periodic solution of the nonlinear Schrödinger equation

$$i\partial_t w + \Delta w + V(x)\,|w|^{p-1}\,w = 0.$$

We discuss also similar conclusions obtained recently by de Bouard-Fukuizumi and Jeanjean-Le Coz. A crucial ingredient is the non-degeneracy of the ground state of

$$\Delta u(x) + |x|^{-b}\,|u(x)|^{p-1}\,u(x) - u(x) = 0.$$

Susanna Terracini (Universit di Milano-Bicocca) *Spectral properties and uniqueness theorems related to optimal partitions*: We consider the spatial segregation in connection with the asymptotics of competition-diffusion systems as competition grows to infinity. We focus of the associated free boundary problem and we prove uniqueness of the partition with respect to the boundary data. Finally, we shall discuss optimal partition problems associated with eigenvalues and related spectral problems.

Juncheng Wei (Chinese Univ of Hong Kong) *Toda system, Allen-Cahn equation and nonlinear Schrodinger equations*: We show that there is a deep connection between the following Toda system

$$(1) \quad -f_j^{''} = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}}, j = 1, ..., K$$

and two well-known scalar autonomous PDES:

$$(2) \quad \Delta u + u - u^3 = 0 \;\; R^2,$$

$$(3) \quad \Delta u - u + u^p = 0, u > 0 \;\; R^2$$

More precisely, given any solution to the Toda system (I), we can find solutions to (2) or (3) of the form

$$u \sim \sum_{j=1}^{K} W(x - f_{\alpha,j}(y))$$

where $f_{\alpha,j} = f_j(\alpha y) + 2\log\frac{1}{\alpha}$. Central to our construction is the use of Dancer's solutions. Other types of solutions to (3) as well as moduli space of solutions to (3) will also be discussed. (Joint work with M. del Pino, M. Kowalczyk and F. Pacard.)

Tobias Weth (Univ of Giessen) *A priori bounds and multiple existence of solutions to a non-cooperative elliptic system*: We report on recent joint work with J.C. Wei and E.N. Dancer on the Dirichlet problem for a non-cooperative two-component elliptic system in a smooth bounded domain. The system arises in the Hartree-Fock theory for a Bose-Einstein (double) condensate with repulsive interaction modeled by a parameter. We completely determine the parameter range for which the system admits a priori bounds for positive solutions. If the underlying domain is a ball, we also construct positive radial solutions with a prescribed number of intersections. If time permits, I will also address the asymptotic shape of solutions as the strength of the repulsive interaction tends to infinity.

Eiji Yanagida (Tohoku Univ) *Solutions with moving singularities for a semilinear parabolic equation*: For the Fujita-type equation, there exists a singular steady state that is radially symmetric with respect to the singular point. In this talk, we consider singular solutions that are obtained as perturbation of the singular steady state. In some parameter range, given any smooth curve and initial data in a wide class, we establish the local existence of a solution with a singularity moving along the curve. This is a joint work with Shota Sato.

Andrej Zlatos (Univ of Chicago) *Speed-up of Reaction-diffusion Fronts by Strong Flows*: We present recent results on speed-up of traveling fronts by strong flows in reaction-diffusion equations. We characterize periodic flows which can arbitrarily speed up fronts for general combustion-type reactions in two dimensions, as well as those which achieve the fastest speed-up rates for KPP reactions in any dimension. Relations to quenching of reactions and to homogenization in the associated passive scalar equations will also be discussed.

List of Participants

Allegretto, Walter (University of Alberta)
Bartsch, Thomas (University of Giessen)
Bates, Peter (Michigan State University)
Berestycki, Henri (EHESS)
Dancer, E. Norman (University of Sydney)
Daners, Daniel (University of Sydney)
Du, Yihong (University of New England)
Fila, Marek (Comenius University)
Ghoussoub, Nassif (BIRS)
Grossi, Massimo (Universita Roma 1)
Gui, Changfeng (University of Connecticut)
Hamel, Francois (Université Aix-Marseille III)
Hilhorst, Danielle (CNRS and Universite Paris-Sud)
Hillen, Thomas (University of Alberta)
Hur, Vera Mikyoung (Massachusetts Institute of Technology)
Jiang, Meiyue (Peking University)
Li, Congming (University of Colorado at Boulder)
Lou, Yuan (Ohio State University)
Matano, Hiroshi (University of Tokyo)
Mawhin, Jean (Université Catholique de Louvain)
Mischaikow, Konstantin (Rutgers University)
Moameni, Abbas (Queen's University)
Polacik, Peter (University of Minnesota)
Quittner, Pavol (Comenius University)
Rabinowitz, Paul (University of Wisconsin, Madison)
Shen, Wenxian (Auburn University)
Smith, Hal (Arizona State University)
Stuart, Charles (Swiss Federal Institute of Technology)
Terracini, Susanna (Universit di Milano - Bicocca)
Vazquez, Juan Luis (Universidad Autonoma de Madrid (Spain))
Wei, Jun Cheng (Chinese University of Hong Kong)
Weth, Tobias (University of Giessen)
Yanagida, Eiji (Tohoku University)
Zhao, Xiaoqiang (Memorial University of Newfoundland)
Zlatos, Andrej (University of Chicago)

Chapter 35

International Workshop on Robust Statistics and R (07w5064)

Oct 28 - Nov 02, 2007

Organizer(s): Claudio Agostinelli (Universita' Ca' Foscari, Venezia, Italy), Peter Filzmoser (Vienna University of Technology), Matias Salibian-Barrera (University of British Columbia), Arnold Stromberg (Department of Statistics, University of Kentucky)

Overview of the Field

Robust Statistics deals with a pressing problem in statistical applications: many classical statistical methods work well only for high-quality data that can be modelled adequately. In many practical applications, however, this is the exception than the rule. When the sample size is moderately small, the sampling variability (i.e. the variation induced by the random nature of the sample obtained from the underlying population) may dominate the error produced from possible model misspecifications. However, the sampling variance decreases when the sample size increases, and for large data sets this variance can be very small, and the overall error may be mainly due to the systematic bias of model misspecification, which is not reduced by larger sample sizes.

In recent years, we have seen an enormous increase in the amount of data being modelled and analyzed. For example automated electronic data collection, and complex data sets, produce data sets for which both the number of cases and variables much exceed the orders of magnitude that were routine only a decade ago. Two problems may arise when analysing these large and complex data sets with classical statistical methodologies: (a) it may not be easy to fit simple and parsimonious models that reflect equally well all the data; and (b) the sampling variability for such large datasets can be very small, to the extent that, as mentioned above, the possible model misspecification bias (which, unlike the variance of most estimators, does not decrease to zero as the sample size grows) dominates the statistical error, and may put into question the validity of the analysis. Furthermore, in some large-scale applications the interest may even be focused particularly in finding whether there is a subset of the data that do not seem to follow the same model as the majority of them.

Robust statistical techniques are natural candidates to deal with the challenges mentioned above. They are designed to perform well both when the data follow the proposed model, but also when a proportion of them may contain "outliers", that is, observations that do not follow the same model as the other points in the sample. The field of Robust Statistics has been extremely active in the last 40 years, with many robust methods proposed and discussed in the best international statistical journals. Unlike the methodologies derived under the assumption that there exists a relatively simple model (at least in terms of mathematically tractability) that fits all data reasonably well, robust techniques are generally characterized by: (a) high computational complexity; (b) relying on asymptotic properties (because their finite-sample properties are typically unknown); and (c) estimators that do not admit a closed-form formula.

After the seminal work of Peter Huber in the early 60's, the field of Robust Statistics enjoyed several years of active development, and faced increasingly complex problems (both of conceptual and computational nature). As mentioned before, the emergence of large-scale applications (where model misspecification becomes either a concern or a feature directly related to the scientific question of interest) has given the field a new period of attention. The rapid evolution of computer technologies has also been instrumental in allowing many robust techniques to be disseminated for wider use in the general scientific community. An example of the renewed research activity in this field includes the International Conference on Robust Statistics (ICORS) that has been held annually since 2001. This conference has consistently attracted the top scientists working in the area and also young researchers, which has resulted in many successful collaborations and developments in the field. Additionally, in 2004 the European Science Foundation established a research network supporting 10 European institutions for the development of robust statistical methods for complex data.

Although recent years have witnessed increased research interest in robust statistics, many of these techniques are still not widely used in practice. This is probably due to several reasons, among which we list their being particularly difficult to compute (even for relatively simple models) and the lack of a generally accepted set of inference principles and methods to follow after point-estimators have been either calculated or approximated. In addition to these considerations, note that there are currently very few easy-to-use, coherently structured and well documented computer programs to calculate or approximate robust estimators, and certainly no freely available ones with these characteristics. Given the intrinsic difficulty of calculating or approximating these estimators it is not completely surprising that the observed properties of some of these methods may depend on the particular algorithm used to find the point estimators. In some cases one may even find that "the algorithm is the estimator" (David Rocke, personal communication). Hence, in this case it is particularly desirable to have a collection of computer programs developed by, or with close collaboration of, the leading researchers in the field.

The computational complexity of these estimators cannot be overestimated. Many of the approximating algorithms are based on non-trivial numerical optimization techniques, and thus a non-specialist may find it very challenging to even approximate these estimators for most real-life applications. A simple example of the type of computational difficulties found in this area is given by the Least Median of Squares (LMS) estimator [30] for linear regression. This estimator is defined as the vector of coefficients that minimizes the median of the squared residuals. To compute the LMS estimator we need to solve an optimization problem in many variables with a non-convex and non-smooth objective function. Although some algorithms exist to find an approximate solution their complexity increases exponentially with the dimension of the problem and become unfeasible for high-dimensional problems (as those commonly found in modern statistical applications). Note that the intrinsic challenge here is not only the lack of smoothness of the LMS objective function being optimized, but also that it is not convex, and thus requires some variation of random search. Furthermore, the objective functions of many robust estimators are non-convex. Intuitively, this follows from the fact that, in order to obtain a robust estimator, the corresponding score equations (which involve the first derivative of the objective function being optimized) need to decrease to zero for large values of their argument. This, together with the requirement that the score equations be similar to the likelihood-based ones near zero, makes the corresponding loss function non-convex. For example, this is the case for smooth high-breakdown regression estimators (as the S-estimators of [13]), and similar problems are found when calculating or approximating robust estimators for other models.

For many years these difficulties were put aside in the hope that improved computer technology would provide faster machines that would allow these approximating algorithms to perform a more exhaustive search of the parameter space. However, as mentioned above, together with faster machines, we have also seen an exponential increase in the complexity and size of the data sets that require analysis. Thus, the need for more efficient algorithms and computer implementations has not diminished, and may have in fact increased. Moreover, the last decades have also seen explosive growth in the routine application of relatively sophisticated statistical analyses by non-statistician in a diverse range of subject-matter disciplines. It is of necessity to provide these practitioners with stable, efficient, scalable and easy to use software. We also believe that this software, being of scientific value, should be provided free of charge, and on an open-source platform to be completely transparent, and contribute to the advancement of science and population welfare.

The existing code for robust estimators is of varied origins, quality, ease of use and accessibility (namely, financial cost, restrictions associated with their licenses, etc.) While many international leaders in robust statistical methodology have written their own computer code to calculate or approximate some estimators, these programs have generally been written individually by different researchers or research teams around the world. In many cases, these programs were originally intended for the private use of their author (as opposed to practitioners in general), and run under one of a variety of different and non-compatible environments, including R [23], S-PLUS, Matlab and SAS.

Objectives of the workshop

One of the main objectives of this workshop was to tackle the problem of the lack of high-quality and easily-available and usable computer implementations for robust statistical methods. An important difficulty in trying to organize and coordinate the atomized efforts of isolated researchers developing code for personal use, is that, until recently, there have been no scientific meetings focused on this problem. Inconsistent (but generally low) scientific credit given to the study of these problems by our peers in academic positions resulted in a lack of international high-calibre meetings devoted to the delicate computational challenges faced by robust statistical methods. Without physically bringing together the people working on these topics, it is extremely difficult to coordinate developments over time.

In the last few years the R-project [23] has established itself as a widely available, powerful and versatile computer program for statistical analysis. In particular, R is distributed under the GNU General Public License [12], it is open-source, and has been already widely accepted and adopted by a broad community of students, practitioners and researchers in many disciplines (not only Statistics). Furthermore, many of the isolated individuals and research teams that have been developing "in-house" computer code, have being using R. Thus, R appeared as a natural candidate for a platform to base the coordinated development of computer code for robust methodologies.

Although efforts towards addressing this issue started to materialize around 2002, they have been sporadic. In 2005 the first International Workshop on Robust Statistics and R (Treviso, Italy) attracted a lot of attention in the Robust Statistics community. In this meeting some initial guidelines were agreed upon, researchers with similar interests discussed their specific ideas, and a few basic communication tools were set up (websites, mailing lists, etc.) Shortly after the Treviso meeting the robustbase package [26] for R was put together building on several existing packages and stand-alone code from a number of different authors.

With this workshop we were intending to facilitate and coordinate further development of the robustbase package for robust statistics tools in R [23] and to promote the interaction and collaboration between researchers interested in the computational aspects of Robust Statistics. Some topics we had identified as being of particular interest included: the desired degree of default accessibility for non-experts, the specifics of hierarchical integration with other packages already existing in R, and a discussion on methods and techniques that may need to be incorporated to the package in the near future. Other topics for discussion were updates to algorithms and estimation and inference methods (recent developments in these areas that might need to be incorporated) and the ability of the current version of the library to manage large scale applications (scalability).

Based on these considerations we decided that the best format for our meeting would be one of a proper workshop, namely: a series of round-table open discussions among relatively small groups of researchers working around a common problem or topic, followed by "plenary" discussions of the main points raised by each of these working groups. In our opinion, we could not have efficiently achieved our goals in a "classical" scientific meeting (i.e. in a conference structured as a long sequence of single-speaker presentations followed by short periods of question-and-answers). We had also anticipated that the development of this package would naturally identify important practical and theoretical challenges and become a driving force for new research and activity in the field.

Recent Developments and Open Problems

In 2001 a high-quality library of robust methods called "robust" was released for S-PLUS. This library contained up-to-date methodologies, and had been developed in consultation with many international experts in robustness. In particular, it included code directly written by many of the researchers who had proposed and investigated the different techniques. At around the same time, R was starting to assert its place in the statistical community as a reliable, open-source and freely available platform for a diverse range of statistical computing. Furthermore, for many individual users and academic units, the price and license restrictions attached to S-PLUS made the migration to R seem a natural choice, particularly after R had attained a certain degree of stability and development maturity.

In 2002, the International Workshop on Computational Methods for Robust Statistics was held as a satellite meeting of the 2002 International Conference on Robust Statistics (ICORS2002) at The University of British Columbia. This workshop was sponsored by the National Science Foundation and brought together for the first time researchers interested in the computational challenges behind many robust statistical methods. A short time later, after a few very successful early ICORS meetings, there was renewed interest in developing a coordinated package of robust methods for R. This lead, in 2005 to the first International Workshop on Robust Statistics and R (Treviso, Italy) which attracted a lot of attention in the Robust Statistics community. Top researchers in the area participated and the critical importance of this type of meetings was stressed. Groups responsible for different parts of the software development process were identified and guidelines to coordinate future work were agreed upon.

One important principle that was adopted in Treviso was that there should exist a “basic” robustness package implementing some “bread-and-butter” functionality and methods, which in turn could be used as a building block for more specific packages. Shortly after this workshop Martin Maechler took the initiative in putting together the robustbase package [26] that initially merged several existing packages and stand-alone code from a number of different authors. Also in this meeting it was agreed to use the book by Maronna, Martin and Yohai [19] as a guideline to select which methods should be included in this “basic” package, keeping in mind that this package was supposed to contain only building blocks, upon which other R packages can be developed. In particular, it was decided that most multivariate methods should continue to be developed in a separate package, currently rrcov [29], and similarly for robust methods for time series (now in the robust-ts package [27]).

To keep track of the developments and progress on the project a website was open at the R Project main site [37], and during the useR!2006 conference [45] a Focus Session on Robust Statistics was organized, mainly devoted to the presentation of the new available software to the R user community.

rrcov package

The rrcov package implements robust estimators for multivariate location-scatter models. Additionally, it is particularly focused on exploiting the functionality of the S4 object-oriented capability of R. The object oriented programming paradigm has revolutionized the style of software system design and development. A further step in software reuse is the object oriented framework (see Gamma et al. [9]) which provides technology for reusing both the architecture and the functionality of software components. The rrcov package provides an object oriented framework for robust multivariate analysis. The goals of this framework include: (i) to provide the end-user with a flexible and easy access to newly developed robust methods for multivariate data analysis; (ii) to allow the programming statisticians an extension by developing, implementing and testing new methods with minimum effort, and (iii) to guarantee the original developers and maintainer of the packages a high level of maintainability.

The objected-oriented nature of this package is better illustrated with a simple example. Starting with the generic object model of a multivariate method with all the necessary interfaces and functionalities we build a class hierarchy to represent it, along with the robust methodologies associated with it. The basic idea is to define an abstract S4 class which has as slots the common data elements of the classical method and its robust counterparts (e.g. principal components analysis, PCA). For this abstract class we can implement standard generic R functions like print(), summary(), plot() and maybe also predict(). Then we can derive and implement a concrete class which will represent the classic method, say PCAClassic. Then we derive another abstract class which represents the associated robust method, e.g. PCARobust. This class is abstract because we want to have a placeholder for the robust methods we are going to develop next. The generic functions that we implemented for the class PCA are still valid for PCARobust but whenever necessary we can override them with new functionality. Thus a platform for building new robust methods for PCA is created.

The framework includes an almost complete set of algorithms for computing robust multivariate location and scatter, which are the cornerstones of most multivariate methods, such as Minimum Covariance Determinant (MCD), different S estimators (SURREAL, FAST-S, Bisquare, Rocke-type), and the orthogonalized GnanadesikanKettenring (OGK) estimator of Maronna and Zamar [21]. The next large group of classes are the methods for robust Principal Component Analysis (PCA) including ROBPCA of Hubert, Rousseeuw and Branden [13], Spherical Principal Components (SPC) of Locantore et al. [14], and the projection pursuit algorithms of Croux and Ruiz-Gazen [7] and Croux, Filzmoser and Oliveira [6]. Further applications implemented in the framework are linear and quadratic discriminant analysis (see Todorov and Pires [44], for a review), multivariate tests (Willems, Pison, Rousseeuw and van Aelst [47]; [43]) and outlier detection tools. As a reference for the multivariate methods implemented in this framework, Chapter 6 of Maronna, Martin and Yohai [19] can be used. The package also includes several examples that illustrate the usage of the framework for data analysis by the end user as well as for development of new methods.

robustbase package

The robustbase package was initially developed after the first workshop held in Treviso, Italy, in 2005. Martin Maechler was particularly active in this project merging several packages already existing and other stand-alone code from a number of different authors. The choice of robust methods included in this package is based on those covered in the recent book by Maronna, Martin and Yohai [19]. The guiding principle is that this package should contain "basic" building blocks, upon which other R packages can be developed.

We can identify three main models for which robust estimation methods are implemented in robustbase: (a) linear models; (b) generalized linear models; and (c) multivariate location-scatter models. In addition, there is also a fit for non-linear regression models, and robust estimators for simple location-scale models. Robust methods currently implemented in robustbase include: robust linear regression: MM-estimators [48] lmrob and LTS estimators [31] ltsReg , robust generalized linear models glmrob (only for binomial and poisson responses, using the robust quasi-likelihood approach of Cantoni and Ronchetti [3]), robust non-linear regression nlrob, location M-estimators with fixed scale huberM, and multivariate location-scatter: the Minimum Covariance Determinant [33], and the Orthogonalized Gnanadesikan–Kettenring [11] and [21].

What follows is a brief description of the functionality implemented in the robustbase package prior to the workshop for each of the main models mentioned above.

- Linear regression models: the main function is lmrob. It currently implements MM-estimators [48] starting from an S-estimator [13]. These estimators are highly resistant to outliers, and also efficient when the errors are normally distributed. By default this function computes a regression estimator with 50% breakdown point and 95% efficiency for normal errors. The S-estimator is computed using the fast-S algorithm proposed in [39], and the local M-solution is found using re-weighted least squares iterations. The standard errors of the estimators reported by summary are based on [5] and valid for the case of both symmetric and asymmetric error distributions (including contaminations). The plot method produces similar plots to those available for the "classical" regression estimators via the plot method for the lm function. Tuning parameters (both for the estimators (controlling their breakdown point and efficiency) and the algorithm (number of random sub-samples for the fast-S algorithm, etc.) are passed using the "control" function lmrob.control.

  The least trimmed squares estimator [31] is implemented in the function ltsReg. This function approximates the estimator using the fast-LTS algorithm proposed in [34] and [35]. Further consistency and finite-sample corrections are applied, see [22].

- Generalized linear models: the main function is glmrob. It implements robust estimators for these models based on the robust quasi-likelihood approach of Cantoni and Ronchetti [3] with monotone score functions. The effect of potential high leverage points can be controlled by downweighting observations according to their Mahalanobis distances using different robust multivariate location-scatter estimators (see [8]). Currently only log-linear and logistic regression models are implemented. As is the case for lmrob, tuning parameters are managed with a control function.

- Multivariate location-scatter estimators are implemented through the functions covMcd and covOGK. The implementation of covMcd uses the Fast MCD algorithm of [32]. The estimator is further corrected applying rescaling factors as in [22]. The Orthogonalized Gnanadesikan-Kettenring is implemented in the function covOGK.

Scientific Progress Made

During this workshop we continued the work started in the first International Workshop on Robust Statistics and R (Treviso, Italy, 2005). In particular, the recent package robustbase was formally introduced, and the discussion focused on what needs to be included in this package in the immediate future. Part of these discussions focused on technical issues (e.g. the use of S4 classes, unified criteria for passing arguments related to both the algorithm and the estimator, guidelines to unify the different implementations, classes, common methods, common graphical displays, etc.)

We took fully advantage of the workshop facilities at BIRS, and decided to divide the participants in focus groups that would meet during the day to have round-table discussions and presentations. Each working group then reported back to the "plenary" session, where we held an open discussion among all the workshop participants. We found the possibility of structuring the work in this way much preferable to the more common style of a series of individual presentations. In our opinion this is the key advantage of BIRS compared with other conference facilities: having several days to interact and work jointly with colleagues. Some short presentations were necessary and very useful, e.g. the working group reports.

There were 31 participants from 11 countries.

Following the structure used in the 2005 Workshop in Treviso, several following working groups were identified around pivotal themes for future development. The workshop participants were allowed to join one of these groups to work throughout the week. The membership of the working groups was

1. Linear models / econometrics: Claudio Agostinelli, Kris Boudt, Christophe Croux, Roger Koenker, Kjell Konis, Guixian Lin, Martin Maechler, Alfio Marazzi, Ivan Mizera, Andreas Rucksthul, Matias Salibian-Barrera, and Stefan Van Aelst.
2. Descriptive / Exploratory data: Rudolf Dutter, Chris Field, Roy Welsh.
3. Time Series: Roland Fried, Ursula Gather, Peter Ruckdeschel, Bernhard Spangl.
4. Multivariate methods: Peter Filzmoser, Heinrich Fritz, Luis Angel Garcia-Escudero, Marc Genton, Justin Harrington, Christian Henning, Ricardo Maronna, Matthias Templ, Valentin Todorov, David Tyler, Gert Willems.

Note that some participants were interested in more than one topic, and thus may have participated in the deliberations of more than one working group along the length of the workshop.

The main recommendations of each working group were:

1. Linear models / econometrics:
   - the function lmrob currently performs re-descending iterations starting from an S-estimator of regression. It was discussed that the choice of initial estimator can be offered as an argument at the user level, so that other options are easily implemented.
   - the current sub-sampling strategy to calculate the S-estimator breaks if the design matrix is particularly sparse. This happens naturally when there are several categorical variables in the regression model. The MS- algorithm of [20] that is already implemented in the robust library for S-PLUS should be included in robustbase to deal with these cases.
   - Although some robust GLM methods are available, it was agreed that it would be important to implement the proposal of Bianco and Yohai [1].
   - Some model selection functionality is desirable. In particular functions like add1 and drop1. The ensuing discussion centered on which criteria could be used (some options include: robust C_p or AIC [28], a robust R^2 [4], or the robust future prediction error RFPE [19].
   - The addition of an anova function to test for nested models was considered. Here one can use Wald-type tests, or R^2 type test statistics ([17] and [18]). Furthermore, valid approximations to the p-values can be computed with the robust bootstrap [40] and [38]. These approximations hold under more general conditions than the usual asymptotic results.
   - Adding distance-distance plots to plot.lmrob was discussed and agreed that it would be useful to have included.

- The accuracy of the estimated standard errors currently reported by summary.lmrob was also discussed.

2. Descriptive / Exploratory data:

   This includes mainly methods that are data based and not model based. Accordingly, methods and plots should focus directly on the description and presentation of the data at hand, but not on the model that may be applied to the data. There already exist useful plots, like boxplot presentations, and plots for outlier detection. However a Data Diagnostics and Visualization (DDV) package for the visualization of outliers in multivariate dataset is still missing.

3. Time Series:

   - Existing packages are on robust Kalman filtering and robust signal extraction. Planned packages are robust counterparts to those included in the stats package, i.e. time series functions and FinMetrics routines.
   - The development and maintenance of packages will be done in R-forge. No mailing list should be used, but R-forge will serve for communication.
   - The contents of the planned ts package includes robust counterparts to acf/pacf (Ma and Genton [16]): quadrant, M-estimator; AIC/BIC (Ronchetti and Staudte [28]); ar/arima: GM, tau, diagnostics; arch/garch (Boudt and Croux [2]); filter; Holt-winters (Croux, Gelper and Fried [10]); spec/spectrum (Spangl and Dutter [41]); methods for plot, print, and summary need to be adapted.
   - The input structure of the functions should consider
     - possible input data are (raw) vectors and irregularly spaced time stamps
     - as far as possible the same arguments as for classical routines
     - for the method argument a (vector of) character(s) or an object of S4 class
     - the control argument is a function generating certain control/tuning objects
   - The output structure of the functions should consider
     - time-stamped elements
     - generally at least S3 classes
     - class should “inherit” from corresponding classical return class
     - in case of several methods computed in parallel: a list of corresponding objects

4. Multivariate methods:

   - General structure: For certain functions and estimators we should make use of overall functions, “superfunctions” (wrapper) handling several single functions for different robust estimation methods. The goal is to design the “superfunction” in a way that it has not to be changed if a new method is available. The design of such a function could be made for each class of estimators by taking a “method” parameter and calling the corresponding estimator function. The method parameter can either be a character string, like “mcd”, “mve”, “M”, “ogk”, “auto”, etc. or a control object. If the method parameter is left empty or is set to “auto”, the overall function tries to call the appropriate estimator.
   - Naming conventions: In rrcov the different robust estimators and methods are implemented as functions returning S4 objects. The names of the function and the corresponding S4 class are identical (i.e. the function CovMcd() returns an object of S4 class CovMcd). The names start with a capital letter, like CovMcd, CovMest, CovOgk, CovMve for location/scatter estimators or PcaHubert, PcaGrid, Pcaproj, PcaMaronna, PcaSpherical for PCA, etc. If a function returns an S3 class, its name starts with a lowercase letter.
     In robustbase, as far as the multivariate methods are concerned, same conventions as in rrcov are used. Currently there are the functions covMcd and covOGK, both returning S3 objects. Later all CovXxx and PcaXxx functions will be moved there. The case of the lmXxx family is more complicated since there are already established conventions for the classical methods, but we hope

that the Regression working group will come up with a solution. It is important to agree on one convention and to be consistent.
In the robust library the names start with lower case letters for the estimator (method) and add upper case "Rob", like lmRob. The problem here is that it inherits from S-PLUS and must maintain comparability with S-PLUS.

- General goals: The idea behind the *robust-library* is to include user-friendly functions with "sensible" defaults and no possibility of fine-tuning for experts. While the user-friendliness is desired any other package, these must grant full control to the experienced user.
- Further "superfunctions": Similar to the design of a general function for robust covariance estimation, such a general "superfunction" could exist for:
  - *PCA:* Here one has to distinguish
    - covariance based PCA
    - projection pursuit based PCA

    For covariance based methods an object from robust covariance estimation can be plugged-in. For projection pursuit based PCA some approaches are already available in R (e.g. package "pcaPP").
  - *LDA:* Here it is important to include
    - different methods for covariance estimation (as plug-in)
    - different ways for computing the common covariance matrix
    - and probably additional features for variable selection
  - *QDA:* This is even simpler than LDA since no pooling of the covariances is needed. Thus we need
    - different methods for covariance estimation
    - functions for plotting the results (cf. Uwe Ligges?)
  - *Factor Analysis:* Already the standard function provides the possibility of using a robust covariance matrix.
- Cluster analysis: This topic is much more complicated. Due to the variety of different approaches, no "superfunction" is possible, only for certain classes of methods this is feasible.
- Further methods:
  - *Logistic Regression:* It is desirable to include the Bianco-Yohai estimator [1]. This should be coordinated with the Working Group on Regression.
  - *Discriminant Coordinates:* for cluster validation Christian Hennig has done work
  - *ICA:* already available in R (package ICS). This should probably not be included in robustbase.
  - *Outlier Detection:* could be implemented using a "superfunction". Some methods already exist in R (e.g. package mvoutlier)
  - *Multivariate Tests:* Hotellings T^2 and Wilks Lambda are already implemented by Valentin Todorov.
- Advances topics:
  - *Robust bootstrap:* This can be used as additional functionality within certain multivariate methods (e.g. selection of the optimal number of PCs for PCA). However, this can be done later.
  - *Missings:* There could be an optional parameter for multivariate methods how to treat data with missing values (e.g. Cov-estimation: impute very large values, apply MCD). Again, this can be done later.
- Where to include the functions: The robust estimators of a given class (e.g. location/scatter, PCA, LDA, QDA, Cluster) are implemented either in package robustbase or in a 'higher level' package which depends on robustbase. The overall function must be in the 'highest-level' package in order to avoid dependence of robustbase on other packages. Later, if considered appropriate, the functions can be moved to robustbase and this will remain transparent for the user.

Kjell Konis confirmed that Insightful Co. was releasing their robust library for S-PLUS under an open-source license, and this library was being converted into an R package by himself. This announcement was well received, and opened the door for re-use of some of these functions in the robustbase package, thus avoiding extensive re-coding. In general, there was general consensus regarding the migration of the package robust originally developed for S-PLUS into R. There was a informative discussion regarding which license applies to this release, and the desired roles of the packages robust and robustbase. It was agreed that, although more modern and efficient algorithms for basic estimators are already included in robustbase, the package robust contains valuable implementations. For example, the fit.model paradigm that allows for side-by-side comparison of different fits (typically robust and non-robust), both graphically and summaries. The concensus was that while robustbase will focus on implementing workhorse functions and allow for direct access to the many tuning parameters and technical options behind each estimator and algorithm, the robust package will provide a layer of user-friendly functionality.

Outcome of the Meeting

One important goal of the workshop was to discuss and agree on specific guidelines for future development of the robustbase package. Based on the working-group structure of our workshop, we had each working group identify and report for plenary discussion what was perceived as the desired next steps. Much discussion centered on which robust methods were sufficiently “mature” to be included in a commercial-quality release of the package. Other issues considered were: scalability of the current algorithms and their ability to deal with different types of data that occur frequently in practice.

After the launch of the R-Forge platform [24] (which is the new platform for the R Community) in late 2007, several projects were open by participants of the workshop: (i) RobKalman [25], which implements several robustifications of the classical Kalman filter. A common filtering iterface for all robustifications is provided as well as S4-classes for state space models and filtering results; (ii) robust-ts [27] which is a collaborative project to provide robustifications to the basic time series procedures from package stats. A target will be chapter 8 in [19]; (iii) libRa [15] which implements robust statistics algorithms from the research group Robust Statistics at the Katholieke Universiteit Leuven and the Universiteit van Antwerpen.

A Task View on Robust Statistics was made available after the workshop at [42] which contains a brief description of all available packages in the field. In order to disseminate to the R users community the new advances in both theory and available software attained after this workshop, we plan to organize a tutorial session during the useR!2008 conference [46].

Both organizers and participants felt that the workshop was a great success. We achieved our goals in a very congenial atmosphere. We have also had very lively and productive discussions. The meeting also gave new incentives for future work and collaborations to many researchers. Although nowadays many possibilities for communication exist, face-to-face meetings allow for a much better and more efficient coordination and motivation.

The relaxed atmosphere contributed a great deal to have long exchanges with several researchers at a time, that would not have been possible in the context of a regular scientific conference / meeting. The workshop-type structure allowed by BIRS during these 5 days greatly contributed to foster interactions, and communication between researchers from different institutions. This seemingly unstructured schedule was identified by many participants as a key factor contributing to the advances made at BIRS. In our opinion, the flexibility offered by BIRS is very valuable and should be preserved, since it was in large part because of it that our workshop exceeded all expectations.

List of Participants

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Chapter 36

Mathematical Methods for Medical Image Analysis (07w5115)

Nov 04 - Nov 09, 2007

Organizer(s): Rafeef Abugharbieh (University of British Columbia), Ghassan Hamarneh (Medical Image Analysis Lab, Simon Fraser University)

Workshop Objectives

The main objective of the workshop was to bring together top-tier international researchers in the field of medical image computing for brainstorming sessions culminating with definition of current research challenges in the area and potential solutions. The workshop's specific goals included:

- Fostering exchange of information, sharing of insights, and seeding of new collaborations between participants in areas of mutual interest.
- Establishing connections between Canadian researchers and well-established internationally-renowned research groups world-wide.
- Improving communication between related disciplines in this multidisciplinary area including Mathematics, Computer Science, Engineering, Medicine, and Physics.
- Enabling in-depth scientific discussions through exciting talks and lively panel discussions on important problems in medical image computing and state of the art analysis techniques that are currently under active research.

Workshop Overview

This workshop was a five-day scientific retreat held at the Banff International Research Station for Mathematical Innovation and Discovery (BIRS). The workshop attracted many of the best known international researchers in the area of medical image analysis who represented top universities and industries.

The workshop provided a forum for in-depth discussions and stimulating interactions among high caliber researchers working on development and application of mathematical methods for solving problems in medical imaging. The workshop helped foster multidisciplinary research in medical image computing by bringing together mathematicians, computer scientists, engineers, physicists and clinicians.

Overview of the Field

Biomedical imaging is revolutionizing medicine in our modern society and the impact of novel computational multi-dimensional data analysis methods on enhancing healthcare is enormous. Applications in clinical and biomedical settings are far reaching, e.g., in computer aided-diagnosis, image-guided intervention, therapy evaluation, monitoring and quantification of disease progression etc. The current research focus of the medical image analysis community is on topics related to image reconstruction, denoising, registration, segmentation, geometric and statistical modeling, and visualization of complex visual medical datasets. The development of algorithms for solving such problems in medical image datasets (such as scalar, vector, and tensor fields) involves various mathematical tools including transforms, spectral analysis, PDEs, nonlinear multivariate statistics, solutions to inverse problems, and optimization methods.

Recent Developments and Open Problems

Over the past two decades, medical imaging has become one of the main pillars of modern healthcare. The increasingly rich spectrum of available structural and functional imaging technologies provides exquisite medical data that offer tremendous opportunities for non-invasive visualization and quantitative analysis of anatomy and physiology. Applications, some of which are clinical realities, vary from basic computer-assisted diagnostics to elaborate image-guided interventions. Exploiting the enormous amounts of information embedded in such medical image data is, however, a very complicated task. In fact, the lack of efficient, accurate, and robust computational analysis techniques poses huge challenges that seriously hinder optimal extraction and use of image information in practice. Manual analysis is mostly not feasible, even impossible in certain cases, as it is prohibitively time-consuming (i.e., expensive), very tedious, can be quite inaccurate, and prone to serious inter- and intra-user variability. This renders automated techniques indispensable for achieving the high accuracy, reproducibility, robustness and efficiency rates needed for real-world medical research and practice. Despite continuous efforts, success in automating medical image analysis tasks has so far been modest and of limited applicability due to numerous complicating factors associated with real medical image data. These factors include extensive variability (normal and pathological) in anatomical and functional image information, serious imaging artifacts and data noise, patient-related artifacts, e.g., motion, and most importantly, the substantial underlying complexity of biological structures and processes.

Sessions and Presentations

Eight scientific sessions were held over five days. Each session included 3-4 short presentations by distinguished participants which were followed by a lengthy panel discussion by a subset group of specialists. A total of 29 presentations and 8 panel discussions were held.

Presentation Highlights

Session 1: Mathematical Methods (Worsley, Miller, Boykov, Staib)

- Worsley, Keith (McGill University): Detecting Sparse Connectivity: MS Lesions, Cortical Thickness, and the ’Bubbles’ Task in an fMRI Experiment

We are interested in the general problem of detecting sparse connectivity, or high correlation, between pairs of pixels or voxels in two sets of images. To do this, we set a threshold on the correlations that controls the false positive rate, which we approximate by the expected Euler characteristic of the excursion set. The first example is a data set on 425 multiple sclerosis patients. Lesion density was measured at each voxel in white matter, and cortical thickness was measured at each point on the cortical surface. The hypothesis is that increased lesion density interrupts neuronal activity, provoking cortical thinning in those grey matter regions connected through the affected white matter regions. The second example is an fMRI experiment using the ’bubbles’ task. In this experiment, the subject is asked to discriminate between images that are revealed only through a random set of small windows or 'bubbles'. We are interested in which parts of the image are used in successful discrimination, and which parts of the brain are involved in this task.

- Miller, Michael (John Hopkins University): Computational Functional Anatomy

Computational Anatomy is the study of the shape and structure of manifolds in human anatomy. This talk reviews results from CA along these lines, including (i) embedding of shapes into a metric structure via flows of diffeomorphisms (ii) conservation laws for geodesics describing metric connection of shapes (iii) statistics on families of shapes encoded via these metrics. The emerging focus in Computational Functional Anatomy is the inclusion of the study of function in the curved coordinates of anatomical manifolds. Methods for performing inference in this setting are examined coupled to morphometric studies.

- Boykov, Yuri (University of Western Ontario): Global Optimization of Geometric Surface Functionals

Optimization of geometric surface functionals is a very common approach to formulating image analysis problems. Combinatorial algorithms (e.g. graph cuts) allow global optimization of a wide class of geometric functionals common in N-D image analysis that were previously addressed mainly via local variational techniques (e.g. level-sets) or other gradient descent methods. In this talk we describe geometric functionals that can be addressed via graph cuts and show applications.

- Staib, Lawrence (Yale University): Models for Biomedical Image Analysis

Biomedical image analysis methods may be enhanced by the incorporation of models, which may come in many forms. Difficulties in the ability to accurately and reproducibly recover quantitative information arise from issues related to image quality and the variability of normal and abnormal anatomy and physiology. Models can provide a crucial constraint on the analysis by providing information related to the manifestation of the relevant anatomy or physiology in the images. Models for shape are particularly useful. Some problems are well suited to the constraints that global geometric information provides, where the shapes of the organs or structures are very consistent and are well characterized by a specific shape model. Other problems involve structures whose shapes are highly variable or have no consistent shape at all and thus require more generic shape information. Biomechanical models, either for regularization or for true modeling, can also be powerful. We describe model-based approaches to a variety of problems illustrating the varying uses of models for image analysis.

Session 2: Image Registration (Christensen, Rueckert, Celler)

- Christensen, Gary (University of Iowa): Non-rigid Image Registration Evaluation Project (NIREP)

Non-rigid image registration is an essential tool for morphologic comparisons in the presence of intra- and inter-individual anatomic variations. Many non-rigid image registration methods have been developed, but are especially difficult to evaluate since point-wise inter-image correspondence is usually unknown, i.e., there is no "Gold Standard" to evaluate performance. The Non-rigid Image Registration Evaluation Project (NIREP) has been started to develop, establish, maintain, and endorse a standardized set of relevant benchmarks and measures for performance evaluation of non-rigid image registration algorithms. This talk will describe the initial work of the NIREP which includes constructing an initial neuroanatomical evaluation database, reports on the initial evaluation measures and gives baseline registration performance. Registration comparisons will be presented as images to correlate algorithm performance spatially with the underlying anatomy and are presented in tabular and graphical formats to compare trends over the population. Finally, this talk will discuss standards for reporting registration results based on the recommendations of the Standards for Reporting of Diagnostic Accuracy (STARD) Initiative and the registration evaluation framework developed by Pierre Jannin et al.

- Rueckert, Daniel (Imperial College): Quantification of Brain Development during Early Childhood Using Medical Image Computing

The majority of brain growth occurs during the first two years of life, much occurring in-utero prior to birth at 40 weeks gestational age, and a full understanding of human brain development must include this early period of rapid development. The characterization of early brain development is particularly important in infants who are born prematurely. Preterm birth affects around 5% of births in industrialised countries and its consequences contribute to significant individual, medical and social problems. The principle morbidity among survivors is neurological, resulting from the profound effect of preterm birth on the developing brain: half of all infants born at less than 25 weeks have neurodevelopmental impairment at 30 months of age, and in less immature infants, neuropsychiatric problems are common in the teenage years. The anatomical correlates of functional disorders are, however, poorly characterized. Using state-of-the-art MR imaging it is nowadays possible to study the brain development and maturation process in infants born prematurely from as young as 25 weeks onwards. Recent advances in MR imaging also enable the acquisition of high-resolution fetal images. This allows us to study the brain development in-utero as well as ex-utero. In this talk we describe the computational challenges in the analysis of these images. In particular we describe how non-rigid registration can be used in cross-sectional and longitudinal studies to quantify brain development and growth between the last trimester of pregnancy and early childhood.

Session 3: Visualization and Clinical Applications (Sonka, Archip, Lee, Möller)

- Sonka, Milan (University of Iowa): Multi-Surface Segmentation of 3D Retinal OCT

3D spectral OCT has become a new imaging modality available in clinical care starting in the fall of 2007. Retinal segmentation and quantitative analysis - both dealing with the total retina and individual retinal layers - is a challenging new direction of ophthalmologic research and clinical care. A novel n-surface segmentation approach will be presented that is applicable to retinal layer segmentation of the macular and peripapillary 3D OCT scans. Validation results and in vivo segmentation examples will be provided.

- Archip, Neculai (Harvard Medical School): Medical Image Analysis for Image Guided Therapy

Image guided therapy is combining advances in imaging and therapeutic technology to develop minimally invasive surgical and interventional techniques. Innovations in imaging and image analysis have enabled key improvements in a) preoperative planning, b) intraoperative targeting, navigation and control, and c) postoperative assessment. We describe recent algorithmic advances that have enabled enhanced surgical navigation, by multi-modality fusion of pre-operative high resolution imaging with low quality intra-procedural images. One of the challenges of developing such algorithms is to meet the hard real-time constraints of intra-operative surgical decision making. In this context, several clinical applications requiring advanced quantitative image analysis will be demonstrated. It will include image guided neurosurgery, where pre-operative DT-MRI, T1w MRI and fMRI are fused with intra-operative MR imaging for enhanced visualization of tumor margin and white matter tractography during brain tumor resection. We will also present new approaches for improved lesion targeting during abdominal intervention procedures (such as liver biopsy and radio frequency ablation), by combining pre-procedural MRI and PET scans, with intra-procedural CT images.

- Lee, Tim (BC Cancer Agency): Analysing Pigmented Skin Lesion Images

The incidence of cutaneous malignant melanoma has been increasing rapidly throughout the world in the last four decades in the countries with the large light-skin population. Statistics have shown that early detection of malignant melanoma plays an important role in the treatment of the disease. Therefore, there are increasing activities in research for computer-aided diagnostic systems to help health professionals in diagnosing the disease and to reduce their workload. Dermatologists and researchers often use white light imaging to diagnose skin lesions, monitor high risk patients, and study the etiology of the disease. In this presentation, I will review some of the development, issues and challenges of automating the macro and micro imaging of skin lesions using white light imaging.

- Möller, Torsten (Simon Fraser University): Graphics and Visualization Approaches to Medical Imaging

In this talk I will summarize some of our research on the visualization of functional medical imaging. Further, I will look into the application of graphics algorithms to tomography algorithms and their mapping to Graphics Programming Units (GPUs).

Session 4: Image-Guided Intervention (Ellis, Rohling, Abolmaesumi)

- Ellis, Randy (Queen's University): From Scans to Sutures: Computer-Assisted Orthopedic Surgery in the Twenty-First Century

Computer-assisted surgery is the process of using medical images, such as CT scans, X-ray fluoroscopy, or 3D ultrasound, to improve patient care. A typical surgical procedure begins by acquiring and processing a CT scan with specially developed image-analysis software. A surgeon then performs a "virtual surgery" on the patient to develop a preoperative plan. In the operating room the medical image is registered to the patients ́ anatomy by finding an optimal rigid-body transformation. This transformation allows an object or motion in one coordinate frame to be represented in the other frame, and thus a surgeon can visualize the location of an instrument deep within concealed anatomy while avoiding structures at risk. The operating surgeon can also use computer-tracked fluoroscopy or ultrasound for 3D guidance. For the past decade, our interdisciplinary research group has been investigating fundamental problems in orthopedic surgery of bones and joints. This talk will be an overview of the problems and solutions that have been tested in a set of pilot clinical trials in which we have treated more than 350 patients for early or advanced arthritis, poorly healed bone fractures, and treatment of deep bone tumors.

- Rohling, Robert (University of British Columbia): Automated Interpretation of Ultrasound Images

The field of ultrasound imaging continues to grow and evolve with each new generation of technology. In recent years, there has been a substantial improvement in image quality due to new image formation and image processing techniques. There has also been a drive towards the development of smaller and lower cost systems. As access to ultrasound is made easier, ultrasound is expanding beyond its traditional place in obstetrics and is found increasingly frequently in other domains such as anesthesiology and emergency medicine. Newer imaging methods such as elastography and high-frequency imaging have also advanced and made their way into clinical use. One of the outcomes of these driving forces is that ultrasound is no longer an imaging method used only be experienced sonographers and radiologists. In many cases, ultrasound is now being used by users who are relatively unskilled in ultrasound interpretation. Ultrasound is also used in larger systems, such as image-guidance systems for interventions. In both cases, automatic ultrasound processing and data extraction is beneficial. For inexperienced operators, automatic processing can provide an aid in interpretation for a specific task. For image-guidance systems, automatic processing can be used for target/tool tracking and validation of a procedure. This talk will cover some examples of automatic processing algorithms and applications. It will also discuss the advent of commercial open-architecture ultrasound systems and the impact they have on the field.

- Abolmaesumi, Purang (Queens University): Ultrasound-Guided Computer-Assisted Orthopaedic Surgery

In recent years, computer-assisted surgery has evolved and developed due to the need to integrate the information derived from different sensory inputs in the operating room and also the necessity to track the surgical operation quantitatively during and after the surgery. The current research in this area envelops all phases of a surgical procedure: preoperative planning, intra-operative registration and guidance, and post-operative follow-ups. A few of the main medical imaging modalities used to plan and perform these operations include ultrasound, X-ray, CT and MRI. This talk will cover the research currently being conducted at the Medical Image Analysis Laboratory at Queen's University regarding the application of medical imaging modalities in both the preoperative planning and intra-operative registration phases of computer-assisted orthopaedic surgery. With regards to the former, the goal is to reconstruct anatomical models of patients from medical images with little computational cost. Regarding the latter, the aim is to register in real time, the patient's anatomy to the pre-operative 3D models extracted from the ultrasound, CT and MRI images. These reconstruction and registration techniques have been incorporated in the design of new user interfaces that will improve intra-operative guidance of surgical tools.

Session 5: Diffusion Tensor Imaging (Siddiqi, Westin, Styner, Whitaker, Lenglet)

- Siddiqi, Kaleem (McGill University): On the Differential Geometry of White Matter Fibre Tracts: Generalized Helicoids and Diffusion MRI

Recovering the differential geometry of white matter fibre tracts from Diffusion MRI is largely an open problem. We develop a characterization of such structures by considering the local behaviour of the associated 3D frame field, leading to the associated tangential, normal and bi-normal curvature functions. Using results from the theory of generalized minimal surfaces we adopt a generalized helicoid model as an osculating object and develop the connection between its parameters and these curvature functions. These developments allow for the construction of parametrized 3D vector fields (sampled osculating objects) to locally approximate these patterns. We apply these results to the analysis of diffusion MRI data via a type of 3D streamline flow. Experimental results on human brain data demonstrate the advantages of incorporating the full differential geometry.

- Westin, Carl-Fredrik (Harvard Medical School): Geodesic-Loxodromes for Diffusion Tensor Interpolation

In algorithms for processing images of diffusion tensors, two common ingredients are interpolating two tensor values, and measuring the distance between them. We propose a new class of interpolation paths for tensors, termed geodesic-loxodromes, which explicitly preserve clinically important tensor attributes, such as mean diffusivity or fractional anisotropy, while using basic differential geometry to interpolate tensor orientations. This contrasts with previous Riemannian and Log-Euclidean interpolation methods that preserve the tensor determinant. From path integrals of the tangents of geodesic-loxodromes, we create a novel measure of over-all difference between two tensors, as well measures for shape difference and orientation difference.

- Styner, Martin (University of North Carolina, Chapel Hill): Automated Fiber-Based DTI in the Developing Brain of Human and Non-Human Primates

Diffusion tensor imaging (DTI) has become increasingly important as a means of investigating the structure and properties of neural white matter, with many applications in our own research of the developing brain in humans and macaque monkeys. Several analysis frameworks have been proposed, such as region based, voxel-based as well as fiber tract based analysis frameworks. I will present our current work in this field using both fiber tract and region based analysis, as well as our developments championing an automated analysis of atlas based, parameterized DTI fiber tracts for the quantitative analysis of the diffusion properties. Additionally the use of DTI based connectivity information is the basis for a novel, population based cortical correspondence computation.

- Whitaker, Ross T. (University of Utah): Volumetric Connectivity: Formulation and Computational Solutions

Diffusion tensor imaging (DTI) offers an opportunity to study, in vivo, the white matter connections between different functionally or structurally relevant regions in the cortex. In this work, we describe a methodology for defining volumetric, white-mater regions from DTI that are associated with the connections between predefined brain regions. The method relies on a Hamilton-Jacobi formulation of white-matter paths and produces a volumetric mask of DTI measurements that contain the optimal path as well as near-optimal paths that provide evidence of a particular connection. This talk will discuss the Hamilton-Jacobi formulation, the mathematical properties of volumetric connectivity, and methods for using these masks to analyze white-matter structure using nonparametric regression. We will also discuss the numerical methods associated with the Hamilton-Jacobi solver, a new algorithm for efficiently solving these equations, and a GPU implementation, which allows user to interactively define and visualize white-matter regions.

- Lenglet, Christophe (Siemens Corporate Research Inc.): DTI Tractography - Applications and Shortcomings

I will introduce a front propagation technique for DTI tractography. I will then illustrate the algorithm through results obtained on the human motor and visual systems. This will be used as a starting point to discuss some open questions for the tractography problem.

Session 6: Knowledge-Based Image Analysis (Lorenz, Warfield, Atkins)

- Lorenz, Cristian (Philips-Germany): Using Domain Knowledge in Medical Imaging

Today, it is not sufficient for a medical imaging device company, to just deliver medical images of good quality. The imaging process itself and further processing such as image segmentation, quantification, computer assisted diagnosis and therapy planning needs to be efficient and highly automated. Currently, mainly knowledge about the human anatomy and about clinical workflow is used to achieve this goal, however, physiological and pathology models will become increasingly important in the future. The talk will try to show current approaches and challenges for the future from the perspective of industrial research.

- Warfield, Simon (Children's Hospital Boston): Algorithms for Quantitative Assessment of Pediatric Brain MRI

In the last trimester before birth, the developing human brain undergoes tremendous changes as it grows. Premature birth during this period is associated with an increased risk of adverse outcomes, with up to 50% of very low birth weight infants going on to develop cognitive and motor deficits. The analysis of magnetic resonance images has a crucial role to play in characterizing normal brain development, and in understanding the impact of early brain injury upon the path of later brain maturation. However, there are a number of unique challenges in quantitatively assessing early brain MRI due to the limited contrast between different types of tissue, the rapid progression of brain maturation, and the logistical challenges of imaging newborn infants. Particular challenges include overcoming the effects of image acquisition artifacts, imaging system noise, and patient-specific normal and pathological variability. Advances in image acquisition and medical image computing algorithms now enable sophisticated characterization of early brain development. We will present recent work in developing and evaluating image analysis algorithms to improve our capacity to characterize early brain maturation and early brain injury and to assess potential therapeutic interventions.

- Atkins, Stella (Simon Fraser University): Role of Eye Gaze Tracking in Medical Applications: A Window into the Mind

I will introduce the basic science involved in eye gaze tracking and the hardware we use, and describe two medical applications: in radiology workstation design, and in surgery.

Session 7: Statistical Shape Analysis (Pizer, Cootes, Larsen)

- Pizer, Stephen (University of North Carolina, Chapel Hill): Robust Estimation of Probability Distributions on One or More Anatomic Objects

Both segmentation and statistical shape analysis of one or more anatomic objects can benefit from estimation of a probability distribution on the geometry of the entity. This geometry will include positional information and sometimes orientational information as well, such as boundary or medial sheet normals. Frequently, training samples are very expensive, so the robustness of the probability distribution estimation as a function of the number of training samples is an important issue. The overall topic of this talk is how to achieve this robustness and how to measure the robustness of estimation. The talk will include: * Definition of application-independent measures of this robustness, measured in both feature space and in the ambient space in which the images are formed. * Discussion of how to use these measures to compare probability distribution estimations on models with different primitives (in particular, b-rep PDMs vs. m-reps). * Means of achieving robust probability distribution estimation by multiscale estimations using successive residues from larger spatial scales together with successive re-alignment with decreasing spatial scale. Application to the segmentation of objects in the male pelvis from CT will be used to illustrate the approach and its results.

- Cootes, Tim (University of Manchester): Automatic Construction of Statistical Shape Models Using Group-wise Non-Rigid Registration

Statistical models of shape and appearance have been shown to be powerful tools for image interpretation, as they can explicitly deal with the natural variation in structures of interest. Such models can be built from suitably labelled training sets. Given a model of appearance we can match it to a new image using the efficient optimisation algorithms, which seek to minimise the difference between a synthesized model image and the target image. To construct such models we require a set of points defining a dense correspondence between every image and every other. This can be very time-consuming to achieve manually, and is potentially error prone. There is considerable demand for algorithms to automatically construct such models from training data, with minimal human intervention. Building on work on registering 2D boundaries and 3D surfaces, we have developed methods for registering unlabelled images so as to construct compact models. This talk will describe the approach, and demonstrate its application to a variety of 2D and 3D datasets.

- Larsen, Rasmus (Technical University of Denmark): Sparse Statistical Models for Relating Anatomical Differences to Clinical Outcome

Natural phenomena can often be explained by a set of few underlying parameters. This property has been used in many years in statistics, e.g. in factor rotation for easier interpretation [1]. In recent years sparsity has been used as design criterion to overcome the problem of the dimensionality of measurements vastly exceeding the number of observations available. Mathematically sparsity is envoked by putting a L0 penalty on the parameters. However, this is computationally intractable. Fortunately, in many situation the L1 penalty - for which computationally feasible solution are available - can work as a proxy for the L0 penalty as is used for instance in LASSO and LARS regression [2,3]. In this presentation we will show how sparse statistical regression and subspace methods can be use to explore hyper-dimensional image data such as shape differences and deformation fields and relate these variables to clinical outcome. The examples will include relating corpus callosum shape as well as whole brain deformations to clinical and cognitive parameters and analysing cranio-facial growth patterns.

Session 8: Functional Imaging and Energy Minimization Methods (Sossi, Salcudean, Ng (Ph.D. student of Rafeef Abugharbieh), McIntosh, Ward (Ph.D. students of Ghassan Harmarneh))

- Sossi, Vesna (University of British Columbia): PET Data Analysis

This talk will review methods of extracting quantitative, biologically relevant parameters from dynamic PET images.

- Salcudean, Tim (University of British Columbia): Imaging Issues in Prostate Brachytherapy

We will outline some of the image processing problems and our approaches for prostate brachytherapy, including boundary segmentation, localization of sources and elasticity modelling based on ultrasound elastography.

- Ng, Bernard (University of British Columbia, Ph.D. student of Rafeef Abugharbieh): Spatial Encoding of Brain Activation in fMRI

Patients with neurological diseases are often found with altered brain activity, and functional magnetic resonance imaging (fMRI) provides a non-invasive means to examine such changes. By far, the most widely used method to analyze fMRI data is statistical parametric mapping (SPM), which involves warping the brain of each subject to a common atlas to draw group inferences. However, due to inter-subject variability in brain shapes and sizes, especially for diseased patients, mis-registrations can easily occur. To avoid warping, the alternative approach is to specify regions of interest (ROIs) for each subject and examine statistical properties of regional brain activation. Conventionally, only magnitude-based features are considered under this ROI-based approach to mitigate the effects of inter-subject variability in brain shapes, brain sizes, and subjects orientation in the scanner. However, the information encoded by the spatial distribution of activated voxels within an ROI is in fact an important attribute of brain activity. Thus, we exploit this spatial information in our analysis by using invariant spatial features. We also extended this spatial analysis to the temporal domain, which provided a spatio-temporal characterization of brain activity. Applying these methods to real clinical data detected brain activity changes that were undetected with traditional means, thus demonstrating the importance of incorporating spatial information in fMRI analysis.

- McIntosh, Chris (Simon Fraser University, Ph.D. student of Ghassan Hamarneh): Learning Optimal Parameters for Medical Image Segmentation

Energy functional minimization is an increasingly popular technique for image segmentation. However, it is far too commonly applied with hand-tuned parameters and initializations that have only been validated for a few images. Fixing these parameters over a set of images assumes the same parameters are ideal for each image. We highlight the effects of varying the parameters and initialization on segmentation accuracy and propose a framework for attaining improved results using image adaptive parameters and initializations. We provide an analytical definition of optimal weights for functional terms through an examination of segmentation in the context of image manifolds, where nearby images on the manifold require similar parameters and similar initializations. Our results validate that fixed parameters are insufficient in addressing the variability in real clinical data, that similar images require similar parameters, and demonstrate how these parameters correlate with the image manifold. We present significantly improved segmentations for a set of 470 clinical examples.

- Ward, Aaron (Simon Fraser University, Ph.D. student of Ghassan Hamarneh): Learning Optimal Landmark Shape and Appearance Features for Point Correspondence Establishment

We propose a highly automated approach to the point correspondence problem for anatomical shapes in medical images. Manual landmarking is performed on a small subset of the shapes in the study, and a machine learning approach is used to elucidate the characteristic shape and appearance features at each landmark. A classifier trained using these features defines a cost function that drives key landmarks to anatomically meaningful locations after MDL-based correspondence establishment. Results are shown for artificial examples as well as real data.

Scientific Progress Made

The workshop focused on investigating and discussing the latest in mathematical methods for solving problems in medical image analysis including segmentation, registration, shape and functional modeling in a multitude of imaging modalities including structural and functional data. Experts in the field exchanged ideas and insights on how to best address the various problems identified especially in image segmentation, registration, feature extraction and object modeling. The challenges in the field were identified and potential approaches for addressing them were debated.

Outcome of the Meeting

The workshop was prominently and internationally recognized as a major event in the field of medical image computing. In fact, it was by chance was held very close to the dates of the premier conference on Medical Image Computing and Computer Assisted Intervention (MICCAI 07 which was held in Brisbane Australia) and many distinguished scholars chose to attend the Banff workshop over MICCAI. Some participants also flew all the way from Australia to Banff to participate despite the gruesome travel requirements.

A number of professional connections and scientific collaborations were facilitated by the workshop. Many researchers met for the first time and a number strengthened ties already established. The organizers received very positive feedback from the participants their appreciation of the scientific atmosphere and the ample opportunities for discussing various challenges in the field of medical image computing. To date, the organizers still receive many inquires whether there will be a sequel event to the first workshop which we hope to be able to hold sometime in the near future.

Summary of Challenges

Participants largely agreed that the challenges facing the area of medical image computing mainly include:

- High dimensionality of image and shape data and low number of training samples.

- Lack of samples and models of pathological cases to train methods.
- Sensitivity of developed methods to parameter setting.
- Designing energy functionals whose minima are correct solutions of medical image and shape analysis problems.
- Encoding expert and high level knowledge in methods.
- Optimizing energy functionals (local minima problems in iterative optimization, discrete subset of search space or simplified functionals in global optimizers).
- Fusion of data from multiple modalities.
- Fusion of data from multiple scales.
- Building Biomechanical and physiological models of anatomy and function.
- Building shape and appearance models of normal and pathological states.
- Need for mechanisms for intuitive interaction methods for highly automated analysis methods.
- Scarcity of viable options for validation of developed methods.
- Need to standardized metrics to compare algorithms.

Workshop Homepage

The workshop home page provides full details including complete presentation slides. The page can be viewed at: http://bisicl.ece.ubc.ca/mmmia2007

List of Participants

Thirty seven participants attended the workshop representing a large number of internationally renowned research labs specializing in medical image computing at top universities, e.g., Harvard, John's Hopkins, Yale, Imperial College, and at giant industries, e.g., Siemens and Philips. Below is a complete list of participants along with their affiliation.

Abolmaesumi, Purang (Queens University)
Abugharbieh, Rafeef (University of British Columbia)
Archip, Neculai (Harvard Medical School)
Atkins, Stella (Simon Fraser University)
Beg, Mirza Faisal (Simon Fraser University)
Boykov, Yuri (University of Western Ontario)
Celler, Anna (Department of Radiology, University of British Columbia)
Christensen, Gary (University of Iowa)
Cootes, Tim (University of Manchester)
Ellis, Randy (Queen's University)
Flores-Mangas, Fernando (University of Toronto)
Hamarneh, Ghassan (Medical Image Analysis Lab, Simon Fraser University)
Huang, Albert (University of British Columbia)
Larsen, Rasmus (Technical University of Denmark)
Lee, Tim (BC Cancer Agency)
Lenglet, Christophe (Siemens Corporate Research Inc.)
Lorenz, Cristian (Philips-Germany)
McIntosh, Chris (Simon Fraser University)
McKeown, Martin (University of British Columbia)
Miller, Michael (John Hopkins University)

Mller, Torsten (Simon Fraser University)
Ng, Bernard (University of British Columbia)
Pizer, Stephen (University of North Carolina, Chapel Hill)
Rohling, Robert (University of British Columbia)
Rueckert, Daniel (Imperial College)
Salcudean, Tim (University of British Columbia)
Siddiqi, Kaleem (McGill University)
Sonka, Milan (University of Iowa)
Sossi, Vesna (University of British Columbia)
Staib, Lawrence (Yale University)
Styner, Martin (University of North Carolina, Chapel Hill)
Tam, Roger (Faculty of Medicine at UBC)
Ward, Aaron (Simon Fraser University)
Warfield, Simon (Children's Hospital Boston)
Westin, Carl-Fredrik (Harvard Medical School)
Whitaker, Ross T. (University of Utah)
Worsley, Keith (McGill University)

Chapter 37

Modern Approaches in Asymptotics of Polynomials (07w5032)

Nov 11 - Nov 16, 2007

Organizer(s): Peter Borwein (Simon Fraser University), Doron Lubinsky (Georgia Institute of Technology), Ed Saff (Vanderbilt University)

Objectives

The focus of the conference is sequences of polynomials, their zeros, and asymptotic behavior as well as related potential theoretic issues, such as distribution of points on a sphere. The aim is to bring together experts who have different approaches to these questions for example those using potential theory, those mixing approximation and number theoretic techniques on integer polynomials, and those using Riemann-Hilbert techniques for asymptotics of sequences of (mainly orthogonal) polynomials. There has not been any meeting focusing on this cross-section of researchers in the past few years. We expect the communication of ideas and methods from these different approaches will encourage new techniques and research across several topics.

We also expect that the young researchers present will benefit from exposure to the leading different approaches.

Overview

RELEVANCE, IMPORTANCE AND TIMELINESS
In recent years, asymptotics of orthogonal polynomials have been used to study random matrices, combinatorial questions such as the longest increasing subsequence in a given sequence, Toda lattices, and weighted approximation. The potential theory that underlies some of these asymptotics has been used in distributing points on spheres and manifolds and in studying the distribution of zeros of sequences of polynomials. Zeros of integer polynomials, and the behavior of integer polynomials has been explored with a view to applications in number theory. The problems within the focus of the conference are widely applied, highly regarded, and very active areas of research. The conference is timely, and has a different focus from any other that we know of. At least 5 of the participants are young researchers (including graduate students and postdocs).

Abstracts of Talks

Jinho Baik, Asymptotics of Tracy-Widom distribution functions in random matrix theory, and total integral of Painleve solutions

The Tracy-Widom distribution functions are certain probability distribution functions expressed in terms of a solution of the Painleve II equation. These distribution functions arise in random matrix theory, statistics, combinatorics and probability. We discuss the asymptotic expansion of these functions at negative infinity, especially the problem of computing the constant term in the expansion. We also discuss the total integral of Painleve II solutions.

Laurent Baratchart, Asymptotic Uniqueness in best-H2 rational approximation to Cauchy transforms of complex measures

Best rational approximation with a prescribed number of poles in H_2 is a classical subject dating back to the work of J. L. Walsh and V. Tikhomirov. One of the most intriguing questions which of major computational interest is that of uniqueness of a best approximant, or better, of a critical point.

We consider in this talk Cauchy transforms of complex measures, that is, functions of the form $f(z) = \int d\nu(t)/(1-zt)$ where ν is a complex measure on a real segment included in (-1,1). Using strong asymptotics for the approximation error as developed in the talk of M. Yattselev, we carry over to the case of complex absolutely continuous ν whose density is Dini-continuous and may vanish only at finitely many points (yet in a restricted manner) the asymptotic uniqueness of a critical point proved by H. Stahl, F. Wielonsky and the speaker for Markov functions (i.e. when ν is positive and satisfies the Szego condition). This previous construction itself did dwell on joint work with F. Wielonsky and E. B. Saff. (Joint work with M. Yattselev.)

David Benko, Uniform approximation by weighted polynomials

Let w(x) be a continuous non-negative weight on the real line which decays faster than 1/—x—. The topic of uniform approximation by weighted polynomials $w(x)^n P_n(x)$ was introduced by Saff. He conjectured that $w(x)^n P_n(x)(n = 0, 1, 2, \dots)$ are dense on the support of the equilibrium measure, assuming $\log w(x)$ is concave. This was proved by Totik. In the talk we give other sufficient conditions on w(x) which imply denseness.

Hans-Peter Blatt, Divergence of Rational Approximants to Non-Analytic Functions

On [-1, 1] we consider best uniform rational approximation with numerator degree n and denominator degree m. E. B. Saff and H. Stahl have shown for $f(x) = |x|^\alpha$ that best approximants diverge everywhere outside [-1, 1] if ray sequences in the lower triangle of the Walsh table are considered, i.e. for sequences $\{(n, m(n))\}_{n=1}^\infty$ with $\frac{n}{m(n)} \to c \in (1, \infty), \qquad n \to \infty$. In contrast to this, for example rational approximants to $f(x) = |x|$ converge in the diagonal of the Walsh table to $f(z) = z$ in the right half-plane and to $f(z) = -z$ in the left half-plane. We investigate the situation in the lower triangle of the Walsh table for more general functions f which are not analytic on [-1, 1].

Sergiy Borodachov, On the asymptotic behavior of the Riesz minimal energy on sets with finite α -dimensional packing premeasure

For compact sets $A \subset \mathbf{R}^d, 0 < \alpha \leq d$ and $s > \alpha$, we study the behavior of the upper limit $g_{s,\alpha}(A)$ of the quotient $\frac{E_s(A,N)}{N^{1+s/\alpha}}$, as N gets large, where $E_s(A, N)$ is the discrete minimal Riesz s-energy of N-point configurations on A. We show that there exists an outer measure $\mu in \mathbf{R}^d$ such that for every compact set $A \subset \mathbf{R}^d$ with finite α-dimensional packing premeasure there holds $g_{s,\alpha}(A) = \mu(A)^{-s/\alpha}$. Earlier, together with D. P. Hardin, E. B. Saff we proved that for integer α and α-rectifiable closed sets A in $\mathbf{R}^d$, the limit of the ratio $E_s(A, N)/N^{1+s/\alpha}$ exists and one can take μ to be a certain constant multiple of the α-dimensional Hausdorff measure in $\mathbf{R}^d$.

Peter Borwein, Littlewood's 22nd Problem: Zeros of cosine polynomials

Littlewood in his 1968 monograph Some Problems in Real and Complex Analysis poses the following research problem, which remained open: If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N}\cos(n_m\theta)$? Possibly N-1, or not much less. Initial computations suggest this might be true. We show however that this is spectacularly false.

Stephen Choi, Mahler's measure and Lp norms of polynomials with restricted coefficients

We compute asymptotic formulas for mean values of Mahler's measure and the L_p norms of several classes of polynomials with restricted coefficients and bounded degree. We study the unimodular polynomials, with complex coefficients of modulus 1, the Littlewood polynomials, with $-1, 1$ coefficients, and the height 1 polynomials, with $-1, 0, 1$ coefficients. We show that both the geometric mean and the arithmetic mean of Mahler's measure of the unimodular polynomials with degree n-1 approach $e^{-\gamma/2}\sqrt{n}$ as n grows large, and that this same result holds for Littlewood polynomials.

Percy Deift, Riemann Hilbert Methods for Orthogonal Polynomials

In this overview the speaker will show how Riemann-Hilbert methods can be used to address a wider variety of algebraic, analytical and asymptotic questions for orthogonal polynomials and related quantities, such as Toepliz and Hankel determinants.

Karl Dilcher, A polynomial analogue to the Stern sequence

We extend the Stern sequence, sometimes also called Stern's diatomic sequence, to polynomials with coefficients 0 and 1 and derive various properties, including a generating function. A simple iteration for quotients of consecutive terms of the Stern sequence, recently obtained by Moshe Newman, is extended to this polynomial sequence. Finally we establish connections with Stirling numbers and Chebyshev polynomials, extending some results of Carlitz. In the process we also obtain some new results and new proofs for the classical Stern sequence. (Joint work with K.B. Stolarsky.)

Zeev Ditzian, Sharp Jackson Inequalities

For trigonometric polynomials on $[-\pi,\pi]\equiv T$, the classical Jackson inequality $E_n(f)_p \le C\omega^r(f,1/n)_p$ was sharpened by M. Timan for $1<p<1$ to yield $n^{-r}\left\{\sum_{k=1}^{n}k^{sr-1}E_k(f)_p^s\right\}^{1/s} \le C\omega^r(f,n^{-1})_p wheres = \max(p,2)$. In this paper a general result on the relations between systems or sequences of best approximation and appropriate measures of smoothness is given. Approximation by algebraic polynomials on [-1, 1], by spherical harmonic polynomials on the unit sphere, and by functions of exponential type on R^d are among the systems for which the present treatment yields sharp Jackson inequalities. Analogous sharper versions of the inequality $\omega^{r+1}(f,t)_p \le C\omega^r(f,t)_p$ are also achieved.

Peter Dragnev, Iterative balayage techniques with applications to Riesz and logarithmic potentials

Iterative balayage techniques have been used before to derive relevant information about the nature of the support of equilibrium measures associated with external fields on the real line. Here we present two recent results, utilizing the technique. The first is a Riesz energy problem on the sphere with axis-supported Riesz potential externa field. The second is a logarithmic energy problem on the real line with external fields, whose signed equilibria has concave positive part. Our emphasis will be on the common approach to the solution of the two problems.

Tamás Erdélyi, On the derivatives of unimodular polynomials

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with complex coefficients. Associated with $\lambda \geq 0 let \mathcal{K}_n^\lambda := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k k^\lambda z^k, \;\; a_k \in \mathbb{C}, \;\; |a_k| = 1 \right\} \subset \mathcal{P}_n^c$. The class $\mathcal{K}_n^0$ is often called the collection of all (complex) unimodular polynomials of degree n. Given a sequence (ε_n) of positive numbers tending to 0, we say that a sequence (P_n) of polynomials $P_n \in \mathcal{K}\lambda_n$ is (ε_n)-ultraflat if $(1 - \varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}} \leq P_n(z)| \leq (1+\varepsilon_n)\frac{n^{\lambda+1/2}}{\sqrt{2\lambda+1}}, z \in \partial D, n \in \mathbb{N}$. Although we do not know in general, whether or not ultraflat sequences (P_n) of polynomials $P_n \in \mathcal{K}_n^\lambda$ exists, we make an effort to prove various interesting properties of them. These allow us to conclude that there are no sequences of (P_n) of conjugate, plain, or skew reciprocal unimodular polynomials $P_n \in \mathcal{K}_n^0$ such that (Q_n) with $Q_n(z) = zP_n'(z) + 1$ is an ultraflat sequence of polynomials $Q_n \in \mathcal{K}_n^1$.

Jeff Geronimo, Some results on bivariate orthogonal polynomials

We will present results on bivariate orthogonal polynomials. In particular we will discuss orthogonal polynomials associated with a class of Bernstein-Szego weights.

Doug Hardin, Asymptotics for discrete minimum energy problems

I will review some recent and some old results concerning the asymptotic properties (as N goes to infinity) of 'ground state' arrangements of N points restricted to a rectifiable d-dimensional manifold A that interact through the Riesz potential $V = 1/r^s$, where $s > 0$ and r is the distance in the embedding Euclidean space. This class of minimal discrete energy problems can be considered as a bridge between logarithmic energy problems and best-packing ones. Many of the new results discussed are joint work with E. B. Saff and S. V. Borodachov.

Thomas Kriecherbauer, Revisiting Szego's curve with Riemann-Hilbert

In this talk I will show how to derive asymptotic results for the zeros of the partial sums of the exponential function using a Riemann-Hilbert approach. The Riemann-Hilbert problem that arises in this situation is of great simplicity and it is only the construction of a local parametrix that is needed to derive asymptotic expansions for the zeros with uniform error bounds.

Andras Kroo, On the density of homogeneous polynomials in the space of continuous functions on convex and star-like surfaces

This talk is related to the Weierstrass theorem on density of polynomials which is one of the most celebrated theorems in Analysis. An interesting related problem here concerns the density of homogeneous polynomials which is a substantially smalller nonlinear subset of the set of all polynomials. A known conjecture here claims that density holds on every 0-symmetric convex surface. A substantial progress was made in last several years towards settling this conjecture. We shall present the corresponding results and also discuss the possible extensions for star-like surfaces.

Arno Kuijlaars, Asymptotic analysis of biorthogonal polynomials arising from a coupled random matrix model

The two sequences of biorthogonal polynomials (p_k) and (q_j) that play a role in a coupled random matrix model with potentials V and W and coupling constant c are characterized by the condition that
$\int\int p_k(x)q_j(y)e^{-V(x)-W(y)+2cxy}dxdy = \delta_{j,k}$. They have been extensively studied by Bertola, Eynard, and Harnad, as well as several other authors. The biorthogonal polynomials are characterized by a Riemann-Hilbert (RH) problem in case V and W are polynomials. However the asymptotic analysis of the RH problem has not been successful yet, which is partly due to the fact that the matrices in the RH problem have size 3×3 or higher. I will outline an approach to the Riemann-Hilbert problem for quartic potentials, which leads to a

4×4 matrix valued RH problem. An essential ingredient is a vector equilibrium problem for three measures involving both an external field and an upper constraint. This is joint work with Maurice Duits.

Xin Li, Functions Whose Moments Form a Geometric Progression

We start with a measure space $L2[R, d\mu]$ and give a lower bound for the norm of functions in this space whose first N moments form a geometric progression. Several consequences are investigated including a new criterion for the determinacy of the moment problem. The corresponding questions on the unit circle are also investigated. In particular we give a lower bound for the $L2$ norm of interpolatory functions in the disk algebra.

Guillermo López Lagomasino, Stieltjes-type polynomials on the unit circle

We present a construction on the unit circle similar to that of Stieltjes polynomials on the real line. We deduce their asymptotic behavior for the more relevant classes of measures supported on the whole unit circle. It is well known that Stieltjes polynomials are intimitely related with Gauss-Kronrod quadrature for algebraic polynomials. For the unit circle, Stieltjes type polynomials also serve to define a Gauss-Kronrod type quadrature on the space of Laurent polynomials. We prove the convergence of such quadrature rules on spaces of analytic functions defined on an annulus containing the unit disk.

Doron Lubinsky, Universality for arbitrary measures on compact sets

We present a new method for establishing universality limits in the bulk, based on the theory of entire functions of exponential type. Let ? be a measure on a compact subset of the real line. Assume that ? is absolutely continuous in a neighborhood of some point x in the support, and that ?? is bounded above and below near x, which is assumed to be a Lebesgue point of ??. Then the following are equivalent: (I) For all real a,

$lim_{n?}((K_n(x + (a/n), x + (a/n)))/(K_n(x, x))) = 1.$

(II) Uniformly for a,b in compact subsets of the plane,

$lim_{n?}((K_n(x + (a/(K_n(x, x))), x + (b/(K_n(x, x)))))/(K_n(x, x))) = ((sin(a - b))/((a - b))).$

Here K_n is the reproducing kernel for ?, while K_n is the normalized reproducing kernel. Thus universality is equivalent to universality ”along the diagonal” , or equivalently ratio asymptotics for Christoffel functions. We emphasize that ? does not need to be a regular measure. In the case that ? is a regular measure, the condition (I) follows from results of Vili Totik. So for regular measures, we provide an alternative proof to recent results of Simon, Totik, and the author. The advantage of the method is that it does not need a ”comparison measure” with similar support, for which universality is known. The method also applies to varying weights, and has been applied by Eli Levin and the author. If time permits, results for varying weights will be presented.

Francisco Marcellan, Asymptotics of contracted Sobolev extremal polynomials on the real line

For a wide class of Sobolev type norms with respect to measures with unbounded support on the real line we will analyze the contracted zero distribution and the logarithm asymptotic behavior of the corresponding rescaled Sobolev orthogonal polynomials. Some questions related with the properties of zeros in the case of Freud weight functions will be also presented. Finally, some open problems about strong and weak asymptotics for such families of orthogonal polynomials will be discussed. References: J. S. Geronimo, D. S. Lubinsky, F. Marcellan, Asymptotics for Sobolev Orthogonal Polynomials for Exponential Weights, Constr. Approx. 22(2005), 309-346. G. Lopez Lagomasino, F. Marcellan, H. Pijeira, Logarithm asymptotics of contracted Sobolev extremal polynomials on the real line, J. Approx. Theory, 143 (2006), 62-73. F. Marcellan, J. J. Moreno-Balcazar, Asymptotics and Zeros of Sobolev Orthogonal Polynomials on Unbounded Supports, Acta. Appl. Math. 94 (2006), 163-192.

Andrei Martinez-Finkelshtein, Squared Bessel process and multiple orthogonal polynomials for modified Bessel weights

We study a model of n non-intersecting squared Bessel processes in the confluent case: all particles start at time t = 0 at the same positive value x = a, remain positive (due to the nature of the squared Bessel process) and are conditioned to end at time t = T at x = 0. In the limit $n- > \infty$, after appropriate rescaling, the paths fill out a region in the tx-plane. In particular, the paths initially stay away from the hard edge at x = 0, but at a certain critical time the smallest paths hit the hard edge and from then on are stuck to it. The phase transition at the critical time is a new feature of the present model. A key fact in the analysis is that in the confluent case the biorthogonal ensemble reduces to a multiple orthogonal polynomial ensemble, corresponding to a Nikishin system of two modified Bessel-type weights. This means that there is a 3x3 RH problem characterizing this model, that we analyze in the large n limit using the Deift-Zhou steepest descent method. There are some novel ingredients in our analysis that might be of independent interest. This is a joint work with A. B. J. Kuijlaars and F. Wielonsky.

Hrushikesh Mhaskar, Spectral approximation of piecewise analytic functions

While classical wavelet analysis is adequate for a characterization of local Besov spaces, we propose a polynomial frame on the unit interval adequate for a characterization of functions analytic at a point on the interval. Thus, at each point on the interval, the behavior of the coefficients in our frame expansion can be used to detect whether the function is analytic at that point or not. The corresponding approximation operators yield an exponentially decreasing rate of approximation in the vicinity of points of analyticity and a near best approximation on the whole interval. In spite of this high localization, the construction of our operators are based on the (globally defined) Fourier coefficients in a general orthogonal polynomial expansion. Previously known results in this direction utilize Chebyshev coefficients, and the techniques to obtain these cannot be used for a similar study of general orthogonal polynomial systems. Another novelty of our paper is that while all the previous estimates for localization of polynomial kernels known to us are deduced using such special function properties of the orthogonal polynomials as asymptotics or explicit formulas for the Christoffel-Darboux kernel, we suggest a very simple idea to obtain exponentially localized kernels based on a general system of orthogonal polynomials, for which the Cesàro means of some order are uniformly bounded. The boundedness of these means is known in a number of cases, where no special function properties are known. The talk is based on joint work with J. Prestin and F. Filbir.

Erwin Mina-Diaz, Asymptotics for Carleman and Szego orthogonal polynomials for analytic curves

We first consider Szego polynomials that are orthogonal over an analytic Jordan curve L with respect to a positive analytic weight. For these polynomials we give a full asymptotic expansion in terms of certain iterated integral transforms that is valid on the whole complex plane. In particular, Szego's strong asymptotic formula follows, but more importantly, the behavior of the polynomials inside the curve L can be finely described under certain natural additional assumptions concerning the singularities of the weight. This result is a generalization of the main result in [1] in which orthogonality is considered over the unit circle. We then consider Carleman polynomials that are orthogonal over the interior domain of the curve L with respect to area measure. For these polynomials we find an analogous integral representation which is valid on the interior of L, and that, roughly speaking, allows us to find a maximal domain U on which Carleman strong asymptotic formula is valid. The set is maximal in the sense that every point of its boundary is an accumulation point of the zeros of Carleman polynomials.

1. A. Martinez-Finkelshtein, K. T.-R. McLaughlin and E. B. Saff: Szego orthogonal polynomials with respect to an analytic weight: canonical representation and strong asymptotics. Constr. Approx., 2006, 24: 319-363.

Hugh Montgomery, Extremal polynomials in number theory

We shall discuss a wide variety of topics, including:
1.Polynomials whose roots are near the unit circle; this involves mention of polynomials with small Mahler measure.
2. Polynomials (with integral coefficients) that are uniformly small on [0,1].
3. Problems concerning polynomials that arise in the discussion of Turan's power sum method.
4. Extremal problems concerning exponential polynomials.
5. Extremal problems concerning Dirichlet polynomials.

Irina Nenciu, On OPUC and the defocusing Ablowitz-Ladik equation In this talk we will describe various properties of a completely integrable system, the Ablowitz-Ladik (AL) equation, through its connection to the theory of orthogonal polynomials on the unit circle. In particular, we will concentrate on its associated Poisson structure using a functional analytic approach; this in turn will allow us to define a multi-Hamiltonian structure for the AL hierarchy. As an important consequence, we show that, in some of the new Poisson brackets, the classical map taking orthogonal polynomials on the unit circle into orthogonal polynomials on the real line becomes a symplectic mapping taking certain of the AL flows into flows from the Toda lattice hierarchy.

Franz Peherstorfer, Extremal problems of Chebshev type

We study the following problem: Let $a \in \mathbb{C}$ be given. Among all polynomials $pfrom\mathbb{P}_n^c$ ($\mathbb{P}_n^c$ denotes the set of polynomials of degree less or equal n with complex coefficients) with $\max_{-1\leq x\leq 1}|p_n(x)| \leq 1$ find that polynomial P_n for which $|P_n(a)|$ is maximal, that is, determine $\sup_{p\in\mathbb{P}_n^c,\ ||p||_{[-1,1]}\leq 1}|p(a)|$. (1) For $a \in \mathbb{R}\backslash[-1,1]$ already Chebyshev has shown that $T_n(x) = \cos n \arccos x$, solves the problem. For purely imaginary a i.e., $a = -\frac{i}{2}(t - \frac{1}{t}), t \in (-1,1)$, Freund and Ruscheweyh discovered the lovely result that $P_n(x) = (T_n(x) - 2itT_{n-1}(x) - t^2T_{n-2}(x))/(1+t^2)$ is a solution of (1). The general case $a \in \mathbb{C}\backslash[-1,1]$ was solved in an elegant way by P. Yuditskii. He gave the solution in terms of elliptic functions, where some parameters (more precisely, harmonic measures of intervals depending on n) are given implicitly only. Therefore there is still a demand for a description in terms of elementary functions. To find such a representation without elliptic functions is in general not obvious as the simpler case of Zolotarev polynomials shows. Here we present an asymptotic expression for the extremal polynomial and for the extremal value in terms of elementary functions. The solution is based on the description of Zolotarev polynomials with respect to square root polynomial weights. Finally minimal polynomials on several intervals and arcs are discussed.

Igor Pritsker, Moments of equilibrium measures and extremal problems

The equilibrium measure of a compact set gives the steady state distribution of charges on the conductor. We study extremal problems for certain moments of the equilibrium measure. Our results have diverse applications to isoperimetric inequalities, and to inequalities for Green's functions and their critical points. We also discuss relations with inequalities for products of polynomials, and with inequalities for means of zeros of various polynomial sequences. Vasiliy Prokhorov, On meromorphic and rational approximation The talk is devoted to results of the theory of Hankel operators, convergence of meromorphic and rational approximants. We will also discuss problems related to asymptotics of Pade approximants, locally the best rational approximants to a given power series.

Thomas Ransford, Computation of capacity

I shall describe a method for computing the logarithmic capacity of a compact plane set. The method yields upper and lower bounds for the capacity. If the set has the Hlder continuity property, then these bounds converge to the value of the capacity. I shall discuss several examples, including the Cantor middle-third set, for which we estimate $c(E) \approx 0.220949102189507$. (Joint work with Jrmie Rostand.)

Avram Sidi, Polynomials biorthogonal to exponential functions with applications to numerical quadrature

TBA

Barry Simon, Fun and Games with the CD kernel

I will present many aspects of the CD kernel - some well known and others less so: Aitken's Theorem, Operator Theoretic Proofs of the CD formula for both OPRL and OPUC, the "half-CD" formula for general OPs, periodic models, Mate-Nevai Upper Bound, weak convergence and applications, Nevai's delta function convergence, Nevai's Comparison Theorem, MNT theorem, Lubinsky's inequality, universality and clock behavior for general OPRL. Herbert Stahl, Rational Best Approximation on $(-\infty, 0]$ We consider rational best approximants $r_n^* = r_n^*(f, (-\infty, 0]; \cdot) \in R_{n,n+k}, on(-\infty, 0], k \in \mathbb{Z}$ fixed, to functions of the form $f = u_0 + u_1 \exp$ (*) with u_0, u_1 given rational functions. As usual in rational approximation with free poles, the study of the approximants r_n^* is inseparably tied up with an investigation of orthogonal polynomials, which in this specific case are orthogonal polynomials with varying weights and a non-Hermitian orthogonality relation that lives on a curve in the complex plane. Starting point of the talk will be the famous solution of the '1/9' problem by Gonchar $\&$ Rakhmanov from 1986, which deals with the uniform rational approximation of the exponential function on $(-\infty, 0]$. This result will be extended to functions of type (*), and further some related questions will be addressed, as for instance, overconvergence throughout the complex plane $\mathbb{C}$, the asymptotic distribution of the poles of the approximants, and the convergence of close-to-best approximants. The renewed and extended interest in the by now classical '1/9' problem is motivated by applications in numerical analysis, where good rational approximants are needed for functions of type (*), as for instance,for 'φ functions' that appear in exponential integrators.

Vili Totik, The polynomial inverse image method

The talk will discuss a method that allows one to go from results on one interval to general compact sets on the real line. The method is based on taking polynomial inverse images with some special polynomials used first by Geronimo and VanAssche. Results that were obtained by this method will be discussed, among them Bernstein inequlity, approximation of —x—, asymptotics for Christoffel functions, and Lubinsky's universality theorem; all these on general sets. Maxim Yattselev, Strong Asymptotics for the Error of Rational Approximation in L2 of the Unit Circle for Cauchy Transforms We derive formulae of strong asymptotics for the error of best rational approximation in the L2-norm of the unit circle for Cauchy transforms of complex measures. The latter are assumed to be supported on a segment and have Dini-continuous complex-valued Radon-Nikodym derivatives with respect the arcsine distribution of that segment that may vanish although in a controlled manner. (Joint work with L. Baratchart.)

Ping Zhou, Some arithmetical results on certain multivariate power series

The arithmetical nature (i.e. irrationality, or even transcendence) of the multivariate power series $i, i, , i_m = 0 f(i + i + + i_m) x^i x^i x_m^{i_m}$ will be discussed. The results very much depend on the arithmetical properties of the one variable projections of the multivariate power series, most of which are proved by using rational or Padé approximants. Some typical applications are also discussed.

A Review of the Meeting and Some Outcomes

There was a lively exchange between researchers in several different topics, including:
(I) Riemann-Hilbert methods for asymptotics of univariate orthogonal polynomials, with applications to random matrices;
(II) Classical operator theory methods for asymptotics of univariate orthogonal and related polynomials;
(III) Multivariate polynomials, with applications to multivariate orthogonal polynomials, distribution of points on manifolds; and multivariate approximation;
(IV) Extremal and integer polynomials with a number theoretic flavor, including applications to Diophantine

approximation, Mahler measure, and nearly unimodular polynomials;
(V) Potential theoretic methods, including old problems like the capacity of the Cantor set, and newer ones, such as Saffs weighted approximation problem involving external fields, and the application of polynomial pullbacks to a host of problems;
(VI) Classical analytic techniques with applications to universality limits;
(VII) Approximation by rational and meromorphic functions in one variable, involving Hankel operators, Cauchy Transforms, and applications to numerical analysis.

Of course, many of these topics intersect, and this was reflected in the talks. Thus, for example, Arno Kuijlaars hour long talk on a coupled random matrix model involved (bi)orthogonal polynomials, random matrices, Riemann-Hilbert asymptotic techniques, and Riemann surfaces. This depth was also evident in Percy Deifts introduction to Riemann-Hilbert techniques, and in Andre Martinez-Finkelshteins treatment of Bessel processes and multiple orthogonal polynomials. Jinho Baik presented asymptotic expansions of Tracy-Widom distributions that arise in random matrix theory, combinatorics, and probability. Thomas Kriecherbauer showed how Riemann-Hilbert methods can be used to give new, and sharper insights into the old problem of the distribution of zeros of partial sums of exp(z). Ken McLaughlin showed how Riemann-Hilbert methods may be extended beyond the case of analytic fields, using the delta bar approximation method: only two derivatives are required of the external field. It was evident that the range of applicability, and the depth and power of Riemann-Hilbert methods is constantly expanding.

Barry Simon displayed the power of operator theoretic and classical analytic methods in obtaining asymptotics in very general situations for orthogonal polynomials on the real line and unit circle. Applications to universality limits were also part of Simons focus. In a related vein, Irina Nenciu delivered a deep talk on the Ablowitz-Ladik equation, its relation to orthogonal polynomials on the unit circle, Toda flows, Poisson brackets, and Hamiltonian structures. One outcome is that the classical relationship between orthogonal polynomials on the unit circle and real line, generates a similar relationship between Ablowitz-Ladik and Toda flows.

Vili Totik explored the power of polynomial pullbacks and potential theory. Polynomial Pullbacks were introduced by Geronimo and Van Assche as a means of studying extremal and orthogonal polynomials on several intervals, and self similar sets. As developed by Vili Totik and others, they have been used to extend Markov-Bernstein inequalities, asymptotics for Christoffel functions, universality limits, asymptotics of best approximation to general compact sets. Franz Peherstorfer provided asymptotics for sup-norm Christoffel functions on one (and several) intervals, involving elementary functions and Zolotarev type polynomials. New methods for establishing universality limits for orthogonal polynomials were outlined by Doron Lubinsky.

An old problem in potential theory is the logarithmic capacity of the Cantor set. Thomas Ransford, author of a classic text on potential theory, presented a method that provides rigorous upper and lower bounds for this capacity, together with an extrapolation to the limit, which is approximately 0.220949102189507. This talk led to several discussions on how to extend and accelerate the current numerical procedures. Potential theory, but with external fields, was the focus of Peter Dragnevs talk. He showed how the iterated balayage algorithm can be used to deduce new information about the support of the equilibrium measure for Riesz energy problems on the sphere, as well as non-standard external fields involving signed equilibrium measures. David Benko discussed Saffs weighted approximation problem, which involves weighted polynomials and delicate potential theory with external fields.

The interplay between number theory and polynomial asymptotics was laid out in a variety of talks. Hugh Montgomery presented several such connections, including: polynomials with small Mahler measure, polynomials with integral coefficients that are small on the entire unit circle, and polynomials that arise in Turans power sum method. Igor Pritsker showed just how close is the connection between classical complex analysis, potential theory, and number theory, through his the solution (joint with A. Baernstein) to an extremal problem involving moments of equilibrium measures. This has important applications to norms of factors of polynomials, Mahler measure, and minimal norm integer polynomials.

Also in a polynomial number theoretic vein, Peter Borwein outlined the recent solution to a very old problem of Littlewood on the number of zeros of cosine polynomials with coefficients 0 and 1. Littlewood conjectured that such a cosine polynomial with N terms cannot have much less than N zeros, but the conjecture turned out be false, there can be a lot less than N zeros. His colleague Stephen Choi continued the number theoretic theme with a penetrating study of Mahler measure and Lp norms of polynomials with coefficients

-1 and 1 or -1, 0 , and 1. Choi and his coworkers determined the asymptotic behavior of the mean value of the Mahler measure and Lp norm (averaged over the finitely many such polynomials of a given degree), as the degree grows to infinity. Karl Dilchers most interesting talk dealt with the Stern sequence, and the polynomials with 0 and 1 coefficients that it generates. Tamas Erdelyi presented his most recent results on ultraflat polynomials, establishing necessary conditions that preclude properties such as conjugacy or skew-reciprocality. Ping Zhou showed an interesting technique to establish irrationality of certain multivariate series.

Multivariate problems and polynomials were the subject of talks by Zeev Ditzian, Jeff Geronimo, Doug Hardin, Sergiy Borodachov, and Andras Kroo. Doug Hardin dealt with the fundamental problem of distributing points evenly on a manifold. This of course has a close connection with Steven Smales problem of distributing points on a sphere. Depending on the parameter in the Riesz potential, the minimal discrete energy used for finding the points, varies between a logarithmic energy problem, and a best packing one. Sergiy Borodachov continued this theme, with a careful analysis of the asymptotic behavior of minimal Riesz energy for arbitrary compact sets, not just rectifiable ones. While there need not be a limit of the scaled discrete energy, the lim sup may be expressed in terms of a certain outer measure that is independent of the specific set.

Jeff Geronimo presented recent results on bivariate analogues of the classical univariate Bernstein-Szego polynomials. These included recurrence relations, positive polynomials, and the factorization of the latter. Andras Kroo took a more approximation theoretic line, examining the density of homogeneous polynomials in spaces of continuous functions on convex and star-like surfaces. Kroo displayed the role of the geometry of the surface in determining density or its failure. Zeev Ditzian presented a unified approach to sharp Jackson inequalities that works for spherical harmonic polynomials on the unit sphere, but also in a variety of univariate contexts.

The classical and difficult topic of rational approximation was discussed by Laurent Baratchart, Hans-Peter Blatt, Vasiliy Prokhorov, Herbert Stahl and Maxim Yattselev. Laurent Baratchart focused on asymptotic uniqueness of best rational approximants in the L2 norm to Cauchy transforms of complex measures. Maxim Yattselev developed this topic further, providing asymptotics for the errors of best rational approximation. Vasiliy Prokhorov displayed the utility of Hankel operator techniques in rational approximation, with additional applications to meromorphic approximants, and the asymptotic distribution of poles of subsequences of Pade approximants. Herbert Stahl analysed best rational approximants on the negative real axis to functions of the form r(z)+s(z)exp(z), where r and s are fixed rational functions. This problem arises in numerical analysis, and leads to a generalization of the famous (1/9)th problem, which was solved by Goncar-Rakhmanov using techniques developed by Herbert Stahl. Hans-Peter Blatt explored the phenomenon of divergence of best rational approximants to a function on [-1,1], outside this domain. The classic example is the function —x—, whose diagonal best rational approximants do converge in the left and right-half planes, but whose off-diagonal rational approximants diverge.

A range of generalized orthogonal polynomials were considered by Erwin Mina-Diaz, Guillermo Lopez Lagomasino, Francisco Marcellan, and Avram Sidi. Erwin Mina-Diaz provided an exact series representation of orthogonal polynomials over an analytic Jordan curve, and Carleman orthogonal polynomials over the interior of that curve. This leads to strong asymptotics for the polynomials, as well as asymptotic information inside the curve, under additional conditions. Guillermo Lopez showed that it is possible to construct an analogue of the Stieltjes polynomials on the unit circle, leading to unit circle analogues of Gauss-Kronrod quadrature. Francisco Marcellan surveyed old and new results on asymptotics of Sobolev orthogonal polynomials. Avram Sidi described asymptotics of polynomials that are biorthogonal to systems of exponentials. The latter arise in convergence acceleration techniques, and in numerical integration of singular integrands.

Xin Li and Hrushikesh Mhaskar examined applications related to the general theory of orthogonal polynomials. Hrushikesh Mhaskar showed how the analyticity of functions can be checked at a point using polynomial frames. The latter use Cesaro means of orthonormal expansions. Xin Li discussed an extremal problem for functions in a weighted L2 space whose first N moments form a geometric progression. This leads to a new criterion for determinacy of the classical moment problem.

Many research problems were raised at the conference. These include:

(i) Methods for establishing universality limits in the bulk and edge of the spectrum, for the most general settings;

(ii) The convergence behavior of subsequences of diagonal Pade approximants that might replace the disproven Baker-Gammel-Wills conjecture, perhaps via Hankel theory methods;

(iii) New directions in asymptotics of Sobolev orthogonal polynomials;
(iv) Convergence acceleration for the sequences of upper and lower bounds for the capacity of the Cantor set, generated by the algorithm of Ransford and his coworkers;
(v) The zero distribution of sequences of polynomials that are biorthogonal to powers of a fixed function;
(vi) Zero distribution of Littlewood and similar polynomials, as well as mean statistics of various quantities for such classes;
(vii) Extensions of unit circle phenomena to the real line, or vice-versa, including flows and various quadratures;
(viii) Riemann-Hilbert analyses of partial sums of entire functions, and Riemann-Hilbert analyses of multi-orthogonal, and bi-orthogonal polynomials;
(ix) The support and behavior of equilibrium measures and related discrete and continuous energies on manifolds.

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Chapter 38

Physics-Based Mathematical Models of Low-Dimensional Semiconductor Nanostructures: Analysis and Computation (07w5057)

Nov 18 - Nov 23, 2007

Organizer(s): Lok Lew Yan Voon (Wright State University), Roderick Melnik (Wilfrid Laurier University), Morten Willatzen (University of Southern Denmark)

Key words: multiscale mathematical models, low dimensional nanostructures, quantum effects, dynamic and transport phenomena, complex coupled systems, control, deterministic and stochastic mathematical models, continuum and atomistic methodologies.

Introduction

In November 2007, some of the world's experts on physics-based mathematical models for nanoscience and nanotechnology met at the Banff Centre, where the Banff International Research Station hosted a workshop on recent developments in the study of the mathematics and physics of nanomaterials and nanostructures. Nanotechnology is the study and application of phenomena at or below the dimensions of 100 nm and has received a lot of public attention following popular accounts such as in the bestselling book by Michael Crichton, *Prey*. It is an area where fundamental questions of applied mathematics and mathematical physics, design of computational methodologies, physical insight, engineering and experimental techniques are meeting together in a quest for an adequate description of nanomaterials and nanostructures for applications in optoelectronics, medicine, energy-saving, bio- and other key technologies which will profoundly influence our life in the 21st century and beyond. There are already hundreds of applications in daily life such as in cosmetics and the hard drives in MP3 players (the 2007 Nobel prize in physics was recently awarded for the science that allowed the miniaturization of the drives), delivering drugs, high-definition DVD players and stain-resistant clothing, but with thousands more anticipated. The focus of this interdisciplinary workshop was on determining what kind of new mathematical and computational tools will be needed to advance the science and engineering of nanomaterials and nanostructures [13].

Thanks to the stimulating environment of the BIRS, participants of the workshop had plenty of opportunity to exchange new ideas on one of the main topics of this workshop - physics-based mathematical models for the description of low-dimensional semiconductor nanostructures (LDSNs) that are becoming increasingly important in technological innovations. The main objective of the workshop was to bring together some of the world leading experts in the field from each of the key research communities working on different aspects of LDSNs in order to (a) summarize the state-of-the-art models and computational techniques for modeling LDSNs, (b) identify critical problems of major importance that require solution and prioritize them, (c) analyze feasibility of existing mathematical and computational methodologies for the solution of some such problems, and (d) use some of the workshop working sessions to explore promising approaches in addressing identified challenges.

Since the main properties of two-dimensional heterostructures (such as quantum wells) are now quite well understood, there has been a consistently growing interest in the mathematical physics community to further dimensionality reduction of semiconductor structures. Experimental achievements in realizing one-dimensional and quasi-zero-dimensional heterostructures have opened new opportunities for theory and applications of such low-dimensional semiconductor nanostructures. One of the most important implications of this process has been a critical re-examining of assumptions under which traditional quantum mechanical mathematical models have been derived in this field. Indeed, the formation of LDSNs, in particular quantum dots, is a competition between the surface energy in the structure and strain energy. However, current models for bandstructure calculations use quite a simplified analysis of strain relaxation effects, although such effects are in the heart of nanostructure formation. By now, it has been understood that traditional mathematical models in this field, applied widely in the physics and engineering communities, may not be adequate for modeling realistic objects based on LDSNs due to neglecting many effects that may profoundly influence optoelectronic properties of the nanostructures. At the same time, precisely these optoelectronic properties are of fundamental importance in a wide range of LDSN applications. Among such effects that may profoundly influence such properties are electromechanical effects, including strain relaxation, piezoelectric effect, spontaneous polarization, and higher order nonlinear effects. Up to date, major efforts have been concentrated on the analysis of idealized, isolated quantum dots, while a typical self-assembled semiconductor quantum dot nanostructure is an array (or a molecule) of many individual quantum dots sitting on the same 'substrate' known as the wetting layer. Each such dot contains several hundred thousand atoms. In order to account for quantum effects accurately in a situation like that, attempts can be made to apply ab initio or atomistic methodologies, but then one would face a task of enormous computational complexity in solving a large-scale many-body problem. On the other hand, taking each quantum dot in isolation would lead to a manageable task for modern supercomputers, but accounting for the wetting layer even in the individual quantum dot model would increase the computational complexity of the problem in several times. Indeed, from a mathematical point of view the resulting model becomes a challenge due to multiple scale effects we have to deal with in such situations. As a result, the entire problem in its generality would be hardly feasible from a practical, routine-based simulation, point of view. Moreover, in calculating atomic positions the definitions of atomic forces that enter the Hamiltonian in such large scale atomic simulations are approximate by nature and a number of important coupled effects, such as piezoelectric, remain frequently outside the scope of the analysis. To attack the problem in hand, one needs to resort to some clever averaging over atomic scales. Such averaging can be achieved by empirical tight-binding, pseudopotential, and $k \cdot p$ approximations. These approximations are very important in further development of mathematical models for LDSNs due to the fact they are well suited for incorporating additional effects into the model, including strain, piezoelectric effects, spontaneous polarization, geometric and materials nonlinearities. These effects, despite their importance, have not been studied with vigor they deserve, in particular in the context of mathematical models for bandstructure calculations. There is a growing interest to such models as they should provide a key to better predicting optoelectromechanical properties of LDSNs which are in the heart of current and potential applications of these structures. With anticipated new discoveries in theoretical and experimental analysis of LDSNs in the coming years, one of the main emphases of the workshop was on the mathematical models that would allow incorporating these effects consistently into the state-of-the-art models for LDSNs. From a mathematical point of view, many such models can be reduced to a large eigenvalue PDE problem coupled to the mathematical models for strain and piezoelectric effects. In its turn, in its general setting the problem of strain and piezoelectric potential calculation requires the solution of a nonlinear system of partial differential equation. A large experience in solving these two parts of the problem separately, independently of each other, has been already accumulated in the distinct communities of the researchers. This BIRS workshop effectively combined expertise of these research communities, summarized the state-of-the-art for modeling LDSNs and key challenges facing these communities, and explored ways to address those challenges in interdisciplinary team settings.

Before the workshop, invited attendees were asked to upload their abstracts electronically to stimulate initial discussions (see (http://www.m2netlab.wlu.ca/ldsn-banff). Following the conclusion of the event, a selected number of refereed extended papers relating to the workshop presentations were published in the Journal of Physics Conference Series (JPCS). A direct link to the 12 JPCS refereed proceeding contributions can be found on the BIRS homepage for this workshop (or from the above site).

State-of-the-art overviews and interdisciplinary efforts in mathematical modelling of low dimensional nanostructures

At the beginning of the workshop, state-of-the-art overviews of the subject from perspectives of experimentalists, physics, applied mathematics and computational science communities were given by key experts in their respective fields. We had four main plenary talks of one hour duration that gave state-of-the-art overviews of the subject from perspectives of applied mathematics (Professor Russel Caflisch of the University of California at Los Angeles), physics (Professor Antti-Pekka Jauho of the Danish Technical University), and computational science and engineering communities (Professor Gerhard Klimeck of Purdue University), as well as from a point of view of experimentalists (Dr Gail Brown of the Materials Lab/Air Force Research Lab at Wright-Patterson AFB). These talks helped identify the areas where joint efforts needed to be directed to, and they set up the scene for further work during the workshop, including discussions at the workshop open problem sessions. All participants presented their own research in LDSNs. At the last day of the workshop, time was allocated for on-site demonstrations of several model-based software tools such as NextNano by Stefan Birner ($\mathbf{k}\cdot\mathbf{p}$ based models) as well as for the NanoHub project by Gerhard Klimeck (models for bandstructure based on tight-binding methodologies), involving LDSN analysis. Good discussions of strengths and weaknesses using these electronic bandstructure methods in determining basic physical properties of quantum dots and computing characteristics of quantum-dot based devices came out of this session. Indeed, it was one of the main ideas to create this discussion atmosphere so as to identify pros and cons in applying various mathematical and computational methods and to explore how they can possibly supplement each another. There is no doubt that large quantum-dot and quantum-wire based structures are still computationally too demanding when assessing most device applications and physical properties. In particular, investigation of carrier dynamics and light propagation in nanodevices with dynamic coupling to electronic state fillings, Coulomb interactions, electromechanical phenomena requires use of less computationally intensive bandstructure methods.

Low dimensional nanostructures as multiscale complex systems

We now discuss some of the main themes in detail. Low dimensional semiconductor nanostructures are multiscale complex systems that require the development of coupled mathematical models for their studies [19]. Parts of these systems (e.g., two bulk materials) are joined together at the atomistic level via interfaces to form a new structure with properties unmatched before. It is an interdisciplinary area where continuum and atomistic, deterministic and stochastic mathematical models go hand in hand. I. Prigogine, a Nobel Laureate in Chemistry, pointed out a while ago that ”complexity is no longer limited to biology or human sciences: it is invading the physical sciences as deeply rooted in the laws of nature”. Low dimensional nanostructures provide one of the most important examples of coupled complex systems in physical sciences with a fascinating range of current and potential applications.

One of the emphases of this workshop was on the development of effective (such as envelope-function) mathematical models for LDSNs, coupled to models of continuum mechanics for strain and electromechanical effects, that mathematically lead to a system of eigenvalue PDE problems. From a physical point of view, electronic bandstructure theory is an important ingredient in the understanding of optical and electronic properties of semiconductors nanostructures. One of the classic tools (since the fifties) for obtaining electronic bandstructures is the envelope-function theory also known as the $\mathbf{k}\cdot\mathbf{p}$ method discussed extensively during the workshop (e.g., [10, 20]). This method has shown its strength in modelling electronic states with remarkable accuracy qualitatively and quantitatively for bulk and quantum-confined structures capturing many of the subtle geometric details as well as microscopic and macroscopic (full nanostructure) symmetry characteristics. The fact that variations in physical fields are larger in smaller quantum-dot structures than in bulk-based semiconductors (due to larger gradients in material-composition in the former as compared to the latter) implies that nonlinear electromechanical coupling effects can play a significantly role not known in bulk applications. In addition, other nonlinear effects, such as nonlinear strain, are becoming increasingly important for such structures [18]. During the workshop, the development of mathematical models for the description of coupled electromechanical effects in low dimensional nanostructures were discussed by Barettin *et al.*, Mahapatra et al, and Lassen et al. Barettin *et al.* presented a method for obtaining accurate strain distributions in cylindrical quantum dots using Navier's equations, Maxwell's equations, and constitutive electromechanical relations derived from free-energy considerations [1]. Lassen et al [10] demonstrated the impact of using the fully coupled electromechanical equations including piezoelectric effect and spontaneous polarization as compared to the semi-coupled approach. Several insightful examples were provided in this context [10].

The governing constitutive equations in the electromechanical fields in the linear case are:

$$\sigma_{ik} = C_{iklm}\epsilon_{lm} + e_{nik}\partial_n V, \quad D_i = e_{ilm}\epsilon_{lm} - \varepsilon_{in}\partial_n V + P_i^{sp}, \tag{38.1}$$

where σ, ϵ, V, D, P_i^{sp} are the stress, strain, electric potential, electric displacement, and spontaneous polarization, C, ε, and e are the stiffness, dielectric, and piezoelectric constants, respectively. Combination with Navier's equation and the Poisson equation:

$$\partial_j \sigma_{ij} = 0, \quad \partial_i D_i = 0, \tag{38.2}$$

in addition to appropriate boundary conditions specify the complete set of model equations. Here the strain is composed of a lattice mismatch part and a part related to spatial derivatives in the displacement vector.

The strength of the model, discussed by Barettin et al, lies in the inclusion of the symmetry and the nanostructure geometry imposing Dirichlet boundary conditions on the mechanical displacement far into the (well) material embedding the quantum dot. They compared four different quantum-dot cases taking into account one or more of the three effects: lattice mismatch between dot and well material, spontaneous polarization, and piezoelectric effects. Their results showed that significant changes in the strain distributions result if any of the three effects are omitted. Hence, since strain modifies qualitatively electronic bandstructures and momentum matrix elements, it is important to always perform a coupled-field investigation when evaluating optoelectronic properties of semiconductor nanostructures, as it was emphasized by Lassen et al [10]. In effect, output results for strain and other physical fields are used as input to envelope-function methods allowing for a computationally fast and accurate determination of optoelectronic properties accounting for electromechanical coupling.

From a mathematical point of view, a system of coupled partial differential equations that needs to be solved in this context (elasticity equations coupled with the Maxwell equation, sometimes referred to as the Navier-Poisson system) leads to non-trivial mathematical difficulties related, in addition to the well-posedness issues [15], to the stability of numerical approximations [16]. The results of the solution of such systems are then used in solving an eigenvalue PDE problem which represents a serious challenge even in the case of single band models, as it was demonstrated in [17] where the issue of correct boundary conditions was addressed. New interesting quantum confinement phenomena, such as existence of critical radius in nanowire superlattices, were studied extensively with such mathematical models [28]. Higher order approximations based on such models both for the strong and weak formulations of the eigenvalue problem were also studied in detail in [11]. However, new challenges appear in attempts to generalize such analyses to the multiband models. Indeed, in the case of 8×8 Hamiltonian, such a system consists of 8 coupled equations where the verification of ellipticity conditions and the issue of spurious solutions lead to another set of non-trivial mathematical difficulties [12].

Nonlinear strain, polarization and diffused interface effects in low dimensional nanostructures were discussed by D.R. Mahapatra et al who emphasized also the importance of the size effects, as well as nonlinear effects, in the modelling of such structures [18]. Their analysis was based on a self-consistent modeling framework within which they developed a variational formulation of the Poisson-Navier-Schrödinger system.

Quantum dot arrays and coupling to other fields

In addition to the importance of electromechanical coupling and the development of associated mathematical models accounting for this coupling, as discussed above, a series of discussions were focused on the importance of other physical fields in the study of low dimensional nanostructures, such as magnetic. In addition, several novel concepts were discussed in the context of quantum dot arrays or molecules where a set of several quantum dots are considered as a coupled system.

In particular, quantum dot molecules have been discussed by D.G. Austing et al who observed striking and unexpected magnetic-field induced intra-dot level mixing and quantum superposition phenomena between two, three and four approaching single-particle states in a quantum dot.

Nonlinear phenomena and mathematical modelling of phase transformations

Coupled nonlinear thermo-mechanical effects were discussed by M. Zhou who focused on the newly identified transformations to novel pseudoelastic behaviors with recoverable strains in nanowires. He pointed out that the transformations also give rise to a coupling between the thermal and mechanical behaviors of the nanowires and offer mechanisms for developing nanocomponents with tunable responses. Problems of structural phase transformations require us to deal with strongly nonlinear effects such as hysteresis and the development of efficient numerical methodologies for such problems represents one of the major challenges in applied mathematics and mathematical modelling [14, 25]. In the context of nanostructures, M. Zhou presented first-principles calculations based on the density functional theory (DFT) and molecular dynamics (MD) simulations that yielded critical conditions for these transformations.

Hysteresis effects without plastic deformation as observed in nanoscale contact experiments were discussed by A. Lew who presented a simple model to explain the observed behavior.

Mathematics of nanocrystal growth and modelling the synthesis of nanocrystals

Nanocrystal growth is an intrinsically multiscale process. One of the examples includes growth of an epitaxial thin film which involves physics on both atomistic and continuum length scales. Indeed, diffusion of adatoms can be coarse-grained, but nucleation of new islands and breakup for existing islands are best described atomistically. These issues were discussed by R. Caflisch who described mathematical modeling, simulation methods and computational results for epitaxial growth, strain in thin films and pattern formation. His growth simulations used an island dynamics model with a level set simulation method. He pointed out that strain computations can be computationally intensive, so that effective simulation of atomistic strain effects relies on an accelerated method that incorporates algebraic multigrid and an artificial boundary condition. R. Caflisch presented simulations that combine growth and strain showing spontaneous and directed self-assembly of patterns (quantum dots and wires) on thin films.

X.B. Niu et al studied the effect of a spatially varying potential energy surface on the self-organization of nanoscale patterns during epitaxial growth. At the workshop they presented their mathematical model and the developed computational approach based on the level set method.

Modelling the synthesis and thermodynamic properties of nanocrystals remains a challenging task for mathematical modellers, scientists, and engineers. D. Chrzan et al demonstrated new results obtained with the developed model for the nucleation and growth based on both the kinetic Monte Carlo simulations of the growth process, and a complementary approach involving the integration of a set of coupled rate equations [29]. They were able to overcome a number of difficulties and shared with the participants of the workshop their experience with modelling nonlinear effects in the confined nanocrystals, including the large hysteresis in the melting point.

Developing computational methodologies for modelling properties of nanostructures

A number of important effects at the mesoscopic level were also discussed at the workshop. H. Guo summarized such effects in the context of mesoscopic spin-Hall effect and provided the audience with details of an efficient numerical technique for solving the 2D quantum scattering problem.

A computational framework for studying the effects of dislocations in semiconductor nanostructures on both electrical and optical properties was presented by H.T. Johnson et al.

A novel hierarchical multiscale model, the surface Cauchy-Born (SCB) model, was presented by H. Park et al. He presented a series of results on numerical methodologies for capturing surface stress effects on the mechanical behavior and properties of nanowires. He also demonstrated how to calculate the resonant frequencies, and thus the elastic properties of nanowires.

Mathematical models for low dimensional nanostructures in biological sciences and medicine

As mentioned above, the quickly progressing technology of low-dimensional semiconductor nanostructures requires and depends on reliable predictive theoretical methods for systematically improving, designing and understanding the electronic and optical properties of such structures. The problem complexity becomes even more pronounced if the nanostructures are combined with biomaterials to form bio-sensors. In this case, new mathematical difficulties are quick to appear. At the same time, commercial applications of nanodevices, like pH, protein, virus or DNA sensors (bio-chips) are gaining in importance in recent years. Therefore, the development of mathematical models and computational tools for their solution are becoming increasingly pressing.

In the paper and presentation by Birner *et al.* [2], realistic models of an electrolyte solution, its interaction with a semiconductor device surface, and of the semiconductor device itself were discussed in the framework of a bio-sensor device based on a silicon-on-insulator structure. Detailed simulations of protein sensors based on silicon allow demonstrations of the applicability of the model approach. The Schrödinger equation and Maxwell's equations (with the Poisson-Boltzmann equation describing ion charge distributions spatially) are coupled and solved for the electrolyte and semiconductor regions. A standard approach was used to calculate the energy levels and wavefunctions based on the one-band envelope function approximation. As aqueous electrolytes for use in bio-sensors are usually buffer solutions they resist changes in H_3O^+ and OH^- ion concentrations (and consequently the pH) upon addition of small amounts of acid or base, or upon dilution. The concentrations of the ions that are contained in the buffer depend on the pH and the dissociation constant. These were calculated using the well-known Henderson-Hasselbalch equation. In addition, the dissociation value depends on temperature and on ionic strength in a self-consistent way. When using a phosphate buffer, the concentrations of the buffer ions at a particular pH are governed by three different dissociation constant values making it extremely difficult to derive concentrations analytically. In order to circumvent this problem, Birner *et al.* followed a Schrödinger-Poisson coupled numerical scheme in an iterative way. This was one out of several examples presented at the workshop where good qualitatitive agreement were found with a multiphysics model employed a fairly simple bandstructure model. It is expected that the use of multiband electronic bandstructure models would lead to improved results and better agreement with experiments.

According to Birner *et al.*, it is necessary to solve the Schrödinger equation in regions where the quantum mechanical density is negligible or zero such as in insulators. Also wavefunction penetration into the barrier materials (e.g. at Si-SiO_2 interfaces) is fully taken into account by including a small region of the barrier material into the Schrödinger equation. Model results were compared with available experimental data and it is found that the Poisson-Boltzmann equation is able to reproduce experimental data in contrast to the widely used Debye-Hückel approximation which faces severe limitations.

One of the applications of the nanoHUB framework presented by G. Klimeck and his collaborators was on biologically active field-effect transistors, or BioFETs. They have been analyzed as potentially fast, reliable, and low-cost biosensors for a wide range of applications as they are direct, label-free, ultrasensitive, and (near) real-time operation. The development of multiscale mathematical models for planar sensor structures and for nanowire sensors requires the drift-diffusion type equations coupled with the mean-field Poisson equation and a Boltzmann model for the ions. The multiscale models are essential in this field due to the large difference in the characteristic length scales of the biosensors: the charge distribution in the biofunctionalized surface layer varies on the Angstrom length scale, the diameters of the nanowires are several nanometers, and the sensor lengths measure several micrometers. It is also important to note that the multiscale models for the electrostatic potential can be coupled to any charge transport model of the transducer. Among others, one of the mathematical challenges that need to be overcome here is the treatment of the boundary layer which cannot be modelled by including its total charge. The authors overcome these difficulties, and on several interesting examples demonstrated the importance of the dipole moment of the biofunctionalized surface layer in addition to its surface charge [7].

Non-conventional nanostructures were reviewed by Y. Zhang who focused on computational challenges for mathematical modelling of such nanostructures properties. Examples included inorganic-organic hybrid nanostructures, core-shell nanowires and others.

Quantum information, photonics, and telecommunication applications

During the last decade, In(Ga)As/GaAs quantum dots (QDs) have received considerable attention due to the predicted improvements in device performance that can be achieved because of their reduced dimensionality and novel behavior enabling development of new technologies such as single photon sources and qubits for quantum information applications. The operating wavelength of GaAs-based optoelectronic devices can be tuned within the important telecom wavelength range (1300-1550 nm) by appropriate material and geometry control of quantum-dot wetting-layer structures. A variety of experimental techniques are known including alternate layer epitaxy, use of low InAs growth rates or by capping the QDs with a thin layer of InGaAs before subsequent GaAs growth or encapsulating the QDs within an InGaAs quantum well (dots-in-a-well or DWELL systems). Clarke *et al.* [4] presented investigations of growth and optical properties of InAs/GaAs quantum dot (QD) bilayers examining the influence of strain. Optical emission-wavelength tuning between 1400 nm and 1515 nm is demonstrated experimentally. The conference included mainly theoretical presentations and methods albeit, ultimately, it is the interplay between theory and experimental methods that most effectively leads to technological improvements and understanding of the important physics involved. One of the issues that certainly needs better understanding is the possibility to control to a higher degree the composition grading and formation of nanostructure shapes and sizes. At present, the use of self assembly such as for, e.g., InGaAs-based quantum-dot structures is a critical point. Theoretical models of growth under temperature, gas inlet control dynamically exist but they are still far from being able to describe qualitatively and, in particular, quantitatively the formation of realistic quantum-dot based nanostructures under laboratory conditions. One of the major conclusions of the workshop is the importance of obtaining in future better theoretical guidance to experimentalists on the growth process and dynamics as well as for theoreticians verification of theoretical models against experimental results. The answer to these questions can open up possibilities for producing new quantum-dot structures and ultimately improved device applications. Secondly, theoretical modelling of the influence of statistical fluctuations, non-ideal nanostructure surface shape- and composition grading effects, which are always present in today's grown structures, was identified at the workshop as an important issue to tackle and address.

Modelling of optical gain as an important task in the applications of quantum dot lasers was discussed by S.L. Chuang.

Slow down of light in a highly dispersive media has gained enormous attention since reportings in the late ninetees on light propagating at 17 m/s through a vapor of ultra cold Na atoms [5]. A large research effort has been put into extending these results to semiconductor nanostructures at room temperature for applications in, e.g., telecommunication. Quantum dot-based devices are promising candidates for applications exploiting quantum coherence phenomena due to their atom-like properties and large dephasing times. For instance, an all optical buffer based on slow light in quantum dots has been proposed. Houmark *et al.* [6] investigated the impact of many-particle interactions on group-velocity slowdown through Electromagnetically Induced Transparency (EIT). At the workshop, a so-called ladder scheme in the active quantum-dot energy levels (effective using an appropriate pump-probe excitation configuration) in the steady-state was demonstrated vs. earlier attempts based on transient schemes. Moreover, the model accounts for Coulomb interaction effects leading to an increase in maximum slow-down as compared to the non-interacting case. The susceptibility was found by computing the microscopic polarizations of the material using density-matrix equations. Emphasis was put on the differences between an atomic model or one that includes many-particle interactions in the Hartree-Fock approximation. Their results also showed that the necessary pump power at which maximum slow down is obtained for EIT remains unchanged. Discussions at the workshop revealed that the use of more realistic electronic bandstructures (envelope-function, empirical, or ab-initio atomistic calculations) and resulting more accurate dipole moments for EIT is expected to be of significant importance for quantitative conclusions on slow down and pump-power requirements. Future investigations along this line and verification against measurements are thus essential.

Advances in quantum dots theory and applications for nanophotonics and quantum information devices were reviewed by Y. Arakawa. Further discussions were centered on quantum dots as promising elements to control single photons and entangled photon pairs.

Dynamic and transport phenomena in low dimensional nanostructures and devices

The development of mathematical models for nanodevice simulation brings a number of serious challenges at theoretical and numerical levels, in particular for the time-dependent models [22]. During the workshop, they were discussed by D. Vasileska et al who emphasized the dominant role of quantum effects in addressing the issue of quantum transport. Among the most commonly used in nanostructure calculations schemes are the Wigner-function approach, the Pauli master equation, and the non-equilibrium Green's functions (NEGF) known also as the Keldysh formalism. The authors showed that the key to the successful application of the NEGF formalism to the 3D quantum transport problem in semiconductor nanostructures is the numerical efficiency of the contact block reduction (CBR) method [9].

A number of new phenomena theoretically predicted with mathematical models were also discussed at the workshop. One of them is spontaneous excitation of coherent nano-particle dipole oscillations through interaction with a quantum-dot two-level system subject to population inversion [21]. Several important observations were made: dipole momentum of nano-particle leads to coherent dipole radiation, optical cavity is not necessary, the size of the dipole laser can be smaller than the optical wavelength (it is effectively a dipole nano-laser). It was demonstrated that the proposed mathematical model allows us to analyze threshold conditions and optical bistability in dipole nano-lasers.

Atomistic and molecular dynamics approaches for low dimensional nanostructures

The importance of the development of predictive modelling tools was emphasized in the presentation by H. Huang who argued that at this stage it is already feasible to model atomistic processes of pico-/nano-seconds using classical molecular dynamics and density functional theory methods. However, the structure evolution processes on the order of minutes remains a challenge once the nanorods or nanowires have variable crystalline orientations. The atomistic methodologies for the structure evolution processes, based on a lattice kinetic Monte Carlo method, were in the focus of this discussion. It was demonstrated that coupling of density functional theory ab initio, classical molecular dynamics, and kinetic Monte Carlo simulations can enable us to achieve a predictive design of nanorods synthesis [8].

A novel multiscale nonequilibrium dynamics (MS-NEMD) model was presented by S. Li who demonstrated that the developed model is capable of simulating coupled thermo-mechanical coupling at small scales, such as nanoscale heat conduction and phonon scattering by defects, e.g. dislocations.

The effect of bandstructure in the atomistic treatment of electronic transport was discussed by N. Neophytou et al who focused on such important effects as non-parabolicities and anisotropies in the electronic structure, strong coupling of bands, degenerate valley splittings due to enhanced quantum interactions, strain, material and potential variations on the nanoscale.

M. Korkusinski et al presented new results on atomistic calculations of electronic and optical properties of semiconductor nanostructures. They found the optimal atomic positions by minimizing the total elastic energy of the system using the Valence Force Field model in the first-nearest-neighbor approximation. Once the equilibrium structure of the sample is established, they proceeded to calculating the single-particle states of the electron and hole confined in the nanostructure using tight-binding approaches.

Atomistic simulation of nanosize electronic devices was discussed by L.-W. Wang who presented a method which uses atomistic pseudopotentials to calculate the electronic structures and electrical properties of the million atom devices. He explained that he uses a linear combination of bulk band (LCBB) method to solve the electron eigenstates of the system, and special formalism to occupy the eigenstates in an nonequilibrium system. He discussed the importance of quantum mechanical effects in the developed modelling framework.

Stochastic mathematical models and control

Another important issue discussed at the workshop was the development of state-of-the-art stochastic models for dynamic problems describing the behavior of low dimensional nanostructures. It is well known that some of the most peculiar features of quantum theory such as the existence of quantum superpositions and of entangled states are typically destroyed by uncontrolled and ultimately inevitable interactions with a surrounding, often incoherent environment [3]. Nevertheless, as it was pointed out during the workshop by Kyriakidis et al most of the recent theoretical analysis done in the area of low-dimensional dynamical quantum systems has been either for open Markovian systems, where the past memory of the system is neglected, or for closed unitary systems, where the dynamics are reversible [24]. He further argued that much of the research done until recently has focused on steady state phenomena where a Markovian approach can be expected to provide reasonable results. However, the transient behavior of the system carries a tremendous amount of information in the form of coherence and relaxation dynamics (examples include, but not limited to ultrafast laser pulse excitations that provide insight into the heterostructure dynamics on a femtosecond timescale). Therefore, the development of a mathematical theory which accounts for quantum dynamical behavior on the same time scale would be of great benefit not only to experiment, but also to basic understanding.

Earlier in the report, we have already mentioned the Keldysh formalism of non-equilibrium Green?s functions (NEGF) and formalisms based on NEGF, which have been very successful in the analysis of many phenomena in mesoscopic systems, including transport through quantum dots. However, as it was argued by Kyriakidis et al, NEGF is inherently a closed-system formalism where the system has Hamiltonian dynamics, and thus does not account for irreversibility arising from interactions with an unseen, unknown, or otherwise intractable environment. Promising attempts have been made to extend NEGF to open quantum system by treating, for example, the environment as a correction to the system?s self-energy, or by calculating two-time correlation functions, effectively separating the time scales into transient and steady-state regimes. Kyriakidis et al investigated the dynamics of bound particles in multilevel current-carrying quantum dots where they looked specifically in the regime of resonant tunnelling transport. Through a non-Markovian formalism under the Born approximation, they analyzed the real-time evolution of the confined particles including transport-induced decoherence and relaxation. In the case of a coherent superposition between states with different particle number, they found that a Fock-space coherence may be preserved even in the presence of tunneling into and out of the dot [24].

During the workshop it was emphasized that to create quantum devices that perform useful functions, we must be able to understand their behaviour, and have effective means to controllably manipulate it [23]. Analysis of system dynamics and the design of effective control strategies require the availability of sufficiently accurate mathematical models of the device as we discussed earlier in the report. New mathematical challenges appear in the task of intrinsic control system identification for quantum devices. The problem of experimental determination of subspace confinement becomes of primary importance in this case. It was argued during the workshop that a fundamental prerequisite for constructing a Hilbert space model for characterizing subspace confinement is knowledge of the underlying Hilbert space. This is a nontrivial problem as most systems have many degrees of freedom, and thus a potentially huge Hilbert space, but effective characterization of the system often depends on finding a low dimensional Hilbert space model that captures the essential features of the system. Schirmer et al proposed simple general strategies for full Hamiltonian identification and decoherence characterization of a controlled two-level system [23].

Physics-based mathematical models and experiments

The development of physics-based mathematical models in this field is closely connected with the state-of-the-art experimental results. In Fig. 1 we schematically presented three main areas that influence significantly the development of mathematical models for nanostructures and nanostructure-based devices. The participants of the workshop identified, in each of these three areas, the significant experimental achievements in recent years. We summarize them below:

- Growth/Nanomechanics (emerging nanoscale experiments; 3D atom precision probe method for positions and compositions; mechanical, thermal, optical and other measurements);

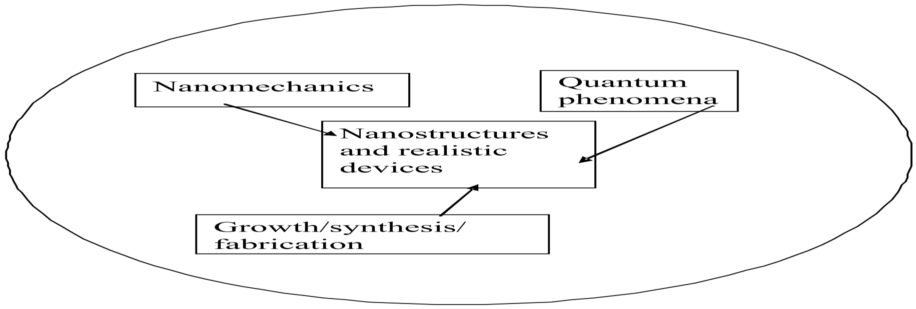

Figure 38.1: Main components to be accounted for in the development of physics-based mathematical models for low dimensional nanostructures.

- Nanodevices and Simulation (multicell junction and core-shell solar cells; entangled photon states; DNA sensitive FETs);
- Nanophysics and Quantum Phenomena (single-photon emission; single-electron spin detection; quantum spin-Hall effect).

Current state-of-the-art modeling achievements were also identified in the same three areas:

- Growth/Nanomechanics (methodologies based on kinetic Monte Carlo (kMC), accelerated MD, longer time scale simulations, multiscale, multiphysics models, atomistic/continuum, density-functional theory (DFT), level set, and others);
- Nanodevices and Simulation (50+ million atoms + Valence Force Field; time-dependent DFT; quantum MC);
- Nanophysics and Quantum Phenomena (DFT relaxation + tight-binding electronic structure; ab initio and temporal behavior; non-equilibrium Green's function, NEGF).

Finally, the feasibility of mathematical and computation methodologies in two of the areas were brought up:

- Growth/Nanomechanics (currently it is possible to simulate 1000 atoms with DFT, while realistic problems often require 1 000 000 for QD and quantum wires; kMC for cluster distribution; MD for diffusion; continuum approach for surface morphology; DFT for sticking coefficients);
- Nanodevices and Simulation (multiphysics modeling of nanostructures; multiscale techniques from atomistic level to the device level; NEGF techniques for transport).

Open problems, points of controversy, new directions, and outcomes

The complexity of present-day laboratory nanostructure growth imposes a strong need for better interaction between theoretical physicists, applied mathematicians, and experimentalists to improve technological development in the field and to advance physics-based mathematical models that would assist this development. Today, only very limited knowledge is available on the influence of growth parameters on the details of nanostructure shape and size, composition determining the important physical properties and ultimately device applications. Finally, an important aspect of this workshop was to remove barriers in communication between theoreticians using different mathematical models and methodologies for the analysis of growth, electronic bandstructures, and device characteristics. This is important for obtaining a better understanding of individual methods and how they can be combined effectively in studying complicated nanostructure phenomena. As we pointed out in the previous sections, current state-of-the-art methodologies cannot be efficiently used to design nanoscale structures and their properties. During this workshop, the study groups were formed to identify some of the key open problems in this rapidly developing interdisciplinary field. Each study group had experts in all three main areas represented at this workshop: experimental physics, theoretical physics, applied mathematics and computational science. The summary of study groups work on the identification of open problems is summarized below:

- Growth/Nanomechanics (reliable surface structure prediction beyond periodicity; reliable growth process control and simulation, including 3D nucleation, defect control; uniformity and quantity of a few nm materials);
- Nanodevices and Simulation (control of carrier injection, and photons and phonons; realistic and reliable device modeling; coherence/decoherence);
- Nanophysics and Quantum Phenomena (reduction for decoherence; quantum system identification; measurement in quantum physics; strongly correlated quantum systems; quantum transport phenomena in complex systems).

There were a number of productive discussions during the workshop that highlighted some points of controversy. For example, given that a range of coupled multiscale effects must be accounted for in the development of mathematical models for LDSNs, how to quantify the quality of approximations and limitations in applicability of main methodologies for bandstructure modelling such as $\mathbf{k} \cdot \mathbf{p}$ envelope function approximations, tight-binding, ab initio methodologies? The workshop generated a substantial interest to these nontrivial issues. It became also apparent that the practitioners who are using one of the above methodologies were not fully aware in the utility and degree of advancement of the other methods.

One of the doctoral students (D. Barettin), working on continuum models, was invited to spend six months visiting one of the plenary speakers working on atomistic models (G. Klimeck). The interest of the participants, who traditionally were working on continuum models, to atomistic first principle analyses has increased as a result of the BIRS workshop which led to several subsequent presentations and publications (e.g., [26, 27]).
The outcomes of this workshop included:

- Networks created across disciplinary borders;
- Recognition of physics and mathematics driven nanoscale modeling and addressing some of the most challenging multiscale problems in mathematical modelling of complex systems;
- Consensus-building towards real-time modelling and simulation for design and control of nanostructures and devices.

The above consensus is being built via the development of physics-based deterministic and stochastic mathematical models for studying low dimensional nanostructures along with the development of coupled atomistic-continuum methodologies for the applications of such models, ranging from growth modelling to the quantum mechanical analysis of electronic properties.

List of Participants

Arakawa, Yasuhiko (University of Tokyo)
Austing, David (National Research Council of Canada)
Barettin, Daniele (Syddansk University)
Birner, Stefan (Walter Schottky Institute/Technical University of Munich)
Brown, Gail (Materials Lab/Air Force Research Lab at Wright-Patterson AFB)
Caflisch, Russel (University of California at Los Angeles)
Chrzan, Daryl (University of California at Berkeley)
Chuang, Shun Lien (University of Illinois at Urbana-Champaign)
Clarke, Edmund (Imperial College London)
Guo, Hong (McGill University)
Houmark, Jakob (Technical University of Denmark)
Huang, Hanchen (Rensselaer Polytechnic Institute)
Jauho, Antti-Pekka (Technical University of Denmark (DTU))
Johnson, Harley (University of Illinois at Urbana-Champaign)
Klimeck, Gerhard (Purdue University)
Korkusinski, Marek (National Research Council of Canada)
Kyriakidis, Jordan (Dalhousie University)
Lassen, Benny (Syddansk University)

Lew, Adrian (Stanford University)
Lew Yan Voon, Lok (Wright State University)
Li, Shaofan (University of California, Berkeley)
Melnik, Roderick (Wilfrid Laurier University)
Neophytou, Neophytos (Purdue University)
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Park, Harold (University of Colorado)
Pedesseau, Laurent (L'INSA de Rennes - France)
Roy Mahapatra, Debiprosad (Indian Institute of Science)
Schirmer, Sonia (University of Cambridge)
Uskov, Alexander (Lebedev Physical Institute)
Vasileska, Dragica (Arizona State University)
Voss, Frands (Syddansk University)
Wang, Lin-Wang (Lawrence Berkeley National Laboratory)
Willatzen, Morten (University of Southern Denmark)
Zhang, Yong (NREL of the U.S. Department of Energy)
Zhou, Min (Georgia Tech)

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Chapter 39

Discontinuous Galerkin Methods for Partial Differential Equations (07w5506)

Nov 25 - Nov 30, 2007

Organizer(s): Bernardo Cockburn (University of Minnesota), Dominik Schoetzau (University of British Columbia), Chi-Wang Shu (Brown University)

The purpose of this meeting was to bring together researchers in a wide variety of areas working on discontinuous Galerkin (DG) methods for partial differential equations, to investigate and identify problems of current interest and to exchange ideas and viewpoints on the most recent developments of these methods. There were 33 participants, mostly from American and Canadian universities, including students and postdoctoral fellows. The program of the workshop consisted of 28 half-hour talks.

Overview of the Field

The origins of discontinuous Galerkin methods can be traced back to the seventies where they were introduced as non-standard discretization techniques for the numerical approximation of linear transport equations [1]. A remarkable adavantage of the original DG method is that the approximate solutions can be computed element-by-element when the elements are suitably ordered along the characteristic directions of the transport field. The success of DG methods for linear equations prompted several researchers to try to extend them to non-linear hyperbolic conservation laws. In the early eighties and beginning of the nineties, Cockburn and Shu introduced the Runge-Kutta discontinuous Galerkin (RKDG) methods for scalar conservation laws, see the review article [3] and the references therein. These methods are based on piecewise polynomial space discretizations, combined with total variation diminishing (TVD) explicit time-stepping algorithms. The resulting schemes have several important advantages as compared to, e.g., finite difference methods. The variational structure of DG methods greatly facilitates the handling of complicated geometries and elements of various shapes and types, as well as the treatment of boundary conditions. Moreover, DG mass matrices are block-diagonal and can be inverted at a very low computational cost, giving rise to very efficient time-stepping algorithms. Soon after their introduction, RKDG methods were extended to non-linear hyperbolic systems and to more general convection-diffusions problems.

Recent Developments

The nineties marked the start of a new era in the field of discontinuous Galerkin methods. The methods started to find their way into the main stream of computational fluid mechanics and began to be applied to a variety of problems for which they were not originally designed. Indeed, nowadays DG methods are successfully used in fields as diverse as meteorology, weather-forecasting, oceanography, gas dynamics, aero-acoustics, turbomachinery, turbulent flows, granular flows, oil recovery simulation, modeling of shallow water, transport of contaminants in porous media, viscoelastic flows, semi-conductor device simulation, incompressible fluid flow, structural mechanics, magnetohydrodynamics, and electromagnetism, among many others. The recent development of DG methods has been accompanied by a sharp increase of publications and workshops in the field. We refer the reader to [2] for the proceedings of the First International Symposium on Discontinuous Galerkin Methods, which took place in 1999 in Newport (Rhode Island), and to [4, 5] for two recent journals devoting special issues on discontinuous Galerkin methods.

Perhaps the main reason for this remarkable development is the fact that discontinuous Galerkin methods are extremely versatile and flexible; they combine elements from classical finite volume and finite element techniques. Their intrinsic stability properties make them extremely well-suited for problems where convection is dominant. Discontinuous Galerkin methods can deal robustly with partial differential equations of almost any kind, as well as with equations whose type changes within the computational domain. Therefore, they are naturally suited for multi-physics applications and for problems with highly varying material properties in complex geometries. Moreover, discontinuous Galerkin methods can easily handle irregularly refined meshes and variable approximation degrees. This property is referred to as hp-adaptivity, and is needed to resolve solution singularities efficiently and accurately.

The goal of the workshop was to discuss and identify the most relevant aspects of the current development of DG methods. One aspect of current interest is the proliferation of degrees of freedom in these methods. That is, on a fixed mesh, the number of degrees of freedom of a DG discretization might be substantially larger than the one of a standard finite element method of the same order. Hence, there is a great need for the development of efficient solvers and implementation techniques, as well as for the design of new methods that have fewer globally coupled degrees of freedom. Another aspect is the devising of DG methods for newly emerging applications in incompressible fluid flow, structural mechanics, electromagnetic wave propagation and in many other areas where convection plays a dominant role. A-posteriori estimation, error control and adaptivity are further aspects that are attracting a lot of attention. The numerical analysis of many adaptive algorithms is only in its beginnings, and many developments can be expected there over the next few years.

Talks

In the meeting, 28 half-hour lectures were given by the world's leading experts in the field of discontinuous Galerkin methods. The abstracts of the talks are listed below. (They were written by the speakers themselves.)

Antonietti, Paoloa (University of Nottingham)
Preconditioning discontinuous Galerkin approximations of elliptic problems
In recent years, much attention has been given to domain decomposition methods for linear elliptic problems that are based on a partitioning of the domain of the physical problem. Since the subdomains can be handled independently, such methods are very attractive for coarse-grain parallel computers. In this talk we shall present in a unified framework some non-overlapping additive and multiplicative Schwarz domain decomposition methods for the solution of the algebraic linear systems of equations arising from discontinuous Galerkin approximations of elliptic problems. In particular, two-level methods for both symmetric and non-symmetric DG schemes will be introduced and some interesting features, which have no analog in the conforming case, will be discussed. For symmetric DG discretizations, optimal convergence estimates will be presented, and we will show that the proposed Schwarz methods can be successfully accelerated with suitable Krylov iterative solvers. A discussion on the issue of preconditioning non-symmetric DG approximations of second order elliptic equations will be included. Numerical experiments to validate our theory and to illustrate the performance and robustness of the proposed two-level methods will be shown. This work has been carried out in collaboration with Blanca Ayuso (Departamento de Matemàticas, Universidad Autònoma de Madrid, Spain).

Ayuso, Blanca (University of Madrid)
Discontinuous Galerkin methods for advection-diffusion-reaction problems
We apply the weighted-residual approach recently introduced by Brezzi et. al., to derive discontinuous Galerkin formulations for advection-diffusion-reaction problems. We devise the basic ingredients to ensure

stability and optimal error estimates in suitable norms, and propose two new methods. The talk is based on joint work with L. Donatella Marini from the University of Pavia.

Brenner, Susanne (Louisiana State University)
Multigrid algorithms for weakly over-penalized interior penalty methods
In this talk we will introduce two weakly over-penalized interior penalty methods that are stable for any penalty parameter and are quasi-optimal in both the energy norm and the L2 norm. We will also discuss multigrid algorithms for these interior penalty methods. Both theoretical and numerical results will be presented.

Celiker, Fatih (Wayne State University)
Adaptive stabilization of discontinuous Galerkin methods for nonlinear elasticity
We introduce a novel approach to stabilizing discontinuous Galerkin methods in nonlinear elasticity problems. The new stabilization strategy possesses the distinguishing feature of allowing the size of the stabilization term to vary throughout the mesh, and to automatically adjust the local level of stabilization according to the solution sought. This stabilization strategy is hence adaptive. The proposed scheme computationally efficient and remains stable for a fairly lengthy quasistatic loading path. This is demonstrated with two and three dimensional numerical examples. We further propose a slight modification of this approach for which we are able to prove theoretical estimates for the minimal values of the stabilization parameters defining the method.

Chen, Yanlai (University of Minnesota)
An adaptive high order discontinuous Galerkin method with error control for the Hamilton-Jacobi equations
We propose and study an adaptive version of the discontinuous Galerkin method for Hamilton-Jacobi equations. It works as follows. Given the tolerance and the degree of the polynomial of the approximate solution, the adaptive algorithm finds a mesh on which the approximate solution has an L^∞-distance to the viscosity solution no bigger than the prescribed tolerance. The algorithm uses three main tools. The first is an iterative solver combining the explicit Runge-Kutta Discontinuous Galerkin method and the implicit Newton's method that enables us to solve the Hamilton-Jacobi equations efficiently. The second is a new a posteriori error estimate based on the approximate resolution of an approximate problem for the actual error. The third is a method that allows us to find a new mesh as a function of the old mesh and the ratio of the a posteriori error estimate to the tolerance. We display extensive numerical evidence that indicates that, for any given polynomial degree, the method achieves its goal with optimal complexity independently of the tolerance. This is done in the framework of one-dimensional steady-state model problems with periodic boundary conditions.

Cheng, Yingda (University of Texas)
Discontinuous Galerkin solver for Boltzmann-Poisson transients
We present results of a discontinuous Galerkin scheme applied to deterministic computations of the transients for the Boltzmann-Poisson system describing electron transport in semiconductor devices. The collisional term models optical-phonon interactions which become dominant under strong energetic conditions corresponding to nano-scale active regions under applied bias. The proposed numerical technique is a finite element method using discontinuous piecewise polynomials as basis functions on unstructured meshes. It is applied to simulate hot electron transport in bulk silicon, in a silicon n^+-n-n^+ diode and in a double gated 12nm MOSFET. Additionally, the obtained results are compared to those of a high order WENO scheme simulation.

Dawson, Clint (University of Texas)
Good and bad aspects of discontinuous Galerkin methods for some geoscience problems
We will discuss the development and implementation of DG methods for two applications arising in geosciences. The first application is the shallow water equations. These equations are a hyperbolic/parabolic system of equations which is well-suited for approximation by the Runge-Kutta DG methods of Cockburn and Shu. We will describe recent successes and lessons learned in the application of hp-adaptive DG methods for shallow water under very complex flow scenarios. The second application is to flow in porous media, in particular flow in the vadose zone described by Richards' equation. Richards' equation is a highly nonlinear parabolic equation and requires implicit time discretization, which leads to the solution of large nonlinear and linear systems of equations. For these problems, DG methods have been observed to give solutions which

are comparable to standard finite element methods, but are much more computationally intensive. Several numerical examples will be presented to illustrate our experiences in this area.

Dong, Bo (Brown University)
Optimal convergence of the original DG method for the transport-reaction equation on special meshes
We show that the approximation given by the original discontinuous Galerkin method for the transport-reaction equation in d space dimensions is optimal provided the meshes are suitably chosen: the L^2-norm of the error is of order $k+1$ when the method uses polynomials of degree k. These meshes are not necessarily conforming and do not satisfy any uniformity condition; they are only required to be made of simplexes each of which has a unique *outflow* face. We also find a new, element-by-element postprocessing of the derivative in the direction of the flow which superconverges with order $k+1$.

Gopalakrishnan, Jay (University of Floriday)
Hybridized DG methods
Discontinuous Galerkin methods have often been criticized for having too many unknowns. However, we show that DG methods can be made competitive with mixed and continuous Galerkin (CG) methods via hybridization. We achieve this by extending hybridization techniques already developed for mixed methods into a unified framework that allows hybridization of a variety of methods. This framework facilitates the discovery of new methods as well as new connections between DG, mixed and CG methods. Moreover, the unified framework allows easy coupling of these different methods, even across non-matching mesh interfaces.

Grote, Marcus (University of Basel)
Discontinuous Galerkin methods and local time stepping for second-order wave equations
The accurate and reliable simulation of wave phenomena is of fundamental importance in a wide range of engineering applications such as fiber optics, wireless communication, radar and sonar technology, and non-invasive testing. To address the wide range of difficulties involved, we consider symmetric interior penalty discontinuous Galerkin (IP-DG) methods, which easily handle elements of various types and shapes, irregular non-matching grids, and even locally varying polynomial order. Moreover, in contrast to standard (conforming) finite element methods, IP-DG methods yield an essentially diagonal mass matrix; hence, when coupled with explicit time integration, the overall numerical scheme remains truly explicit in time. To circumvent the severe stability (CFL) condition imposed on the time step by the smallest elements in the mesh, we propose local time-stepping schemes, which allow arbitrarily small time steps where small elements in the mesh are located. When combined with the symmetric IP-DG discretization, the resulting fully discrete scheme is explicit and exactly conserves a discrete energy. Starting from the standard second order “leap-frog” scheme, time integrators of arbitrary order of accuracy are derived. Numerical experiments illustrate the usefulness of these methods and validate the theory.

Guzmán, Johnny (University of Minnesota)
Superconvergent discontinuous Galerkin methods for second-order elliptic problems
We identify discontinuous Galerkin methods for second-order elliptic problems having superconvergence properties similar to those of the Raviart-Thomas and the Brezzi-Douglas-Marini mixed methods. These methods use polynomials of degree k for both the potential as well as the flux. We show that the approximate flux converges with the optimal order of $k+1$, and that the approximate potential and its numerical trace superconverge, to suitably chosen projections of the potential, with order $k+2$. We also apply element-by-element postprocessings of the approximate solution to obtain new approximations of the flux and the potential. The new approximate flux is proven to have normal components continuous across inter-element boundaries, to convergence with order $k+1$, and to have a divergence converging in with order $k+1$. The new approximate potential is proven to converge with order $k+2$. This is joint work with Bernardo Cockburn and Haiying Wang.

Hesthaven, Jan (Brown University)
Nodal DG-FEM for the modeling of free surface flows using high-order Boussinesq approximations
We shall discuss the modeling of free surface flows and fluid-structure interactions using high-order Boussinesq approximations. These sets of equations are characterized by being purely dispersive and strongly non-linear, with additional complications introduced by high-order spatial derivatives. We shall discuss the

key elements of the formulation and some of the properties of the Boussinesq system. These properties shall be used to argue why DG-FEM may be a suitable approach for the solution of these equations. We shall continue to develop the basic elements required for solving this system, discussing a number of subtleties and addressing practical concerns of performance and efficient solvers. The computational approach will be extensively validated with both benchmark tests and experimental data. This is work done in collaboration with A.P. Engsig-Karup (DTU, Denmark), P. Madsen (DTU, Denmark), H. Bingham (DTU, Denmark) and T. Warburton (Rice).

Kanschat, Guido (Texas A&M University)
Convergent adaptive algorithms for the interior penalty method
We discuss an adaptive strategy producing a sequence of meshes with a guaranteed error reduction in each step. Mesh refinement is based on a bulk criterion employing an energy norm a posteriori error estimate. The convergence proof relies on a modified discrete local efficiency. Numerical experiments show the feasibility of our approach.

Li, Fengyan (Rensselaer Polytechnic Institute)
A second order DGM based fast sweeping method for eikonal equations
The original fast sweeping method for solving static Hamilton-Jacobi equations is in the finite difference framework. By incorporating the causality of the equation into the discretization and combining the Gauss-Seidel iteration with alternating sweeping orderings, the method defines an efficient solver for these nonlinear equations with linear computational complexity. That is, the number of iterations needed for the error to be reduced to certain threshold is independent of the number of unknowns. On the other hand, this finite difference based fast sweeping method is only first order accurate. In this talk, I will discuss our recent progress in developing a second order fast sweeping method based on the discontinuous Galerkin discretization for an important family of static Hamilton-Jacobi equations - Eikonal equation. The main new components in our algorithm are the properly chosen numerical Hamiltonian in the DG formulation and the procedure to consistently enforce the causality. Numerical examples will be presented to demonstrate the performance of the proposed algorithm.

Peraire, Jaume (MIT)
A discontinuous Galerkin formulation for Lagrangian dynamic analysis
Dynamic analyses in solid mechanics are typically carried out by integrating the second order system expressing the balance of momentum. In this work, we follow a different approach. The non-linear governing equations are first cast as a system of first order conservation laws for momentum (and energy, if required by the constitutive law). In addition, a first order time evolution equation is also written for the deformation gradient tensor, $\mathbf{F}$. This results in a coupled system of 6 first order equations in 2 dimensions and 12 first order equations in 3 dimensions. The resulting system is found to possess involutions for the form $\nabla \times \mathbf{F} = \mathbf{0}$, which must be accounted for in the solution process in order to preserve stability. We develop an approximate Riemman solver and discretize the resulting equations using a high order discontinuous Galerkin method and integrate in time using an explicit Runge-Kutta algorithm. The method preserves momentum exactly and numerical results show excellent energy conservation properties for long time integrations. It is well known that for non-dissipative materials the solutions may develop shock waves which need to be handled numerically by the explicit addition of artificial viscosity. Shock capturing techniques developed in the context of non-linear conservation laws are then readily applicable. Unlike other formulations, the method is purely Lagrangian and therefore is formulated in the reference configuration. We also consider an alternative formulation in which the deformation gradient tensor is eliminated form the equations at the expense of introducing second order spatial derivatives. These are treated in the discretization using the CDG method. This formulation has a significantly reduced number of unknowns (4 in 2D and 6 in 3D) but seems to be less adequate to deal with strong discontinuities. This is work done in collaboration with P.-O. Persson (MIT) and J. Bonet (Swansea).

Persson, Per-Olof (MIT)
Preconditioning of Newton-GMRES solvers for discontinuous Galerkin problems
A fundamental problem with Discontinuous Galerkin methods is their high computational and storage cost. This is partly because they require more degrees of freedom than other methods, but mainly due to the wide nodal stencils which result in very large Jacobian matrices in implicit solvers. In this work we address the high cost of solving the corresponding linear systems of equations and propose a new preconditioner for implicit

solution of stationary or time dependent Discontinuous Galerkin problems. The viscous terms are discretized using the Compact Discontinuous Galerkin (CDG) method, but the results should be representative for other schemes as well. We consider several existing preconditioners such as block-Jacobi and Gauss-Seidel combined with multi-level schemes which have been developed and tested for specific applications. While our results are consistent with the claims reported, we find that these preconditioners lack robustness when used in more challenging situations involving low Mach numbers, stretched grids or high Reynolds number turbulent flows. We propose a preconditioner based on a coarse scale correction with post-smoothing based on a block incomplete LU factorization with zero fill-in (ILU0) of the Jacobian matrix. Our block-Minimum Discarded Fill algorithm numbers the elements such that the error in the ILU0 factorization is minimized in a greedy fashion, a step which turns out to be critical for convection dominated problems. While little can be said in the way of theoretical results, the proposed preconditioner is shown to perform remarkably well for a broad range of representative test problems. These include compressible flows ranging from very low Reynolds numbers to fully turbulent flows using the Reynolds Averaged Navier Stokes equations discretized on highly stretched grids. This is joint work with J. Peraire (MIT).

Rivière, Béatrice (University of Pittsburgh)
Coupling of surface and subsurface flows
The study of surface/subsurface interaction is important in the environmental problem of groundwater contamination. A mathematical weak formulation of the coupled Darcy flow with Navier-Stokes flow is presented. A numerical approach coupling discontinuous Galerkin and continuous finite element methods is analyzed. Extensions to time-dependent problem are given.

Ryan, Jennifer K. (Virginia Tech)
Local Post-processing for discontinuous Galerkin methods: Mesh, derivatives, and applications
In this presentation an overview of aspects of post-processing for discontinuous Galerkin methods will be given. Specifically, we will examine extensions to non-uniform meshes, derivative calculation, and applications to streamlines. Improving the accuracy in the solution and its' derivatives is important for many scientific applications in such areas as fluid mechanics and chemistry. The specific technique that we will examine was shown to improve the order of accuracy from k+1 of the DG approximation to 2k+1 for the post-processed solution over a uniform mesh where k is the highest degree polynomial used in the approximation. We will explore two techniques for extending the applications to smoothly varying and non-uniform mesh structures. Additionally, we will focus on extension of the existing kernel to improve the accuracy in the derivatives of the numerical solution. Lastly, we will present an application that uses one-dimensional filtering to aid in calculation of a streamline, regardless of the dimensionality of the solution.

Sherwin, Spencer J. (Imperial College London)
DG Methods using spectral/hp element discretisations: Balancing utility and efficiency
Given the growing interest in discontinuous Galerkin methods from both the academic and university sectors, it is of interest to ask why and when it is useful to apply the DG formulation in the context of high-order spectral/hp element discretisations. The spectral/hp element discretisation methodology originated from the confluence of two family of high order finite element methods: p-type finite elements and spectral element methods. Mimicking the p-type finite element techniques, spectral/hp elements employ hierarchical polynomial functions as trial and test functions as opposed to nodal Lagrange polynomials that are traditionally used in spectral element methods. A potential draw back of hierarchical expansions is that they lead to non-diagonal mass matrix systems which can be costly to invert in hyperbolic problem discretised explicitly in time. The discontinuous Galerkin method provided a significant advantage for this problem over the standard continuous Galerkin formulation particularly when using upwind fluxes which advantageously penalize the dispersive high frequencies. When we consider, however, parabolic or elliptic problems, the advantages of the DG formulation are not so apparent. For instance, the DG formulation is arguable more complicated to implement since it encapsulates many (but not all) of the continuous Galerkin components. DG formulations typically lead to a greater number of degrees of freedom compared to classical Galerkin formulations, and do not necessarily provide significant improvement in accuracy or compactness, at least for problems with constant coefficients. However, the ability to adapt a DG formulation and the absence of a direct stiffness (or global assembly) process can lead to greater flexibility/adaptivity. In addition, if the solution is required to exist in a discontinuous space then there may not be many obvious alternative. In summary if one already has a continuous Galerkin implementation the benefits of converting to a DG formulation are questionable

for many standard problems, especially when using low order polynomial spatial approximations. In this presentation we will explore the above competing arguments. Finally, we will also outline some of the potential benefits of the recently proposed LDG-Hybrid approach where the issue of additional degrees of freedom have been addressed and which may offer additional advantages in massively parallel implementations. This is joint work with Robert M. Kirby (University of Utah).

Shu, Chi-Wang (Brown University)
L^2-Stability analysis of the central discontinuous Galerkin method and a comparison between the central and regular discontinuous Galerkin methods
We give stability analysis and error estimates for the recently introduced central discontinuous Galerkin method when applied to linear hyperbolic equations. A comparison between the central discontinuous Galerkin method and the regular discontinuous Galerkin method in this context is also made. Numerical experiments are provided to validate the quantitative conclusions from the analysis. This is joint work with Yingjie Liu, Eitan Tadmor and Mengping Zhang.

Van der Vegt, Jaap (University of Twente)
Discontinuous Galerkin finite element methods for hyperbolic nonconservative partial differential equations
In this presentation a new space-time discontinuous Galerkin finite element (DGFEM) formulation will be discussed for partial differential equations containing nonconservative products, such as occur in dispersed multiphase flow equations. Standard DGFEM formulations cannot be applied to nonconservative partial differential equations. We therefore introduce the theory of weak solutions for nonconservative products into the DGFEM formulation leading to the new question how to define the path connecting left and right states across a discontinuity. The effect of different paths on the numerical solution is investigated and found to be small. We also introduce a new numerical flux that is able to deal with nonconservative products. The scheme is applied to two different systems of partial differential equations. First, we consider the shallow water equations, where topography leads to nonconservative products. Second, a simplification of a depth-averaged two-phase flow model is discussed. This model contains more intrinsic nonconservative products. This is joint work Sander Rhebergen and Onno Bokhove.

Wang, Wei (Brown University)
The discontinuous Galerkin method for the multiscale modeling of dynamics of crystalline solids
We present a multiscale model for numerical simulation of dynamics of crystalline solids. The method couples nonlinear elastodynamics as the continuum description and molecular dynamics as another component at the atomic scale. The governing equations on the macroscale are solved by the discontinuous Galerkin method, which is built up with an appropriate local curl-free space to produce coherent displacement field. The constitutive data are based on the underlying atomistic model: it is either calibrated prior to the computation or obtained from molecular dynamics as the computation proceeds. The decision to use either the former or the latter is made locally for each cell based on suitable criteria. This is joint work with Xiantao Li (Penn State) and Chi-Wang Shu (Brown University).

Warburton, Timothy (Rice University)
Advances in wave propagation with the discontinuous Galerkin method
A range of important features relating to the practical application of discontinuous Galerkin method for wave propagation will be discussed. Recent investigations of the spectral properties of the discrete discontinuous Galerkin operators have revealed important connections with their continuous Galerkin counter parts. Theoretical and numerical results will be shown which demonstrate the correct asymptotic behavior of these methods and controls spurious solutions under mild assumptions. Given the suitability of DG for solving Maxwell's equations and their ability to propagate waves over long distance, it is natural to seek effective boundary treatments for artificial radiation boundary conditions. A new family of far field boundary conditions will be introduced which gracefully transmit propagating and evanescent components out of the domain. These conditions are specifically formulated with DG discretizations in mind, however they are also relevant for a range of numerical methods. There is an Achilles heel to high order discontinuous Galerkin methods when applied to conservation laws. The methods are typically constructed with polynomial field representations and unfortunately these suffer from excess maximum gradients near the edges of elements. I will describe a simple filtering process that allows us to reduce these anomalous gradients and provably yield a dramatic increase in the maximum allowable time step. Finally, I will discuss progress in using a posteriori

error estimates for mesh adaptivity and demonstrate guaranteed error reduction on refinement for some model (static) problems. These results indicate a need for very fine local refinement of meshes to accurately capture solution singularities. I will show a very simple approach for local time stepping with discontinuous Galerkin methods in order to practically use such meshes in time-domain computations.

Wheeler, Mary (University of Texas)
Coupling discontinuous Galerkin and mixed finite element discretizations using mortar elements
Discontinuous Galerkin and mixed finite element (MFE) methods are two popular methods that possess local mass conservation. In this paper we investigate DG-DG and DG-MFE domain decomposition couplings using mortar finite elements to impose weak continuity of fluxes and pressures on the interface. The subdomain grids need not match and the mortar grid may be much coarser, giving a two-scale method. Convergence results in terms of the fine subdomain scale h and the coarse mortar scale H are established for both types of couplings. In addition, a non-overlapping parallel domain decomposition algorithm is developed, which reduces the coupled system to an interface mortar problem. The properties of the interface operator are analyzed. Computational results are presented.

Wihler, Thomas P. (McGill University)
A-posteriori error estimation for hp-DG time-stepping for parabolic PDEs
We will present an hp-version a posteriori error analysis for the time discretization of parabolic problems by the discontinuous Galerkin time-stepping. The resulting error estimators are fully explicit with respect to the local time steps and approximation orders. Their performance within an hp-adaptive time stepping procedure shall be illustrated with a series of numerical experiments. This talk comprises joint work with Dominik Schötzau (University of British Columbia).

Xu, Yan (University of Science and Technology of China)
A local discontinuous Galerkin method for the Camassa-Holm equation
In this paper, we develop, analyze and test a local discontinuous Galerkin (LDG) method for solving the Camassa-Holm equation which contains nonlinear high order derivatives. The LDG method has the flexibility for arbitrary h and p adaptivity. We prove the L^2 stability for general solutions and give a detailed error estimate for smooth solutions, and provide numerical simulation results for different types of solutions of the nonlinear Camassa-Holm equation to illustrate the accuracy and capability of the LDG method. This is joint work with Chi-Wang Shu (Brown University).

Zhang, Yongtao (University of Notre Dame)
Application of a discontinuous Galerkin finite element method to reaction-diffusion systems in developmental biology
Nonlinear reaction-diffusion systems which arise from mathematical modeling in developmental biology are usually highly stiff in both diffusion and reaction terms, and they are built on high dimensional complex geometrical domains due to the complex shape of embryos. Computational challenges come from both the temporal direction and the spatial direction. The stiffness of these reaction-diffusion equations demands efficient temporal numerical discretization. Complex geometrical domain can be handled by finite element spatial discretization. Based on the recent work by Cheng and Shu, we combined the DG scheme with Strang's operator splitting and the Crank-Nicholson discretization to solve the nonlinear reaction-diffusion systems, and overcome the difficulties from high stiffness of the system and the complexity of the geometrical domain. Directly solving a coupled nonlinear system like the standard implicit schemes was avoided. We applied the method to various reaction-diffusion models in developmental biology, including a system from the skeletal pattern formation in developing chick limb, to show the accuracy and efficiency of the method. This is a joint work with Jianfeng Zhu, Mark Alber and Stuart A. Newman.

Zhang, Zhimin (Wayne State University)
Discontinuous Galerkin methods for singularly perturbed problems
We study DG methods for convection diffusion equations. When the diffusion coefficient ϵ is small, the underling equation is singularly perturbed. Compared with continuous Galerkin methods, i.e., the traditional finite element methods, DG methods have the advantage to avoid non-physical oscillations. Our main concern is the convergence independent of the singular perturbation parameter ϵ. Furthermore, we shall apply recovery techniques to the DG methods by identifying some superconvergence points and using them to reconstruct local solutions in achieving higher order convergence on the whole domain or some sub-domains. Both the

one-dimensional and two-dimensional model problems are discussed. Numerical examples will be presented. This is joint work with Huiqing Zhu (Wayne State).

Outcome of the Meeting

In the meeting, the most pressing and relevant issues of the current development of discontinuous Galerkin methods were discussed and identified: the devising of new methods with fewer degrees of freedom, the devising of new methods for fluid mechanics, solid mechanics, electromagnetism, and other applications of engineering practise, the analysis of new adaptive algorithms, and the development of new preconditioning and solution techniques.

Since the workshop concentrated on discontinuous Galerkin methods, it was quite efficient in terms of exchange and presentation of ideas. On the other hand, the invited researchers represented quite different directions of research so that quite different aspects and viewpoints on discontinuous Galerkin methods were presented. Our mix of junior and senior researchers created an atmosphere that was very inspiring and stimulating for all the participants. We received many positive comments from the participants about the quality of the talks, the facilities and how the workshop was run. As a result of this, Chi-Wang Shu announced the organization of a special issue on DG methods in the Journal of Scientific Computing, with contributions from the researchers invited to the workshop.

Finally, we would like to thank the staff at BIRS for their help and hard work to make this a very enjoyable workshop.

List of Participants

Antonietti, Paola F (University of Notthingham)
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Guzman, Johnny (University of Minnesota)
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Chapter 40

First Nations Math Education I (07w5504)

Dec 2 - Dec 4, 2007

Organizer(s): Melania Alvarez-Adem (Pacific Institute for the Mathematical Sciences), Genevieve Fox (First Nations Adult and Higher Education Consortium), Sharon Friesen (University of Calgary), Joanne Nakonechny (Director Science Centre for Learning and Teaching)

Workshop

Mathematicians and math educators from Western together with First Nations elders, teachers and bands representatives have met twice at the Banff International Research Station; the first time in June 2006 and the second time in December 2007. The purpose of these meeting was set in motion a variety of initiatives to move forward in promoting mathematical opportunities for Aboriginal and First nation students.

The workshop was based on the impression that First Nations/Aboriginal student participation and success in school math programs is limited. This impression was readily confirmed by data presented. The performance of Aboriginal students in BC for the last five years has been significantly lower than the performance of non-Aboriginal students :

As early as grade 4, Aboriginal students lag behind their non-Aboriginal classmates by about 20 percent in their performance on the Foundation Skills Assessment in numeracy.

By grade 10, the gap widens and only 47 percent of Aboriginal students fulfill the expectations in numeracy, compared to 77 percent of non-Aboriginal students.

Over the last seven years, only 5-7 percent of Aboriginal students have written and passed the Principles of Mathematics12 provincial exam, compared to 25-27 percent of non-Aboriginal students.

In the same period, 38 percent of Aboriginal students completed their grade 12, and graduated from secondary school, while 77 precent of non-Aboriginal students graduated. We find the same pattern on Aboriginal students performance all over Canada, however there are not exact figures since the test data from different provinces are not available.

It was agreed by participants that successful achievement in math programs is critical for Aboriginal students if the outcome of cultural, political and economic equity for Aboriginal peoples is ever to be realized. Several participants in the workshop described barriers to success for Aboriginal students and identified the shortcomings of current approaches. Most prominent among these were: the cultural and social dissonance between school and one's Aboriginal society; the inhospitable nature of public education systems for Aboriginal students in that their history is ignored and their potential unrecognized; the absence of math programs that lead to success, for all students; and the lack of teachers trained to successfully teach math. These factors, when considered together, constitute insurmountable barriers to success in math for Aboriginal students.

The workshop presenters provided examples of inspiring initiatives that are overcoming one or more of the above barriers. Other experts shared information about the powerful and effective traditional mathematical knowledge of First Nations peoples, which should become part of every teacher's lexicon when presenting mathematics.

In our discussion we found that one of the main factors that prevent aboriginal students to have access to a good math education is teacher's preparation. All teachers of aboriginal peoples in Canada need themselves to be properly prepared to teach mathematics, and to teach it effectively within the culture of their students. Teachers must have the necessary knowledge of mathematics in order to be able to teach effectively, and to increase their expectations about aboriginal students. First Nations students can do mathematics, and they must be given the confidence to become successful in this area.

It was very useful to have a collection of the people who have actually worked on these problems, and experienced those challenges first-hand, share those experiences with all of us and engage in a dialog about solutions. Mathematicians, Elders, Math Educators and teachers were involved in this workshop.

Although much work remains to be done in this arena, there are models for addressing the challenges that have achieved some success, both in Canada and the U.S., and having some models that have worked available for presentation, analysis, and development.

What are the outcomes you would like to see with respect to math education for First Nations?

The ultimate goal would be to have all First Nations people have access to quality mathematics education, and be well-prepared to take advantage of it. There are a number of pieces to this puzzle that need to be addressed. Here is a partial list, certainly quite incomplete.

- All teachers of aboriginal peoples in Canada need themselves to be properly prepared to teach mathematics, and to teach it effectively within the cultures of their students.

- The mathematics curricula in First Nations schools must be up to the standards of curricula in other schools, and provide students with a realistic chance of being prepared to continue on to college if they so choose (and they should be strongly encouraged to so choose). This does not necessarily mean having, say, calculus available by the end of the twelfth grade, but students going on to college should be within one precalculus course of being ready to engage calculus.

- It is important for students to understand that mathematics is an important element of their own living cultures, and not something that is solely white peoples knowledge, to put it bluntly.

- It is also important for students to see mathematics as something that can be interesting and rewarding, rather than as a dry subject whose only value is in its utility. The latter is an approach that has too often been taken elsewhere, for example in the mathematical education of American Indians in the U.S., and has not created very many enthusiastic students of mathematics.

How do you think we can get there?

That, of course, is the tough question. Though volumes could be written about different pieces of this puzzle and possible solutions, here are at least some items that could be addressed by PIMS, colleges and universities throughout Canada, and other institutions in the nation.

- Changing the attitudes of both teachers and students, was agreed by the participants, was a good way to start. Teachers could be the harbingers of academic and cultural change, and we as a community should support them. One way to do it is by empowering teachers in providing them with the necessary knowledge of mathematics in order to be able to teach effectively. Teachers should also have high expectations for their aboriginal students, and a good way for this to happen is by learning more about First Nations culture and traditional ways of knowing. First Nations students can do mathematics, and they must be given the confidence to become successful in this area, a good teacher can give them the confidence and the skills.

- Teachers, mathematicians and elders should come together to create a workshop program for improvement in math education among aboriginals, where the standards of mathematics learning will be high, and the cultural context will be acknowledge. The powerful and effective traditional mathematical knowledge of First Nations peoples, should become part of every teacher's lexicon when presenting math. It is important for students to understand that mathematics is an important element of their own living cultures, and not something that is solely white peoples knowledge, to put it bluntly. Teachers need to teach math in the cultural context of the students; recognize the historical and practical role of math in the traditional and current lives of First Nations/Aboriginal people and introduce the rich history and its current significance in the field of math. Elders should be invited to talk to teachers and mathematicians about traditional ways of knowing, and how mathematics is and was part of their traditional culture. The goal is that teachers will have a greater appreciation of the aboriginal cultures and this will be reflected in their teaching. Mathematicians on the other hand, can learn from elders different approaches to mathematical thought. In addition, mathematicians could help Elders by making them aware of the mathematics in their traditional knowledge.

- Teachers of aboriginal peoples in Canada need themselves to be properly prepared to teach mathematics, and to teach it effectively within the cultures of their students. We should provide opportunities for teachers of aboriginal schools to develop their math knowledge and math teaching skills. Mathematicians and math educators should find ways to provide math workshops for teachers, in areas where we find that many teachers have difficulties understanding a concept and delivering it in class.

- In-service teacher education programs should be designed that can deliver training to teachers in remote areas, and not require them to travel great distances to central locations. Creative distance learning models could be examined for their potential in this area.

- Where culturally relevant mathematics materials exist (e.g., those designed by Jerry Lipka and his group at the University of Alaska), they should be made widely available to teachers of First Nations students, and workshops in their use should be conducted. These could be designed to be delivered using distance education models; see the point immediately preceding this one.

- A course could be designed for teachers on the history of First Nations education, particularly its mathematical components. A similar course, on the history of American Indian education in the U.S., has proved to be a popular and interesting portion of the education of American Indian teacher aides in the ENACT program at Southwestern Indian Polytechnic Institute in Albuquerque, New Mexico.

- More Math Circles types of activities could be designed for First Nations students, with the goal of introducing the students to truly challenging but interesting mathematics. College and university faculty should be instrumental in the design and delivery of these activities, and they should be available to students throughout the regular academic year, ideally on a weekly basis.

The ultimate goal would be to provide all First Nations people with access to quality mathematics education, and be well-prepared to take advantage of it. Knowing the math is key! First Nations people must be given the opportunity to obtain a solid foundation in mathematics and science. They must be given the opportunity to participate fully and equitably in a world now increasingly dependent on technology.

By working together, success is attainable. It is critical that those involved in the circle of education; primary, elementary, secondary, post secondary coordinate their efforts and create a smooth transition from one level to the next, and the only way this can happen is by communicating and learning from each other.

The workshops at Banff set in motions some of the initiatives we described above. Meeting between elders, mathematicians, teachers and math educators have been happening resulting in the development of materials that bring together mathematical and traditional knowledge.

List of Participants

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Chapter 41

Minimal submanifolds and related problems (07w5059)

Dec 09 - Dec 14, 2007

Organizer(s): Jingyi Chen (University of British Columbia), Ailana Fraser (University of British Columbia), Richard Schoen (Stanford University), Yu Yuan (University of Washington)

Minimal Surfaces and General Relativity

The portions of Mathematical Relativity which were covered by the workshop include the study of the constraint equations, especially the study of black hole solutions. In the initial value formulation of the Einstein equations, the initial data are specified on a three dimensional manifold, and the data consists of a Riemannian metric (initial position) and a symmetric (0,2) tensor (which will be the second fundamental form of the spacelike hypersurface in the evolved spacetime) which plays the role of the initial velocity of the gravitational field. The Einstein equations impose an underdetermined set of nonlinear partial differential equations on the initial data called the constraint equations. This set of equations is entirely geometric in nature and arises from the Gauss and Codazzi equations for a spacelike hypersurface in a spacetime satisfying the Einstein equations. The content of the initial value problem is that any initial data set which satisfies the constraint equations evolves under the hyperbolic equations of motion to a local solution of the Einstein equations. Of course there are many unsolved questions concerning the long time behavior, but these do not fall under the topic of the workshop.

The minimal surface and mean curvature theory enters most directly in the Riemannian geometry of the constraint equations. The notion of trapped surface is defined in terms of mean curvature. When one has a trapped surface in the initial data it follows from theorems of Penrose and Hawking that the resulting spacetime will be singular, so such data is referred to as black hole initial data. It turns out that in an asymptotically flat data set which contains a trapped surface, there is always an outermost trapped surface which is often called an apparent horizon (or marginally outer trapped surface). Such a surface is a stable minimal hypersurface in the special case of initial data with zero initial velocity (often called the Riemannian case). The lecture of Galloway considered the question of the possible topologies which can occur for apparent horizons. He described a proof of the theorem (see [7], [8]) that in any dimension an apparent horizon is Yamabe positive in the sense that the induced metric can be conformally deformed to a metric of positive scalar curvature. This theorem generalizes a theorem of Hawking to the higher dimensional case. In three dimensions Hawking's theorem is a key step in the proof of the classical black hole uniqueness theorems. Galloway also described the recent work of Andersson/Metzger [1] and Eichmair [6] which solve the existence problem for the apparent horizon equation. An important question in this connection is whether the singularities which are known to arise in volume minimizing hypersurfaces of dimension 7 or more can arise generically in Einstein initial data.

Another important theorem which has been partially proven over the past decade is the Penrose inequality. This is a remarkable extension of the Positive Mass Theorem which provides a sharp lower bound for the mass when black holes are present. More precisely, this lower bound is given in terms of the area of an apparent horizon, and equality is achieved only for Schwarzschild metrics. The Riemannian Penrose inequality was first proven in three dimensions in 1997 by G. Huisken and T. Ilmanen [9] for the case of a single black hole. In 1999, H. Bray [2] extended this result to the general case of multiple black holes using a different technique. In his talk Dan Lee described his recent extension (joint with H. Bray [3]) of the Riemannian Penrose inequality to higher dimensions; precisely the inequality has been extended up to dimension less than 8. An interesting feature of the proofs of the Penrose inequality is that both proofs use weak solutions of evolution equations which are closely tied to the theory of volume minimizing hypersurfaces. The Huisken/Ilmanen proof which works only in the three dimensional case for a connected apparent horizon uses the $1/H$-flow while the Bray proof (which Bray and Lee have extended to higher dimensions) works for apparent horizons with multiple connected components, and uses a novel conformal flow of metric. The major outstanding problem in the area is the general Penrose inequality; that is, the corresponding inequality for arbitrary asymptotically flat initial data sets. The Riemannian Penrose inequality in dimensions 8 or more is also open and the difficulty is related to the possibility of singularities in volume minimizing hypersurfaces which may occur in this case.

The talk by Corvino described his work (see [4], [5]) on constructions of new solutions of the constraint equations which can be obtained by localized gluing methods; that is, gluing methods which produce smooth solutions which agree with the original solutions outside the gluing region. Such constructions are possible for the constraint equations because of its particular underdetermined structure. Corvino showed that any solution of the vacuum constraint equations with appropriate asymptotics can be deformed outside an arbitrarily large ball to a new solution which is identical with a Kerr (Schwarzschild in the Riemannian case) solution. Corvino also described his recent proof that the center of mass defined by Huisken and Yau [10] agrees with that defined by Corvino [4]. The Huisken/Yau definition is associated with a special foliation of the exterior region of the initial data by constant mean curvature spheres.

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Partial Differential Equations

Micah Warren from University of Washington spoke on “A priori estimates for special Lagrangian equations”.

Abstract. We discuss recent a priori interior Hessian estimates for solutions of the special Lagrangian equation, when the equation has phase at least a certain value, or when the solution is convex. These equations include the sigma-2 equation in dimension three. The gradient graph of any solution is a minimizing Lagrangian surface. While Heinz showed in the 1950's that similar estimates hold for the sigma-2 (Monge-Ampere) equation in dimension two, Pogorelov showed that such estimates cannot hold for the sigma-3 (Monge-Ampere) equation in dimension three. This is joint work with Yu Yuan, partly also with Jingyi Chen.

The fully nonlinear special Lagrangian equation

$$F\left(D^2u\right)=\sum_{i=1}^{n}\arctan\lambda_i=\Theta \tag{41.0.1}$$

where λ_i are the eigenvalues of the Hessian D^2u, arises from the special Lagrangian geometry of Lawson and Harvey. The gradient graph $(x, Du\,(x))$ of the potential u is a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$. The Lagrangian graph is called special when the phase, which is the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$, is constant Θ; that is, u satisfies equation (41.0.1). A special Lagrangian graph is a volume minimizing minimal submanifold in $\mathbb{R}^{2n}$.

In the 1950's, Heinz derived a Hessian bound for the two dimensional Monge-Ampère type equation, including (41.0.1) with $n = 2$. In the 1970's Pogorelov constructed irregular solutions to $\sigma_3 \left(D^2u\right) \det(D^2u) = 1$ in dimension three, which were generalized to σ_k equations with $k \geq 3$ by Urbas. Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k equation with $k \geq 2$ were obtained by Pogorelov and Chou-Wang respectively. Pointwise Hessian estimates in terms of certain integrals of the Hessian for σ_k equations and for special Lagrangian equation (1.1) with $n = 3, \Theta = \pi$ were produced by Urbas and by Bao-Chen, respectively. Recently, for (41.0.1) Hessian estimates have been obtained: for convex solutions with a certain smallness assumption on the height in [4]; for a sharper bound when $n = 2$ in [5]; when $n = 3$ and $|\Theta| \geq \pi/2$, including the equation $\sigma_2 \left(D^2u\right) = 1$ in dimension three, in [6], [7]. More recently, Hessian estimates for general convex solutions have been obtained in [1].

Open problems:

a) Whether one has Hessian control over the solutions to the special Lagrangian equation (41.0.1) with general phases in dimension three and higher, including $\triangle u = \det D^2u$ corresponding to $\Theta = 0$ and $n = 3$?

b) Derive a Hessian bound for the solutions to the quadratic Hessian equation $\sigma_2 \left(D^2u\right) = \lambda_1\lambda_2 + \cdots = 1$ in dimension four and higher.

Joel Spruck from Johns Hopkins University spoke on "A half-space theorem for complete embedded cmc 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ "

Abstract. The famous half-space theorem of Hoffman-Meeks says that a properly immersed minimal surface in $\mathbb{R}^3$ that is contained in a half-space must be a plane. We improve (in joint work with L. Hauswirth and H. Rosenberg) an analogous result for a complete properly embedded cmc $1/2$ surface in $\mathbb{H}^2 \times \mathbb{R}$ (possibly with compact boundary).

Theorem 1. Let Σ be a complete properly embedded constant mean curvature $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$. Suppose Σ is asymptotic to a horocylinder C, and on one side of C. If the mean curvature vector of Σ has the same direction as that of C at points of Σ converging to C, then Σ is equal to C (or a subset of C if $\partial\Sigma \neq \emptyset$).

A strong motivation for the half space theorem is that it is used to prove the following result.

Theorem 2. Let Σ be a complete immersed constant mean curvature $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$. If Σ is transverse to the vertical Killing field $Z = \frac{\partial}{\partial t}$. Then Σ is an entire vertical graph over $\mathbb{H}^2$.

Such entire vertical graphs are plentiful. In fact using Theorem 2 and a construction of Fernandez-Mira, we have

Theorem 3. For each quadratic holomorphic differential on $\mathbb{C}$ or the unit disk, one associates an entire $H = 1/2$ graph.

The proof of Theorem 1 is based on the study of "horizontal graphs" over horocylinders, which satisfy the strange looking equation

$$(g^2 + g_t^2)g_{xx} - 2g_xg_tg_{xt} + (1 + g_x^2)g_{tt} = -g(1 + g_x^2) + \frac{W^3}{g^2} \qquad (*)$$

Theorem 3 follows from the existence of catenoid-like solutions as in the proof of the Hoffman-Meeks half-space theorem.

Theorem 4. Let U be the annulus $U = B_{R_2} \backslash B_{R_1}$ with $R_2 \geq 2R_1$. Then for $\epsilon > 0$ sufficiently small (depending only on R_1), there exist constant mean curvature $H = 1/2$ horizontal graphs g^+ and g^- satisfying (*) on U with Dirichlet boundary data $g^\pm = 1 \pm \epsilon$ on ∂B_{R_1}, $g^\pm = 1$ on ∂B_{R_2}. Moreover $g^\pm$ is unique and varies continuously with the parameters ϵ, R_1, R_2 and $g^\pm$ tends to $1 \pm \epsilon$ uniformly on compact subsets as R_2 tends to ∞.

Bo Guan from Ohio State University spoke on "Complete conformal metrics with negative Ricci curvature on manifolds with boundary"

Abstract. Let $(\bar{M}^n, g)$, $n \geq 3$, be a compact smooth Riemannian manifold of dimension n with smooth boundary ∂M, $M = \bar{M} \setminus \partial M$ be the interior of $\bar{M}$, and let Ric_g denote the Ricci tensor of g. We are interested in the question of whether there exists a complete metric $\tilde{g}$ on M with negative Ricci tensor in the conformal class of g satisfying the equation:

$$\det(-Ric_{\tilde{g}}) = 1. \qquad (41.0.2)$$

More generally, let f be a smooth symmetric function defined in a open convex symmetric cone $\Lambda \subset \mathbb{R}^n$ which contains $\Lambda_n^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$ satisfying certain ellipticity structure conditions. Let $\Lambda^-[g]$ denote the collection of metrics $\tilde{g}$ on M in the conformal class of g such that $\lambda(Ric_{\tilde{g}}) = (\lambda_1, \cdots, \lambda_n)$, the eigenvalues of $\tilde{g}^{-1}Ric_{\tilde{g}}$, belongs to $-\Lambda$ everywhere on M.

Problem1. Find a complete metric $\tilde{g} \in \Lambda^-[g]$ on M with

$$f(-\lambda(Ric_{\tilde{g}})) = 1 \quad \text{in} \quad M. \tag{41.0.3}$$

In this talk we present some recent results from joint work with Huaiyu Jian and discuss open questions. Our result implies, in particular, that on any smooth domain in $\mathbb{R}$ contained in a half-space there exists a complete conformally flat metric with negative Ricci tensor satisfying equation (41.0.2).

Jaigyoung Choe from Korea Institute for Advanced Study spoke on “Capillary surfaces in a convex cone”.

Abstract. Some capillary surfaces are known to be rigid, i.e., part of a sphere. Here are two known examples: a disk type capillary surface in a ball, and a disk type capillary surface with at most three edges in a domain bounded by planes and spheres. We have found more examples as follows. Let C be a convex cone in $\mathbb{R}^n$ a hypersurface in C which has constant higher-order mean curvature and is perpendicular to the boundary of C. Then S is a spherical cap. We will prove this using the Reilly formula.

Moreover, let C be a polyhedral cone in $\mathbb{R}^3$ and S a capillary surface in C with constant contact angle(not necessarily 90 degrees) and with at most 5 edges. Then S is a spherical cap. This can be proved by using Poincare-Hopf’s theorem and Bonnet’s parallel surface.

Finally Nitsche’s result about a minimal disk in a ball which is perpendicular to the boundary sphere motivates the following problem: Let h be a harmonic function on a ball whose boundary values are equal to their normal derivatives. Show that they must be linear functions. We will prove this using Rellich’s identity.

One of Choe’s posed problems on harmonic functions with equal boundary Dirichlet and Neumann data was solved during a discussion in the workshop.

Pengfei Guan from McGill University spoke on “Isoperimetric inequality of quermassintegrals for star-shaped domains”

Abstract. I will describe a recent joint work with Junfang Li. We give a simple proof of the isoperimetric inequality for quermassintegrals of non-convex starshaped domains. The proof is based on work of Gerhardt and Urbas for an expanding geometric curvature flow of hypersurfaces of $\mathbb{R}^{n+1}$ and the observation of a certain monotonicity property of isoperimetric constants.

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Calibrated submanifolds

Spiro Karigiannis spoke on "Moduli spaces of calibrated cycles in G_2 manifolds ".

G_2 manifolds [10] are a class of Ricci-flat manifolds with special holonomy, occurring in 7 real dimensions. They are analogous to Calabi-Yau 3-folds in many respects, and are of interest to physicists in M-theory and supergravity [1]. They admit two natural classes of calibrated submanifolds: the 3-dimensional associative submanifolds, and the 4-dimensional coassociative submanifolds. These are both absolutely volume-minimizing in their homology class. In joint work with Naichun Conan Leung [11], we prove that these submanifolds, together with unitary connections on them satisfying some special condition, are critical points of a naturally defined functional of Chern-Simons type. Specifically, the pair of an associative submanifolds together with a flat connection is such a critical point, as well as the pair of a coassociative submanifold together with a connection for which the trace of the curvature form is self-dual. Additionally, there is a special type of connection (called deformed Donaldson-Thomas connections) on the ambient 7-manifold which can also be interpreted as a critical point of the same functional.

An interesting and important open problem is to study the stability of these critical points. That is, are they non-degenerate, and if so, are they minima? This non-degeneracy is likely related to the smoothness of the moduli space of such objects. For example, it is known that the moduli space of coassociative submanifolds is smooth and unobstructed, whereas the situation is much worse for associative submanifolds [12]. They are in general obstructed and their infinitesimal deformations are the kernel of a twisted Dirac operator. The deformation theory of Donaldson-Thomas connections has not yet been analyzed.

Another important question that needs to be addressed is how the moduli of pairs of submanifolds and connections as described above relates to the analogous situation for Calabi-Yau 3-folds [3], [4]. If N is a Calabi-Yau 3-fold, and S^1 is a circle, then the product $M = N \times S^1$ is a G_2 manifold. Associative submanifolds include products of S^1 with a holomorphic curve in N, and coassociative submanifolds include products of S^1 with a special Lagrangian submanifold in N. The exact relationship between the two geometries is non-trivial, however, because the G_2 moduli involve a complicated mixing of the complex and Kähler moduli of the Calabi-Yau 3-fold.

Marianty Ionel talked about “Constructions of special Lagrangian submanifolds”.

Abstract: Special Lagrangian submanifolds are a particular class of minimal submanifolds, introduced by Harvey and Lawson in the wider context of calibrated geometries. In this talk, I will describe the cohomogeneity one special Lagrangian 3-folds in both the deformed and the resolved conifolds. Our results give an explicit construction of the families of $SO(3)$ and T^2-invariant special Lagrangian submanifolds in these conifolds and describe their asymptotic behavior. The families of special Lagrangian submanifolds in the two conifolds approach asymptotically the same special Lagrangian cone. We will also describe a family of T^3-invariant coassociative 4-folds in the total space of the spin bundle of S^3, with the Bryant-Salamon G_2-metric.

Some open problems: 1. By moding out by appropriateS^1-actions on the spin bundle of S^3, one obtains the deformed and the resolved conifold. The relationship between the family of T^3- invariant coassociative 4-folds and the T^2-invariant special Lagrangian submanifolds constructed in the conifolds could be explored further using these actions. 2. Construct some symmetric explicit Cayley submanifolds in the spin bundle of S^4 endowed with the Bryant-Salamon $Spin(7)$-metric.

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Geometric flows

Curvature estimates are significantly important for the geometric evolution equations, such as the Ricci flow equation, etc. Gang Tian spoke on "Curvature estimates for Ricci flow in dimension 4". The following curvature estimates along the Ricci flow when the dimension of the manifold is 4 is stated and a proof is outlined.

Theorem. Suppose the Ricci flow equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

in $M \times [0,T)$, and $\dim M = 4$. Given $K > 0$, there exist $\epsilon, C = C(K)$ such that if

$$\int_{B_r(x,t_0)} |Rm|^2 \leq \epsilon$$

and

$$|Ric(g_0)| \leq K,$$

then

$$\sup_{B_{r/2}(x,t_0)} |Rm|_t \leq \frac{C}{t - t_0}$$

for $t \in (t_0, t_0 + r^2]$.

This theorem has a direct corollary as follows.

Corollary. Under the conditions

$$\int_{B_r(x)} |Rm|^2 dg \leq \epsilon$$

and $|Ric(g)| \leq K$, then $B_{r/4}(x)$ is diffeomorphic to U/Γ in $C^{1,\alpha}$ sense, where $U \subset \mathbb{R}^4$ is an open subset and $\Gamma \subset Iso(\mathbb{R}^4)$ is a finite group acting freely on $\mathbb{R}^4$.

This result can be viewed as a generalization of a theorem by Cheeger-Tian in [3], where the 4-manifolds are assumed to be Einstein.

G. Tian also proposed the following problem relevant to the the theorem:

Do all 4-manifolds with bounded Ricci curvature and finite Euler characteristic have finite topological type?

By the theorem for any sequence of 4-manifolds with bounded Ricci curvature and finite Euler characteristic, it will converge outside finitely many singularities. And if one could understand the topology around the singularities, then one could answer the above problems.

Yng-Ing Lee spoke on “Special solutions to Lagrangian mean curvature flow”

Abstract: In this talk, I will report on some special solutions to Lagrangian mean curvature flow constructed by me and my collaborators. The first category is eternal Brakke solutions. Recall that Brakke flow is a generalization of mean curvature flow, which is defined for varifolds, and an eternal solution is a solution defined for any t from negative infinity to infinity. Our solutions are (smooth) Lagrangian self-shrinkers for $t < 0$, Lagrangian cones for $t = 0$, and Lagrangian self-expanders for $t > 0$. Moreover, some of our solutions satisfy the additional property that every time slice is Hamiltonian stationary. When $n = 2$, our solutions resolve Schoen-Wolfson cones and can distinguish a $C_{2,1}$ cone from other cones. In higher dimension, we find Hamiltonian stationary cones which generalize Schoen-Wolfson cones, and eternal Hamiltonian stationary Brakke solutions which resolve these cones.

I will also talk about two other types of solutions. One type is translating solutions which include examples with arbitrarily small oscillation of the Lagrangian angle. These examples will play an important role in developing regularity theory in the Lagrangian mean curvature flow. Another type is other self-similar solutions. We have self-expanders with arbitrarily small oscillation of the Lagrangian angle and which are asymptotic to a pair of planes. Conversely, given any pair of Lagrangian planes with sum of characteristic angles less than $\pi/2$, we can always find such self-expanders asymptotic to this pair of planes. Examples of compact Lagrangian self-shrinkers are also obtained.

Jiayu Li from ICTP spoke on “Symplectic surfaces in K-E surfaces”.

Abstract: In this talk I will review our recent results in symplectic translating solitons and symplectic critical surfaces in K-E surfaces.

In a Kähler surface M, one can define the Kähler angle α of an oriented surface Σ in M by $\omega|_\Sigma = \cos\alpha d\mu_\Sigma$, where ω is the Kähler 2-form of M and $d\mu_\Sigma$ is the area form of Σ in the induced metric. Σ is symplectic if $\cos\alpha \geq 0$ on Σ. Li-Han considered the critical points of the functional $\int_\Sigma 1/\cos\alpha d\mu_\Sigma$ in the space of symplectic surfaces. They derived the Euler-Lagrange equation for this functional and showed that it is an elliptic equation. Properties of this equation are then studied, for example, a formula on the number of the complex points which is similar to that in the minimal surface case . The gradient flow of this functional and evolution equations of various geometric quantities along this flow are also considered. In particular, if the initial surface is symplectic and closed, then it remains so along the flow as long as the flow exists.

For symplectic mean curvature flows, Li-Han show that $\sup|\alpha| > \frac{\pi}{4}\frac{|T|}{|T|+1}$, where α is the Kähler angle of a symplectic translating soliton with $\max|A| = 1$ and A is the second fundamental form and T is the direction in which the surface translates.

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The classical theory of minimal surfaces

New developments in the area of the classical theory of minimal surfaces were presented and discussed at the workshop - the topics included: new constructions of minimal submanifolds in Euclidean space and in spheres, classification of minimal surfaces, comparison between the second variation of area and energy for minimal surfaces, curvature estimates, etc... For example, Meeks presented recent work with Perez and Ros which settles the old question of classifying all properly embedded genus zero minimal surfaces in $\mathbb{R}^3$. Several of the mealtime group conversations revolved around discussions with Meeks on issues about and open questions on the classical theory of minimal surfaces [6]. Below is a highlight of the workshop talks.

Leobardo Rosales from UBC spoke on "Minimal immersions with prescribed boundaries". Recently L. Simon and N. Wickramasekera [8] introduced a PDE method for producing examples of stable branched minimal immersions in $\mathbb{R}^3$. This method produces q-valued functions u over the punctured unit disk in $\mathbb{R}^2$ so that either u cannot be extended continuously across the origin, or G the graph of u is a $C^{1,\alpha}$ stable branched immersed minimal surface. The present work gives a more complete description of these q-valued graphs G in case a discontinuity does occur, and as a result, we produce more examples of $C^{1,\alpha}$ stable branched immersed minimal surfaces, with a certain evenness symmetry.

Adrian Butscher of Stanford University spoke on "New constructions of submanifolds of the sphere which are critical points of the volume functional". If one searches for k-dimensional submanifolds with critical k-dimensional volume in a Riemannian manifold, then one is led towards elliptic partial differential equations involving the mean curvature vector of the submanifold. This talk presented new constructions ([1], [2]) of volume-critical submanifolds of the sphere in two contexts: hypersurfaces with constant mean curvature in spheres of any dimension; and Legendrian submanifolds in spheres of odd dimension that are stationary under variations preserving the contact structure. These are constructed by solving the associated elliptic PDE using singular perturbation theory. The analytic and geometric similarities between these two contexts was highlighted.

Mario Micallef of University of Warwick spoke on "Comparison between second variation of area and second variation of energy of a minimal surface". The conformal parameterisation of a minimal surface is harmonic. Therefore, a minimal surface is a critical point of both the energy functional and the area functional. This talk described joint work [3] with N. Ejiri which compares the Morse index of a minimal surface as a critical point of the area functional with its Morse index as a critical point of the energy functional. The difference between these indices is at most the real dimension of Teichmuller space. The methods for this comparison also allow Micallef and Ejiri to obtain surprisingly good upper bounds on the index of minimal surfaces of finite total curvature in Euclidean space of any dimension. They also bound the index of a minimal surface in an arbitrary Riemannian manifold by the area and genus of the surface, and the dimension and geometry of the ambient manifold.

Y.L. Xin of Fudan University spoke on "Curvature estimates for minimal submanifolds of higher codimension". Estimates of the Hessian of several smooth functions defined on Grassmannian manifold were derived. Based on these, curvature estimates for minimal submanifolds of higher codimension in Euclidean space were obtained, via the Gauss map ([9]). Thus, Schoen-Simon-Yau's results and Ecker-Huisken's result for minimal hypersurfaces are generalized to higher codimension. In this way, the results for Bernstein type theorems done by Hildebrandt-Jost-Widman and Jost-Xin could be improved.

William Meeks of the University of Massachusetts, Amherst, spoke on "The classification of embedded minimal planar domains in $\mathbb{R}^3$" (joint work with Joaquin Perez and Antonio Ros). Recently William Meeks, Joaquin Perez, and Antonio Ros [7] have succeeded in classifying all properly embedded genus 0 minimal surfaces in $\mathbb{R}^3$. Based on their previous results it remained to prove that the examples of infinite topology are the examples discovered by Riemann in 1860, called the Riemann minimal examples. The proof of the uniqueness of the Riemann minimal examples is in part related to the holomorphic integrability of the classical Shiffman function $S(M)$ which is a nonzero Jacobi function on a possible counterexample M. They relate the integrability to an evolution equation for the Gauss map of M which in turn can be related to the integrability of the KdV equation on the complex plane with initial Cauchy data. The proof of integrability depends on the theory of KdV hierarchy and algebro-geometric potentials. In the end they prove that $S(M)$ vanishes which means that M is foliated by circles and lines in a family of parallel planes, which by Riemann's earlier results implies M is a Riemann minimal example. Some related theoretical results were also discussed.

David Hoffman of Stanford University spoke on "Embedded Helicoidal minimal surfaces in $\mathbb{R}^3$ and $\mathbf{S}^2 \times \mathbb{R}$". In joint work [4] [5] with Brian White, they construct embedded genus-one helicoids in $\mathbb{R}^3$ by variational means without recourse to the Weierstrass representation or other function-theoretic methods. They are also able to show that important geometric properties of the examples they construct are shared by all other examples with sufficient symmetry. The talk described their construction of examples of embedded helicoidal minimal surfaces in $\mathbf{S}^2 \times \mathbb{R}$ of arbitrary genus.

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List of Participants

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Two-day Workshop Reports

Chapter 42

Third Northwest Functional Analysis Symposium (07w2136)

Mar 30 - Apr 01, 2007

Organizer(s): Berndt Brenken (University of Calgary), Juliana Erlijman (University of Regina), Alexander Litvak (University of Alberta), John Phillips (University of Victoria)

Functional Analysis plays an important role in mathematics. It is a major strength of Canada's, particularly in western Canada, where, whithin the PIMS network, there are several strong groups with internationally recognized standing.

In our workshop such groups were well-presented. We had also participants who work in related areas where methods of Functional Analysis are used. Altogether 6 western Universities were represented, and many postdoctoral fellows and advanced graduate students participated. There were 11 talks on various aspects of Functional Analysis and related topics. There was also time for informal communication between participants.

Geometric Functional Analysis

Geometric Functionals Analysis deals with geometry of Banach spaces in both finite-dimensional and infinite dimensional cases. In particular, it studies asymptotic properties of high dimensional convex bodies and related topics in convex and combinatorial geometry.

More than 60 years ago Paul Erdos raised two questions which became the best known problems in combinatorial geometry. (i) How often can the same distance occur among N points in the plane? (ii) At least how many distinct distances must occur among N points in the plane? He conjectured that for (i) the upper bound is $N^{1+\epsilon}$ and in (ii) the lower bound is $N^{1-\epsilon}$. The first conjecture would imply the second one, however both conjectures are widely open. Even less is known for point sets of general normed spaces. **József Solymosi** described some related results and listed several problems and conjectures.

Márton Naszódi presented his joint work with Károly Bezdek and Balázs Visy on Petty numbers of normed spaces. In 1983, P. $Erd\mathbb{H}$ os raised the following question: What is the largest number $g(n)$ with the property that every system of n non-overlapping unit disks in the Euclidean plane has an independent subsystem with at least $g(n)$ members; i.e., there are $g(n)$ members no two of which are tangent. In 1971, C.M. Petty proved that the maximum number of pairwise touching non-overlapping translates of a convex body in d-space is 2^d. Motivated by $Erd\mathbb{H}$ os' problem and Petty's Theorem, Márton with his coauthors defined two quantities in a real normed space M^d of finite dimension d. For $m \geq 2$, the mth *Petty number* $P_{\text{pack}}(m, M^d)$ of M^d is the largest cardinality of a set A in M^d with the property that among any m points of A there are two at distance one. The *packing number* $P_{\text{pack}}(m, M^d)$ of M^d is the largest integer k such that there are k non-overlapping unit balls in M^d with the property that among any m of them, there are two touching ones. The main challenge is to find upper bounds for the Petty- and the packing numbers. Márton showed general upper bounds in terms of m and the dimension of M^d, and bounds specifically for ℓ_p spaces, and for the space $l^d_\triangle$ with unit ball $T-T$, where T is the regular simplex. These problems, apart from being intriguing questions in the theory of finite dimensional normed spaces, also have applications in coding theory.

It was shown by Milman in 1970 that the product of any two strictly singular operators on $L_p[0,1]$ is a compact operator. In his talk **Vladimir Troitsky** presented his joint work with G.Androulakis, P.Dodos, A.Popov, and G.Sirotkin. It was shown by Milman in 1970 that the product of any two strictly singular operators on $L_p[0,1]$ is a compact operator. Vladimir with coauthors show extension of this approach to other Banach spaces. It was proved that if X is a Banach space such that the number n of pair-wise non-equivalent seminormalized weakly null Schreier spreading basic sequences is finite, then the product of any $n+1$ strictly singular operators on X is compact. Further, for every countable ordinal ξ the Schreier family S_ξ (introduced by D. Alspach and S.A. Argyros) was used to define the class of S_ξ-singular operators. It was proved that the classes of S_ξ-singular operators are nested increasing in ξ and exhaust the class of all strictly singular operators. These classes are norm-closed, and stable under left and right multiplications by bounded operators. It was also shown that if the number n_ξ of pair-wise non-ξ-equivalent seminormalized weakly null Schreier spreading basic sequences is finite, then the product of any $n+1$ S_ξ-singular operators on X is compact. However, in general, S_ξ-singular need not even be polynomially compact. Finally, analogously to HI-spaces, HI$_\xi$-spaces we introduced. It was shown that the classes of HI$_\xi$-spaces are nested increasing in ξ and exhaust the class of all HI-spaces, and that every operator on an HI$_\xi$-space is of the form "scalar plus S_ξ-singular". Here are some related open questions: (i) Whether there exists an HI$_\xi$-space X such that $n_\xi(X)$ is finite? A positive answer to this question would imply a positive answer to the following long-standing question: does there exist a Banach space on which every operator has an invariant subspace? (ii) Whether the classes of S_ξ-singular operators are operator ideals?

Abstract Harmonic Analysis

Abstract Harmonic Analysis concerns investigations of Banach algebras and spaces of measures or functions associated with a locally compact group. Two talks were given on this subject.

Hung Le Pham delivered a talk on the structure of (discontinuous) homomorphisms from $C_0(X)$ into Banach algebras. In particular, it was shown that for most locally compact metrizable space X the continuity ideal of a homomorphism from $C_0(X)$ (the largest ideal on which the homomorphism is continuous) is not always a finite intersection of prime ideals. The examples of such spaces include all uncountable σ-compact locally compact metrizable spaces. The method of construction is algebraic, involving results from algebraic geometry. Another consequence of the construction is a general result on norming commutative semiprime algebras; extending the class of algebras known to be normable. The problem of describing the detailed structure of (the continuity ideals of) homomorphisms from $C_0(X)$ into Banach algebras is still open.

In his talk **Faruk Uygul** proved that every completely contractive dual Banach algebra is completely isometric to a w^*-closed subalgebra the operator space of completely bounded linear operators on some reflexive operator space.

C^*-algebras and non-commutative geometry

Three talks on current research projects within several areas of non-commutative geometry were delivered.

Heath Emerson reported on aspects of joint work with R. Meyer concerning a noncommutative extension to an operator algebra context of the classical Lefschetz invariant associated with a group acting on manifold or simplicial complex. This is accomplished by connecting equivariant KK theory to Lefschetz fixed point theory.

The irrational rotation algebras were among the first significant examples of noncommutative operator spaces associated with topologically intractable dynamical systems. **Zhuang Niu** reported on progress of recent joint work with Elliott concerning C*-algebras generated by the irrational rotation algebra and certain spectral projections, so called extended rotation algebras. This is the first step in a programme to find a canonical C*-algebraic way to embed the rotation algebra in an AF algebra. In certain cases these are shown to be simple and nuclear, so of interest to the Elliott classification programme. The Elliott classification programme is one of the major threads of research in C*-algebras over the past 15 years, with a goal of finding a classification functor on nuclear, separable, simple C*-algebras with range involving K-groups with order structure, and topological convex cones with natural pairings between them.

E_0 semigroups of endomorphisms and its attendant formal classifying structure of product systems is a significant area of current research interest. **Remus Floricel** gave a talk on classes of essential representations of product systems: Essential representations of a product system E give rise to E_0-semigroups whose associated concrete product systems are isomorphic to E. He first constructed essential representations of a product system $E = \{E(t)|t > 0\}$ with respect to a given unital vector $u_t \in E(t)$, $t > 0$. His constructions employes techniques developed by M. Skeide, W. Arveson and V. Liebscher. He then showed that the E_0-semigroups associated with the essential representations of E constructed out of two unital vectors $u_t \in E(t)$, respectively $u_s \in E(s)$, are conjugate if and only if $s = t$, producing in this way uncountably many non-conjugate, cocycle conjugate E_0-semigroups. Secondly, he constructed essential representations of E with respect to measurable sections of E, and shows that the associated E_0-semigroups are conjugate if and only if the group of automorphisms of E acts transitively on the set of measurable sections. He also described the structure of the tail algebras of these E_0-semigroups.

Ergodic Theory

Two talks we delivered on Ergodic theory and relations to Analysis and Functional Analysis.

Ergodic theory provides a number of powerful tools for analysis of the asymptotic behavior of measure preserving dynamical systems. Some systems come with a natural invariant measure; many more do not. In his talk **Chris Bose** discussed both classical and new results about how to find invariant measures in some of these cases, and how to compute explicit approximations where there may be no tractable formula for the measure.

Many authors have considered "subsequence ergodic theorems", where measurements (i.e. the value of an L^p function) of a measure-preserving dynamical system are taken at a fixed sequence of times. A sequence of times is called good if the averages converge. In his talk **Anthony Quas** considered bad sequences of times and asked for the maximal rate of badness.

Applications of Operator Theory

Pedro Massey gave a talk on The Schur-Horn theorem for operators and frames with prescribed norms and frame operator (joint work with J. Antezana, M. Ruiz and D. Stojanoff). In recent years, the concept of frame in s Hilbert space has gained attention due to its important applications to data transmission in channels with noise. Roughly speaking, an ordered set of vectors $\mathcal{F} = \{f_k\}_{k\in\mathbf{N}}$ is a frame for a separable (real or complex) Hilbert space $\mathcal{H}$ if linear combinations with coefficients in $\ell^2(\mathbf{N})$ generate $\mathcal{H}$ in a stable way. It turns out that the canonical way in which the vectors in $\mathcal{H}$ are generated by a frame $\mathcal{F}$ depends on inverting the so called frame operator of $\mathcal{F}$, which is typically a hard problem. A way out to this problem is to consider frames with simple frame operators, but then the existence of such frames is no longer evident. Given a bounded positive definite operator $S \in B(\mathcal{H})$, and a bounded sequence $c = \{c_k\}_{k\in\mathbf{N}}$ of non negative numbers, the pair (S, c) is called *frame admissible* if there exists a frame $\{f_k\}_{k\in\mathbf{N}}$ for $\mathcal{H}$ with frame operator S and such that $\|f_k\|^2 = c_k$, $k \in \mathbf{N}$. We relate the existence of such frames with the problem of determining the principal diagonals of an operator in $B(\mathcal{H})$. This is a key tool, since this last problems has been recently considered by several researchers. Massey gave a reformulation of the extended version of Schur-Horn theorem due to A. Neumann that characterizes the (closure of the set of) possible diagonals of selfadjoint operators in $B(\mathcal{H})$ and used it to get necessary conditions, and to generalize known sufficient conditions, for a pair (S, c) to be frame admissible. We also describe some results for diagonals of finite rank operators determining completely when a pair (S, c) is frame admissible when dim $\mathcal{H}$ is finite.

List of Participants

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Chapter 43

Symbolic computer algebra in theoretical physics (07w2153)

Apr 13 - Apr 15, 2007

Organizer(s): Andrzej Czarnecki (University of Alberta)

Overview of the Field

Computers have assisted many fields of theoretical research as a tool for numerical calculations and simulations. However, it is a relatively recent development that computers can also manipulate algebraic expressions, and in particular that they can handle really huge formulas. This has become possible thanks to an increased power of the hardware, in particular the availability of large memory. Computer algebra requires larger resources than most numerical calculations because it needs, in general, more complex data structures.

Atomic and subatomic physics have benefited from this progress perhaps more than other fields. This is so because they study very clean systems where experiments can reach very high precision which, in principle, can be matched by the theory. However, the theoretical studies are limited by the complexity of calculations which can be carried out by humans.

One measure of the complexity of such calculations is the number of “loops”. Represented by so-called Feynman diagrams, subatomic processes have closed loops of interacting particles – a purely quantum phenomenon. Although the first successful one-loop calculations were performed already in late 1940s, the number of loops that can be studied today rarely exceeds three. This slow progress on the side of theory is due to challenges that one has to overcome in each order in the number of loops.

On the other hand, the experimental progress has been huge. New technologies in laser spectroscopy and electronics have enabled studies of increasingly complex atomic systems with ever larger precision. A recent illustration is the 2005 Nobel Prize in physics awarded for the development of the so-called frequency comb, a new tool for optical frequency measurements. In Canada, Eric Hessels at York University has achieved the highest accuracy in helium fine structure measurements. If only a better theoretical knowledge of this spectrum can be achieved, his results will lead to a new determination of the fine structure constant, a fundamental characteristic of electromagnetic interactions.

Symbolic computation allows us now to close the gap between theory and experiment and take full advantage of the recent progress in measurements. Computer algebra has been successfully applied to determine properties of simple atoms and elementary particles. At the University of Alberta the Centre for Symbolic Computation has been created: a laboratory equipped with powerful hardware and software dedicated to manipulating large algebraic expressions.

However, to make further progress it is not enough to rely on the increasing power of computers. To meet the requirements of experiments, four-loop accuracy has to be achieved for a variety of subatomic observables, for example the Lamb shift and the anomalous magnetic moment of the electrons. This next level can be reached only with new mathematical methods.

A typical multi-loop calculation requires an evaluation of a large number of very difficult multiple integrals. The difficulty consists in part in divergences with plague individual integrals and cancel only in a the final sum of partial results. Thus, numerical evaluations are not a good option. An approach which has become standard consists in constructing a system of recurrence relations which, in principle, enable a reduction of all needed integrals to an irreducible basis of so-called master integrals.

Two aspects of this program are challenging:

(a) the solution of the recurrence relations; and

(b) the evaluation of master integrals.

The purpose of this short workshop was to review recent progress in mathematical approaches to the evaluation of Feynman diagrams, with the emphasis on methods of reducing the large number of diagrams to a set of master integrals. Some promising methods of evaluating master integrals were also reviewed.

Recent Developments and Open Problems

Recent work of many talented people has resulted in an emergence of new links between areas of mathematics such as perturbation theory, differential equations, and special functions. Indeed, a whole new class of special functions has been identified, harmonic polylogarithms [1]. These functions help to express many master integrals in an exact analytical form.

The real breakthrough in the approach to multi-loop diagrams was the discovery of the so-called Laporta algorithm [2]. It allows to use a computer algebra system to reduce the number of integrals which must be evaluated. Whereas in the past a problem was considered intractable analytically if it involved say a thousand Feynman integrals in total, now we attack problems which contain a similar number of master integrals. In practice, this means that the cutting edge has shifted, roughly speaking, from two- to four-loop problems. This is a huge progress.

However, the experimental techniques have made similar or even greater advances, and we really need to develop methods to tackle the four-loop challenge. The bottleneck turns out to be the evaluation of master integrals, for which an algorithm has yet to be developed.

Presentation Highlights

The opening talk of the workshop was given by Matthias Steinhauser (University of Karlsruhe, Germany). It was devoted to automatic generation of Feynman diagrams and their asymptotic expansions. The Karlsruhe group has created a powerful system of programs [3, 4] which can handle a wide class of processes. It is however important to extend it to other cases, especially to the practically important threshold kinematics. Such an extension would also enable one to solve a variety of atomic physics problems.

We had a special talk by a young and very talented PhD student, Alexey Pak. The intention of that talk was to be somewhat provocative, namely to present a number of new ideas from a newcomer to the field, how master integrals can be evaluated. This was a great success and a lively discussion followed (the talk was scheduled as the last one in a session). In addition, Alexey presented new ways of visualizing Feynman diagrams, which will be very helpful when connected with automatic generators of diagrams.

Andrey Grozin (Russian Academy of Sciences, Novosibirsk) reviewed applications of Groebner bases for the reduction of Feynman integrals. The original work by Buchberger provides a prescription for a unique reduction of any multi-variable polynomial. Very recently this has been extended to handle also *non-commuting* objects, such as operators raising and lowering powers of terms in the integrand of a Feynman integral [5]. Unfortunately, the implementation of that new algorithm is not a trivial task, and the one existing program is not publicly available.

Outcome of the Meeting

Much of the discussions at this workshop were devoted to various new ideas for evaluating master integrals. The most important outcome is an emerging collaboration between Alberta and Karlsruhe, whose purpose is to determine master integrals for the threshold production of heavy particles. We identified two promising approaches. One is based on differential equations, the other on asymptotic expansions. There is a very strong experimental motivation for this progress and we are confident that it will stimulate development of new mathematical methods.

List of Participants

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Chapter 44

Math Fair Workshop (07w2134)

Apr 20 - Apr 22, 2007

Organizer(s): Tiina Hohn (Grant MacEwan College), Ted Lewis (University of Alberta), Andy Liu (University of Alberta)

Report of the Math Fair Conference in BIRS in April 2007, 07w2134.

This was the fifth BIRS math fair workshop, which is becoming a popular annual event. The participants came from elementary schools, junior-high and high schools, from independent organizations, and from universities and colleges. The thirty-eight participants at this year's workshop were educators of all types, from teachers to grad students to expert puzzle and game designers. Our key note speaker this year was Scott Kim. SNAP Mathematics Foundation officially changed the name from BIRS Math Fair Workshop to Ted Lewis Workshop on SNAP Math Fairs.

The purpose of the workshop was to bring together educators who are interested in using our particular type of math fair, called a SNAP math fair, to enhance the mathematics curriculum. (The name SNAP is an acronym for the guiding principles of this unconventional type of math fair: It is student-centered, non-competitive, all-inclusive, and problem-based.) The projects at a SNAP math fair are problems that the students present to the visitors. In preparation, the students will have solved chosen problems, rewritten them in their own words, and created hands-on models for the visitors. At a SNAP math fair, all the students participate, and the students are the facilitators who help the visitors solve the problems. This process of involving students in fun, rich mathematics is the underlying vision that makes the SNAP program so unique and effective. No first prize! No arguments about judging! Everyone is a winner!".

At the BIRS workshop, the participants learn about and try math-based puzzles and games that they can use in the classroom. They have a chance to see how other teachers have organized math fairs at their schools, how the SNAP math fair fits the curriculum, and what some schools have done for follow-ups. And then they go back to their schools and change the culture of mathematics in their class-room.

This year we learned from Ontario teacher Tanya Thompson how to introduce the math fair to the students and tie it together with a theme. Scott Kim, puzzle designer in his company Shufflebrain, presented how to help students to design their own puzzles. Gordon Hamilton from Calgary brought variety of games and problems.

Cathy Campbell shared her excellent presentation how to introduce math fair to students who have no previous experience of it. Bill Ritchie from Thinkfun explained math resource rooms in a school. Grad students from University of Alberta.

Shawn Desaulniers and Trevor Pasanen shared their experiences with math fairs and how to teach difficult concepts in a puzzle environment. Dr. Jim Timourian connected the math fair concept with other benefits of student learning and then on Sunday we also touched a very hot topic for teachers: Chris Stroud from West Point Grey Academy in B.C. opened a whole new conversation with assessment ideas on math fairs. That will definitely be one of the key issues in future workshops. The concept of the SNAP math fair originated in Edmonton with Andy Liu and Mike Dumanski, and it has proved so successful that it led to the formation of a non-profit organization, the SNAP mathematics foundation, which has helped promote mathematics in schools around the world. As well as the SNAP foundation, the Calgary-based Galileo Education Network Association (GENA) helps schools organize math fairs, and provides valuable lesson-study follow-ups.

The BIRS math fair workshops have contributed greatly to the proliferation and popularization of the SNAP math fair. In some places, the use of a SNAP math fair to change children's attitudes about mathematics has almost become a ”grass-roots” movement, and so it is difficult to pin down exactly how many schools are now doing them. We have a fair idea about the numbers in Edmonton and Calgary - for example over 60 percent of the elementary schools in the Edmonton catholic system now hold regular math fairs, and as far as we can gauge, the numbers are high in the public system as well. GENA reports similar figures for the Calgary area.

SNAP and CMS are also providing some support for the launch of a similar math fair workshop in the Fields institute in Toronto, and PIMs is providing math fair booklets for the participants. The Fields workshop is being organized by Tanya Thompson who has been a valuable participant at past BIRS workshops.

Altogether, the BIRS math fair workshops are having a noticeable impact on mathematics education.

Regards,

Ted Lewis,

Department of Mathematical and Statistical Sciences,

The University of Alberta

Tiina Hohn

Mathematics Department

Grant MacEwan College

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Hamilton, Gordon (Masters Academy and College)
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Chapter 45

Statistical science for ‘omic research in Canada (07w2140)

Jun 29 - Jul 01, 2007

Organizer(s): Mayi Arcellana-Panlilio (University of Calgary), Jennifer Bryan (University of British Columbia), Robert Gentleman (Fred Hutchinson Cancer Research Center), Karen Kopciuk (University of Calgary/Alberta Cancer Board)

Overview of the Field and Workshop Motivation

Rapid advances in genome research over the past decade have engendered a unique and inventive interface between biology and statistics. It’s a burgeoning field that has quickly grown into an entirely new specialty discipline tentatively called statistical genomics.

The rapid growth of statistical genomics has been driven by the vast amounts of data generated by today’s genomics research that needs to ask increasingly complex questions in order to uncover relationships and interactions, and shed light on biology and meaning. Cutting-edge statistical research to develop novel analytical methods is now in great demand. It has become clear that genomic and statistical research must go hand in hand, each informing the other based on sound principles with the common goal of obtaining answers closer to the objective truth than either could have achieved alone.

Challenges have arisen in step with these advances. Funding agencies have not kept pace in recognizing that development of these critically required methodologies merits specific, independent financial support. University departments struggle to provide meaningful pathways for their graduate students and faculty in the new field. There is a strong push both from the statistical genomics discipline and the genetic research community for action on these issues.

The workshop allowed this very small community of Canadian researchers to develop and strengthen research connections and generate strategies for impacting this research area even more in the future. We wrote a white-paper type document for distribution to the statistical genomic community, which was based on the issues and solutions identified from this workshop. Hence, this proposed workshop was crucial to accelerating knowledge of genomic research and to the continuing development of Canadian interdisciplinary statistical genomic researchers.

Presentation Highlights

Rapid development of new high throughput technologies and accumulation of complex biological information from different perspectives are demanding ever more powerful and sophisticated statistical methods, creating the new specialized discipline of statistical genomics. Banff workshop participants identified a number of challenging problems that face the new discipline and that require novel and complex solutions.

Potential solutions for three primary challenges have been identified. Firstly, it was agreed that statistical genomic research does not fit the mandate of most genomic research funding agencies and generally, traditional statistical funding agencies are not providing adequate funds to carry out necessary empirical validations. Strategies to improve the funding environment have been suggested. Although there have been some positive signs, implementation of these strategies is needed now.

Secondly, the workshop evaluated problems and suggested solutions for creating a workable training and career environment that will attract and retain new members in the discipline. These include the need to develop collaborations within and amongst universities and to provide training opportunities with adequate time, for example by off-loading some teaching responsibilities during training. National networks and strategies to strengthen interdisciplinary research were suggested as requirements to advance the field; as well as making a natural home for it within university departmental structures.

Finally, promising career paths are needed to generate the critical mass necessary to grow and sustain our new discipline in Canadian universities and institutions. The current situation has failed to attract large numbers into the field, even for those with masters degrees in statistics with an emphasis on genetics or genomics. To remedy this, we have made recommendations ranging from reduction of undergraduate teaching loads to establishment of a Canada Research Chair in Statistical Genomics.

Outcome of the Meeting

A White Paper on the Status of Statistical science for ’omic research in Canada was the main outcome of this two-day workshop. Eighteen of Canada’s leading researchers (from BC, AB, ON and QC; see Appendix) active at the interface of statistics and genomics were brought together for the first time in a series of facilitated sessions to explore challenges and opportunities facing ’omic research in Canada. This report summarizes those discussions and presents key recommendations.

The White Paper was shared with and used by national leaders from the statistics field involved in the NSERC Restructuring exercise and in discussions with CIHR leadership. K. Kopciuk presented the results at the 3rd Annual Canadian Genetic Epidemiology and Statistical Genetics Meeting at the Fields Institute in Toronto on May 1, 2008. The White paper in its entirety was published in the meeting program.

List of Participants

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Chapter 46

PHAC-MITACS Joint Symposium on Modeling Sexually Transmitted and Blood-Borne Infections (07w2156)

Aug 10 - Aug 12, 2007

Organizer(s): David Fisman (Hospital for Sick Children and Ontario Public Health Laboratories), Tom Wong (Public Health Agency of Canada), Jianhong Wu (York University)

The symposium brought together a group of modelers and public health experts in sexually transmitted blood borne infections (STBBI), to identify significant public health knowledge gaps in modeling STBBI in Canada, and to develop opportunities for interdisciplinary collaboration.

The workshop was held shortly after the International STD Research Meeting in Seattle (July 29–August 1, 2007), to ensure maximal impact of the workshop on the Canadian policy towards the study of sexually transmitted and blood-borne infections. One of the objectives of this symposium, set before it took place, is to form a national focus group on modeling sexually transmitted and blood-borne infections that is capable of ”making recommendations to research funding agencies”. Also, it was anticipated that an immediate goal of this multidisciplinary network is to develop research proposals and apply for seed funding from the Public Health Agency of Canada (PHAC) and to make recommendation to various agencies about the role of mathematical modeling for the management of sexually transmitted and blood-borne infections. The great facility already in place at BIRS facilitated communication between group members of markedly differing backgrounds, to a degree that enhanced the success of the symposium. The quality of the meeting and beauty of the setting led to very enthusiastic support for subsequent meetings[1], which will ensure the long-term impact of this gathering.

The symposium started with remarks by Dr. Tom Wong (PHAC), which described the meeting objectives as 1). to identify and prioritize public health knowledge gaps in modeling STBBIs; 2). to develop STBBI collaborative modeling research teams to draft high public health priority proposals to address gaps; and 3). to give advice to various agencies on the role of mathematical modeling for STBBIs. His open remarks were followed by a brief introduction of Dr. Jianhong Wu (MITACS Centre for Disease Modeling at York University) to the growth of Canadian industrial and applied mathematics facilitated by BIRS, and in particular to the research and outreach activities of a MITACS team on modeling infectious diseases.

Five invited talks were carefully selected in order to stimulate the general discussions for fruitful collaboration. Dr. David Fisman (Ontario Public Health Laboratories Branch) talked about High School-Based Chlamydia Screening: Projected Health and Economic Impact in Philadelphia. In 2002, Philadelphia had the

[1] Indeed, there were multiple sub-projects developed. In addition, a few meetings have been organized to follow-up the agendas of the BIRS Symposium such as the PHAC-MITACS-INSPQ workshop ”Genital Herpes and HPV Modelling for Public Health Workshop” held at the MITACS Centre for Disease Modelling on May 29-30, 2008; and a NAHO workshop in Ottawa (May 22-23, 2008) to address the importance with regard to sexually transmitted and blood borne infectious for First Nations, Inuit and Merits Communities.

2nd highest rate of reported Chlamydia infections among major urban areas in the United States. 38% of these cases occurred among 15-19 year olds. Consequently, with the support of the Philadelphia Schools and Health Commissioners, the Philadelphia high-school screening program for gonorrhea and Chlamydia was initiated in January 2003. Because both economic and political forces have the potential to threaten Philadelphias high school screening program, it became important to evaluate the economic attractiveness of the Chlamydia screening component of this program; in short, to answer the question of whether the program represented money well spent. The study presented by Dr. Fismen chose to base cost, effectiveness and cost-effectiveness estimates on a mathematical dynamic model that takes into account the fundamental transmissibility of a sexually transmitted infection such as Chlamydia. This approach allowed the projection of the health and economic benefits of such important components of the Philadelphia screening program as inclusion of males. Indeed, few available economic analyses of Chlamydia screening incorporate the effects of screening males on downstream health consequences in females. This approach also allowed the assessment of the potential impact of rebounding Chlamydia prevalence on the economic attractiveness of screening. The study projected that school-based screening for Chlamydia in Philadelphia public high schools is likely to reduce both disease and net medical costs. When program administrative costs are incorporated into the analysis, the program remains extremely cost-effective relative to commonly accepted preventive interventions. These findings are robust in the face of wide-ranging sensitivity analyses. It was noted that the use of a model that incorporates transmissibility permits identification of non-linear health and economic effects unlikely to be seen with fixed risk models, including the attractiveness of screening males for Chlamydia, and delineation of the time required for health agencies to break even on their investment in population health. In this analysis, screening of both males and females, or males alone, emerge as likely more efficient and more effective than screening limited to females in a high-school population.

Pauline van den Driessche (University of Victoria) talked about her recent work on spread of gonorrhea among the Ariaal. After some detailed description of the unique culture in the studied region, she depicted the Rendille/Ariaal HIV networks and discussed the available survey data, based on which four subpopulations in Ariaal culture were identified (single males (moran); single females (nykeri and others); married males (wazee); married females (mamas)). She pointed out that the 1996 survey in Ariaal settlement of Karare quantified a high level of heterosexual mixing, and discussed the preliminary goal of modeling analysis as to testing the hypothesis that gonorrhea can spread in this population without a core group (i.e. without CSWs). For this purpose, a further survey was conducted in October 2003 (based on a UNAIDS questionnaire). In this Ariaal 2003 data, 100 individuals in each of four subpopulations responded to questions on behavior and number of partners in the past month and year. This permitted the calculation of the mean and variance of numbers of partners. She then formulated the compartmental model under the assumption that gonorrhea confers no immunity on recovery and has a very short latent period (ignored). Her model takes the form of SIS structure in each subpopulation, and she focused on the 11 year period so there was no change in marital status and no entry into sexually active population. The calculated basic reproduction number is always less than 1 under a variety of biological realities, indicating that gonorrhea is not expected to persist in the population with age-structured mixing. Therefore, persistence of disease must be due to factors not included, e.g. a core group of CSWs and long distance truck drivers, and/or concurrency. That motivated the 2005 survey of CSWs in order to obtain more data for an extended model study.

Tom Wong (PHAC) presented a global picture about Canadian surveillance of STIs and hepatitis C. He summarized the disease features and challenges in terms of surveillance and modeling, for a long list of diseases important for Canadian public health including Chlamydia, Gonorrhea, Syphilis, LGV, HSV, Genital Herpes, Human Papillomavirus and Hepatitis C. He then provided evidences to support the research that takes into account population lens such as street youth and vulnerable populations including immigrants and aboriginal, and these evidences directed greatly the afternoon discussions and the development of a few projects (see below). His talk touched on the interaction of public health, data collection, dissemination and analysis and interpretation, and demonstrated the great opportunity to use modelling to answer high priority public health questions, and the importance of integrating surveillance, research, prevention, treatment & care across sexually transmitted bloodborne infections to reduce disease burden through addressing: common risk factors and common risk populations.

Marie-Claude Boily (Imperial College) gave an overview of the potential impact of male circumcision on HIV and other sexually transmitted infections. She started with a brief discussion of the compelling evidence that male circumcision (MC) reduces male susceptibility to HIV infection from ecological studies, metaanalysis of observational studies, randomized controlled trials of MC, to plausible biological mechanisms. She then discussed various issues related to male circumcision as a prevention tool against HIV(population-level impact as who, when, where; acceptability, safety, cost/feasibility and risk replacement). She then listed a few key modelling questions, and formulated a stochastic model to simulate the MC trial in Kisumu, for the purpose of determining if MC efficacy against STI and HIV can independently and validly be estimated; to determine whether MC efficacy against STI alone can produce large effectiveness against HIV; to estimate the fraction of all HIV infections prevented in the trials that are attributable to efficacy against STI when both efficacies combine; to look at the population level impact of MC as an intervention. Model based analysis shows that STI are unlikely to explain the large protective effect of MC against HIV, but at the population level male circumcision has the potential to be an effective prevention and additional impact of MC against STI could be quite modest. She concluded with a number of questions emerging from her modeling work such as when the protection against STI can have a substantial incremental impact at the population-level, and what the best circumcision programme are in terms of age and risk group to target, and the potential role of negative outcomes.

Robert Smith? (University of Ottawa) discussed the issue of evaluating HPV vaccination for children vs. adults. His talk covered a wide range of issues related to HPV: the epidemiology of HPV, the details of the vaccine(s), some relevant research questions, the mathematical model appropriate to address these questions, the model based calculation of eradication threshold and relevant recommendations. The issues addressed specifically in his modeling study include: Can a childhood-only vaccination program eradicate HPV? Should an adult vaccination program supplement childhood vaccination? What happens for vaccines with suboptimal efficacy? Can a childhood-only vaccination program eradicate HPV? The model is based on the assumptions that men do not get vaccinated, children progress to the sexually active after 10 years, women and men are in the sexually active pool for 4 years (after this time, they cannot be vaccinated), and the vaccine may not confer 100% protection. He formulated a compartmental model that takes into account the question to be addressed. Using this model, he determined a threshold for eradication of the disease, the amount of vaccination for a childhood only program, the amount by which childhood-only vaccination will be offset by adult vaccination, and the outcome dependence upon the vaccine efficacy and vaccine immunogenicity. His analysis shows that eradication of HPV is feasible, childhood vaccination programs should be supplemented by adult vaccination and adult vaccination is actually more efficient, but less logistically likely. In addition, he shows that there is a critical vaccine efficacy (82%) below which eradication is not possible, and there is a critical vaccine immunogenicity (83%) below which even 100% childhood vaccination cannot eradicate the epidemic. As 77% of parents were in favour of vaccinating their children, which is less than required for eradication ($> 85\%$) if only children are to be vaccinated, he concluded that voluntary adult vaccination should be covered by Canadian health care.

The Saturday afternoon session began with group discussions, followed by three breakout parallel sessions to identify rapporteur and use powerpoint template to answer distributed questions in order to develop ideas for research projects and networks. The whole group then met again on Sunday morning to hear reports from the breakout group discussions, and to finalize the plan to identify a focal point, explore study design, determine next steps and set a follow-up teleconference date. Three projects/themes were identified and are expected to be followed up closely: the Vulnerable Populations & Co-infections; Immune Issues in Modeling STI Modelling; Public Health Interventions. For each of these themes, the following were identified: Public health issues with high priority for modeling; research objective and work plan; Current controversies and policy questions; Current data/surveillance resources; Current work in progress/Existing models; Barriers to modeling, dissemination, or knowledge translation; Key partners; Potential funding agencies; and detailed work plans. There were a few themes that participants felt should be addressed further, but did not receive sufficient discussions due to the tight schedule, these include issues related to blood-borne infections, viral STIs and genital ulcerative disease migration. Blood-Borne Infections, Viral STIs and Genital Ulcerative Disease Migration.

Overall, it was a quite productive symposium that sets a valuable precedent for interaction between mathematicians and non-mathematicians in meeting current public health challenges in Canada. The symposium and its follow-up activities are expected to help to build a strong group as a major public health asset in this country, and this is likely to inspire similar efforts elsewhere.

The workshop was held shortly after the International STD Research Meeting in Seattle (July 29-Aug 1, 2008) to ensure maximal impact of the workshop on the Canadian policy towards the study of sexually transmitted and blood-borne infections.

One of the objectives of the workshop was to form a national focus group on modeling sexually transmitted and blood-borne infections that is capable of making scientific recommendations to governments at provincial and national levels in the subject area. An immediate goal of this focus group is to develop research proposals for funding support from Public Health Agency of Canada and make recommendation to funding agencies about the role of mathematical modeling for the management of sexually transmitted and blood-borne infections.

The symposium brought together a group of modellers and medical experts in sexually transmitted diseases with colleagues involving public health policy study and surveillance design from relevant Canadian provoncial and federal agencies to identify issues of great significance to the public health of Canadians, and to develop opportunities for interdisciplinary collaboration.

We believe the one of the objectives of this workshop, to form a national focus group on modeling sexually transmitted and blood-borne infections that is capable of making scientific recommendations to governments at provincial and national levels in the subject area, was very well achieved. In fact, such a focused group also identified research projects that will lead to proposals for funding support from Public Health Agency of Canada and make recommendation to funding agencies about the role of mathematical modeling for the management of sexually transmitted and blood-borne infections.

The great facility already in place at BIRS facilitated communication between group members of markedly differing backgrounds, to a degree that enhanced the success of the symposium. The quality of the meeting and beauty of the setting led to enthusiastic support for subsequent meetings, which will ensure the long-term impact of this gathering. We hope to have your continuing support for this growing group, as this meeting set a valuable precedent for interaction between mathematicians and non-mathematicians in meeting current public health challenges in Canada. The group will be seen as a major public health asset in this country, and is likely to inspire similar efforts in elsewhere. Topics covered by the general and group discussions included: Overview of the Field, Recent Developments and Open Problems, Presentation Highlights, Scientific Progress Made, and Outcome of the Meeting.

List of Participants

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Chapter 47

Intuitive Geometry (07w2144)

Aug 31 - Sep 02, 2007

Organizer(s): Ted Bisztriczky (University of Calgary), Gabor Fejes Toth (Alfréd Rényi Institute of Mathematics), Ferenc Fodor (University of Szeged), Wlodzimierz Kuperberg (Auburn University)

Summary

This two-day workshop was organized to provide a much desired opportunity to share research findings in the interconnected fields that are represented in Intuitive Geometry. The term *Intuitive Geometry* was coined by László Fejes Tóth to denote those geometric disciplines in which the unifying theme is that their problems themselves can be explained fairly easily, even to an advanced high school student, however, the solution of these problems require diffucult and very deep methods of modern mathematics. This Workshop is also part of a series of Intuitive Geometry conferences the first of which was organized in 1975 in Tihany, Hungary, and the last one was in 2000 in Balatonföldár, Hungary. This workshop was the sixth such meeting. The Intuitive Geometry Workshop was immediately followed by the Intuitive Geometry Day in Calgary held at the Department of Mathematics and Statistics of the University of Calgary. The Intuitive Geometry Day was a direct continuation of the Intuitive Geometry BIRS workshop. Its main purpuse was to provide an extension to the BIRS event and thus make attendance of the workshop more desirable to colleagues from overseas. In this regard, the event was a great success, out of the 30 participants 11 were from outside North America. The 30 participants of the meetings gave 24 high quality research talks on their recent results of which 16 were 30-minute and 8 were 20-minute presentations. Subject of talks covered the broad areas of general convexity, iterative geometric processes, the theory of packing and covering both in Euclidean and hyperbolic spaces, polytopal approximation of convex bodies, Minkowski geometry, combinatorial geometry, the theory of geometric transversals, extremal problems for convex sets, and abstract and convex polytopes.

The workshop was a resounding success, it brought together researchers from many different fields of Geometry, and among them, 3 advanced graduate students and several postdoctoral fellows. New collaborations among participants are already noticable, especially among the graduate students and postdocs.

Results presented at the Workshop and the Intuitive Geometry Day in Calgary will be published in a special Intuitive Geometry volume of the journal Periodica Mathematica Hungarica. In summary, the future directions for research in Intuitive Geometry are plentiful and the area is very much alive being a central part of modern geometric research.

The Intuitive Geometry Day in Calgary was generously supported by the Pacific Instutite for the Mathematical Sciences, the Faculty of Science, and the Department of Mathematics and Statistics of the University of Calgary.

List of Participants

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Chapter 48

Mathematical Modelling of Water Resource Allocation Strategies (07w2003)

Sep 07 - Sep 09, 2007

Organizer(s): Collins Ayoo (University of Calgary - Economics), Ted Horbulyk (University of Calgary - Economics), Marian Weber (Alberta Research Council)

LESSONS LEARNED

The workshop provided the participants with an opportunity to discuss a diverse range of issues in the area of water allocation models and to share experiences and perspectives from different jurisdictions. From the presentations and discussions some of the key lessons learned were:

1. That more research needs to be done in the area of water economics to provide answers to questions such as the optimal allocation patterns, the choice of irrigation technologies, the prospects of using water pricing and trading to address water scarcity, the role of uncertainty in water allocation decisions, and how to build intertemporal economic optimization models.

2. The type of model used depends on what kind of policy issue is being investigated. Several management questions can be addressed using a hydro-economic model like CALVIN or a modified WRMM. These include:

a. Testing the welfare effects of changing water institutions (e.g. introducing water markets, water pricing, and water banks);
b. Evaluating the social and private benefits of storage expansion/contraction and who should pay;
c. Evaluating the benefits of conjunctive management.
d. Evaluating the costs of instream flow objectives and constraints

3. That there is considerable scope for collaboration in research in which the different participants can make valuable contributions according to their unique strengths and comparative advantage. This however requires a shared solution-driven research focus.

4. The WRMM model developed by Alberta Environment is capable of being adapted to use an explicitly economic objective function with multi-period solution values, perhaps augmented by a nonlinear optimiza-

tion algorithm. This model is already available in the public domain for use by diverse researchers, but since there is no firm budgetary commitment or timeline to install these enhancements, it is not clear how the potential power and functionality of this modeling platform can be fully realized in the near term.

5. That a system needs to be developed to assemble detailed data on various aspects of water use and management. The data needs to be collected and organized in a way that it can be shared. Similarly a mechanism needs to be developed for sharing modeling approaches and to harness the complementarities between the models being developed by the various participants.

6. That links need to be established between researchers and policy makers to ensure that the research being carried out addresses important practical questions and contributes in concrete ways to an improvement in water management.

INTRODUCTION

The workshop Mathematical Modeling of Water Resource Allocation Strategies was held from September 7-9, 2007 at the Banff International Research Station for Mathematical Innovation and Discovery. The purpose of the workshop was to bring together a small group of researchers and government and industry participants who are active in the use or development of water policy models to understand the state of the art methods in water policy modeling and to explore opportunities to advance water economics research in Alberta. The participants were drawn from both the United States and Canada. The specific objectives of the workshop were:

1. To identify the principal modeling groups and approaches currently active on Alberta water issues and to identify the potential for diverse types of collaboration among the researchers;

2. To understand the current state of practice and the current state of the art in quantitative water modeling including alternative applications, techniques and methodologies (e.g., scope and scale - farm-level to basin-scale, model structures, assumptions, calibration, data requirements, software), strengths and weaknesses, and to identify potential areas for improvement in future work;

3. To identify whether there are relevant approaches and lessons from other jurisdictions about processes for interdependent quantitative analyses, including collaboration in such areas as data collection and data sharing, model verification and validation;

4. To identify the areas of opportunity for broadening or coupling economist's models of water quantity and quality to other physical models of land use policy and land use change, for example, and /or for introducing or integrating economic analysis to existing physical models.

The workshop was organized into five sessions with each session involving a presentation or set of presentations followed by questions and discussions by participants. The first four sessions were held on Saturday and explored various dimensions of water policy modeling. The fifth session was held on Sunday and focused on a series of questions raised by the topics covered on Saturday. This was followed by a discussion of opportunities to advance water policy modeling in Alberta. The next section of this report summarizes each of the five sessions without attempting to record or to present fully the wide range of technical issues and discussion surrounding each session.

PRESENTATIONS

Session (A): State of the Art Approaches to Water Modeling

This session was led by Prof. Richard Howitt, Department of Agricultural and Resource Economics, University of California, Davis. The title of Richard's presentation was State of Art Approaches to Water Modeling. The presentation provided an extensive survey of computational techniques and approaches. Some important points in Richards presentation were:

- That both simulation and optimization water models have a role - often complementary. Simulation models help us understand the physical properties of the system while optimization models are necessary to evaluate the costs and benefits of policy options such as water pricing and trading, establishing instream flow objectives, investment in storage capacity, and conjunctive management. Where values, preferences, or the resource base is changing, optimization will have a greater role because of the need to evaluate the public and private values from new management approaches.

- That both spatial and intertemporal allocation problems are important, but usually have to be solved at different scales due to complexity.

- That inductive parameter estimation is theoretically preferable, but calibration methods may yield better disaggregated fits for a given data set.

- That large hydroeconomic models are generally easier to run and debug if they are constructed as modular, but formally linked, units rather than fully integrated models. There is greater flexibility in terms of questions that can be addressed, and it is easier to manage model development. Someone needs to be responsible for integration i.e., ensuring that data inputs and outputs are compatible.

Session (B): Modeling Agricultural Water Use

This session was led by Prof.Chokri Dridi, Department of Rural Economy, University of Alberta. The title of Chokris presentation was Water Modeling Challenges: Irrigation Technology Choice and Political Economy of Water Pricing. Chokri discussed modeling approaches that involve farm heterogeneity and technology choice. Many mathematical models used in water economics are principally planning models based on assumptions about profit maximizing behavior of representative agents. Another class of models considers the individual behavior of farmers, recognizing that farmers are heterogeneous and may use private information about their farm characteristics and practices to behave strategically. These models can be used to understand policy feedbacks including the heterogeneous response of producers to policy variables as well as behavior that influences the design of policy. Some of the main points made in this session were:

- Pricing and trading influences the adoption of better irrigation methods and the retirement of less efficient lands for water conservation.

- That the political economy dimension of water pricing is an important factor in the success of water pricing reforms. In particular, volumetric water rates may be subject to manipulation by producers who determine whether or not the members of the Water Users Association are re-elected. The final water pricing structure depends on farm characteristics.

- Modeling these types of problems for a basin requires disaggregated approaches that consider individual water consumers and their interactions (for example agent-based approaches). These types of modeling approaches are complementary to the hydro-economic approaches discussed above. Outcomes from the agent based models can be fed into hydro-economic models to better understand policy outcomes.

Session (C): Empirical Issues in Water Value Estimation and/or Model Calibration

This session was led by two presenters, Professors Diane Dupont and Steven Renzetti, both of the Department of Economics, Brock University. Their presentation was titled, Canadian Water Valuation and Demand Modelling: State of the Art and Future Directions. Diane's presentation focused on valuation and its role in water management. She noted that historically there have been high rates of water consumption in Canada due to low prices and lack of metering, and that valuation is important if efficient water use decisions are to be made. Some of the other points that she made were:

- That the valuation of water is problematic because many goods/services provided by water were not bought or sold in markets. The total economic value approach could however be employed to derive estimates of the value of water. This entails aggregating the use, non-use/passive and existence values. The various values can be captured. Non market valuation techniques can be used to capture these values.
- That since consumers typically combine water with other market goods to provide useful services, use values can be revealed indirectly by observing consumers' choices of market goods and water.
- That to capture non use/passive use and existence values an artificial market can be constructed that describes change in water services/attributes. By directly asking respondents to state their preference for change or status quo, values can be inferred through stated preferences.
- Indirect and direct methods can be used to value water services. Indirect services include the travel cost method, hedonic price method and the defensive expenditure approach.
- That the direct methods that can be used to value water services are referendum style CVM, and attribute-based stated choice methods (ABSCM) or choice experiments (CE).
- That there are very few current Canadian studies that have employed modern nonmarket valuation techniques to estimate the value of water services.
- That context is very important in carrying out nonmarket valuation studies.
- That there are big knowledge gaps in areas such as passive use values, the values of ecosystem services and how values are affected by user characteristics. Although benefit transfer has been widely used to estimate the values of ecosystem services, considerable care needs to be exercised in its use since values are to a large extent context dependent. Discussion after Dianes presentation highlighted the importance of getting some of these parameters into hydro-economic models being used to evaluate policy.

Prof. Steven Renzetti continued the presentation by discussing water demand modeling. He pointed out that there are few studies of Canadian residential water demands due to limited data and the governments low demand for specific parameter estimates as part of their own policy analyses. The few available studies were based on single equation models, many of which do not adequately address price endogeneity and are now out of date. However, based on US studies it could be inferred that water demands were generally inelastic but more elastic for outdoor use. With regard to industrial demand, microdata from the Industrial Water Use survey are available to support studies that use the KLEMW cost functions model. Studies on industrial water demand show that the price and output elasticities were higher in the manufacturing sectors than in the industrial or agricultural sectors.

As for agricultural demands: lack of data on water use and lack of volumetric pricing makes it difficult to understand producer behavior in response to increased water scarcity or water costs. Most studies on agricultural water demands in Canada have used the engineering/ agronomic approach where water demand is determined by crop type, precipitation, temperature, soil type and irrigation system efficiency. This is the approach currently used in the WRMM irrigation demand model. Some drawbacks of this approach are that output choices and investment are not modelled so it only captures changes in water use at the intensive margin, rather than considering how water use also changes at the extensive margin through changes in land use. A behavioral approach would examine the farmers choice of inputs and outputs as a function of prices and preferences. A greater level of responsiveness would be captured since more substitution among inputs and land uses would be allowed. Finally, assumptions about profit maximizing behavior could be tested rather than assumed. The engineering and behavioral approaches should be treated as complements for each other since the weaknesses of one approach tend to be the strength of the other. Some future directions in water demand modeling that Steven identified were estimating residential demands using microdata and applying more advanced econometric techniques to estimate these demands. Discussion after Stevens presentation highlighted the need to collect better information about agricultural water use.

Session (D): Collaborative Modeling Exercises

This session was led by Prof. Jay Lund, Department of Civil and Environmental Engineering, University of California, Davis. Jays presentation was titled Insights from Optimization: In and Out of the Ivory Tower. Jay began his presentation by discussing why we do computer modeling. Modeling is used to:

- integrate empirical and deductive knowledge
- generate complex testable hypotheses;
- improve and test intuitive understanding
- identify gaps in our understanding
- explore and compare solutions to problems
- avoid costly trial and error associated with testing policy in the field; and
- reduce uncertainty and provide assurances.

In sum, the value of collaborative modeling efforts between economists and engineers is that they can help one structure complex problems, identify important gaps, explore novel solutions, demonstrate detailed thought, and provide and support insights that would be otherwise unavailable for complex systems. These models also allow us to explore complexities beyond our intuitive limits.

Professor Lund then discussed the CALVIN model which is an economically driven optimization model of Californias water supply. He pointed out that the CALVIN model includes both surface and groundwater systems, urban and agricultural values of water, operating costs, and environmental flow constraints, and prescribes a monthly operation system over a 72-year representative hydrology. Important insights from CALVIN for Californian water policy include:

- water markets lead to large efficiency improvements.
- storage expansion in California was less valuable than conveyance expansion.
- groundwater and conjunctive use has large benefits
- Dismantling the OShaughnessy dam would not significantly change streamflows and result in environmental benefits because users would consume from substitute sources.

Session (E): Where to from here?

The session was introduced by reference to the following questions posed by the organizers and added to from the floor:

1. Would water policy research/researchers gain from having access to a common integrated modeling platform, such as augmented versions of WRMM, WUAM, or CALVIN? If so, which one, and why?
2. What about a common modeling platform at the farm-scale or cellular automata level?

3. What are researchers data needs, data gaps, and data sharing possibilities?
4. What processes, funding, approaches and coordination could best make this happen?
5. What about the potential for greater interaction with scientists and researchers in other disciplines whose research methods, models and data may be relevant?
6. What are the three top policy questions regarding water use and management in Alberta that we need to be addressing in our modeling work?

After clarification of the questions, the discussion focused on Questions 1, 3, and 4.
Question 1 -

- Most of the discussion focused on question 1. Participants noted that lack of an optimization model for Alberta/Canada is a gap.
- Discussion focused on the Water Resource Management Model (WRMM) developed by Alberta Environment and whether or not it could be modified as an optimization model. It was pointed out that the WRMM was originally developed as a planning tool for surface water resources utilization, with the river basin being the unit of analysis. WRMM has the capability of computing a steady state water balance over a sequential period of user-defined time steps.
- Conclusion - WRMM is not an economic optimization model but could easily be modified since it currently assigns water based on priority when constraints are binding. Priorities are assigned using penalty functions which could be converted into water demand functions without changing underlying model structure. Water demands would come from other modules (similar to how the irrigation demand model and WRMM currently interact).
- The discussion of the WRMM was followed by a discussion of the Water Use Analysis Model (WUAM) developed by Environment Canada. The WUAM is an interactive computer simulation model designed primarily to provide projection of multisector water uses in a drainage basin context. Water use in the model is a function of price, so it is possible to test the effects of changes in price on water balance in the basin. However WUAM is not an optimization model. Therefore it is not possible to identify the costs and benefits of management options, nor could the model be modified to function as an optimization model.

Question 2 In the second half of this session the participants discussed the data problems that are commonly faced in modeling water allocation issues. Concern was expressed about data gaps, data reliability, and accessing the data that were available. The participants also noted the need for detailed documentation of data (metadata) so that the users of the data know the details about how the data were gathered and the manipulation carried out on the data. Possible micro data sources for agriculture were noted, including data held by the irrigation districts. Various individuals agreed to follow up with David Hill on this.

Question 4. Participants would like to collaborate but not formally. David Hill noted that stakeholders should be part of research design and that poor communication between science community and stakeholder groups is a problem.

ACKNOWLEDGMENT

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- The Banff International Research Station for Mathematical Innovation and Discovery
- Alberta Environment
- Alberta ingenuity Centre for Water Research and
- Institute for Sustainable Energy, Environment and Economy at the University of Calgary.

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Chapter 49

Affine Schubert Calculus Workshop: Design and Implementation of Research Tools in MuPAD-Combinat (07w2157)

Sep 14 - Sep 16, 2007

Organizer(s): Anne Schilling (University of California, Davis), Nicolas M. Thiery (Univ Paris-Sud)

Background

This meeting was part of the NSF Focus Research Group DMS-0652641:

”Affine Schubert Calculus: Combinatorial, geometric, physical, and computational aspects”

This project concerns the development of a vast extension of Schubert calculus to affine Grassmannians and affine flag varieties, called ”affine Schubert calculus”. Classical Schubert calculus, a branch of enumerative algebraic geometry concerned with counting subspaces satisfying certain intersection conditions, is the outcome of the solution to Hilbert’s Fifteenth problem. In the modern formulation, Schubert calculus is usually interpreted in cohomology theories of homogeneous spaces, most notably flag varieties. The full development of affine Schubert calculus will solve long-standing open problems in Macdonald theory and have an impact on physical questions, such as generalizations of Wess-Zumino-Witten conformal field theory models and extensions of Calogero-Sutherland quantum mechanical models whose eigenfunctions are k-Schur functions. The new approach to affine Schubert calculus is made possible by the recent discovery of certain explicitly defined symmetric functions called k-Schur functions. The k-Schur functions, which arose in the study of the seemingly unrelated Macdonald theory, were recently shown to be connected to the geometry and topology of the affine Grassmanian. The novel combinatorics of k-Schur functions can be exploited to deduce formulae for various multiplicities, including intersection multiplicities in the affine Grassmannian and the affine flag manifold. Some of these multiplicities are known to occur in Macdonald theory and as Verlinde fusion coefficients for the WZW model.

This many-faceted project involves and ties together various problems from combinatorics, geometry, representation theory, physics, and computation. The main questions to be addressed can be viewed from several points of view: a geometric perspective (questions such as ”how many lines are there satisfying a number of generic intersection conditions?”), a combinatorial perspective (”how many elements are in given sets and what properties do these sets have?”), a physics perspective (”how do fields correlate?”), and computational aspects (”are there efficient algorithms for calculating these numbers or objects?”).

The project is an international cooperative research venture, with core group members located in Canada, the United States, Chile, and France, and interdisciplinary, involving mathematicians, physicists, and computer scientists.

The investigation is largely fueled by extensive computational experimentation. The robust implementation of algorithms derived from the project, is leading to the development of major extensions to the algebraic combinatorics open source project *-Combinat. It has a major impact on this project's life and growth. Reciprocally, the dissemination of this new software through an open-source development model, not only advances the proposed research program but also has an outreach impact on the mathematics, physics, and computer science communities.

Purpose and outcome of the workshop

This meeting served as a start-up meeting for the participants of the FRG ”Affine Schubert Calculus”, focusing on computational aspects of the project. All participants attended the BIRS workshop 07w5048 ”Applications of Macdonald Polynomials” on September 10-14 2007 prior to the meeting.

The MuPAD-Combinat developers Florent Hivert, Nicolas Thiéry and Francois Descouens presented the main features of the existing package MuPAD-Combinat and introduced the participants to the technical details of the programming language and the distributed development tools (svn, wiki, etc.). All participants wrote small combinatorial algorithms appearing in their research as an exercise and committed them to the svn repository. For example, affine Stanley symmetric functions were implemented for types A and C. Further work was done on root systems, the k-shape poset, and the integration of the graph drawing tool graphviz.

In addition, the meeting served for discussions and code brainstorm about the design, development schedule, implementation, and integration of the required research tools:

- Kac-Moody algebras
- Representation theory (how to interact with GAP for example)
- k-Schur theory
- Integration of external C/C++ programs ...

The future of the software and the MuPAD platform was also largely discussed. This was an important step in the later decision (June 2008) of migrating from MuPAD-Combinat to Sage-Combinat. Sage is an open-source mathematics software package developed by a worldwide community of over one hundred people. It was started in 2005 by William Stein (now at the University of Washington) and it consists of over two million lines of code. It incorporates several of the best free, open-source mathematics software packages available (GAP, Singular, Macaulay, GMP, MPFR just to name a few), as well as a huge original library, including several new algorithms not yet found elsewhere (for more information on sage, visit www.sagemath.org). The merging of all code from MuPAD-Combinat to Sage-Combinat is in progress.

List of Participants

Bandlow, Jason (University of California, San Diego)
Descouens, Francois (York University)
Hivert, Florent (University of Rouen)
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Zabrocki, Mike (York University)

Focused Research Group Reports

Chapter 50

The Xi-transform (07frg138)

Mar 25 - Apr 01, 2007

Organizer(s): Maciej Dunajski (Cambridge University), George Sparling (University of Pittsburgh)

Most of the participants of the meeting had strong background in conformal geometry in one form or other. The Ξ–transform of Sparling was a unifying theme, and a lot of progress has been made in understanding the geometry of ordinary differential equations either in twistor terms, or in the language of Cartan connection.

During the meeting I made progress towards understanding the metricity of projective connection. The discussions with Gover were most useful. I have also made some connections between the theory of ASD structures in neutral signature, and the systems of second order ODEs. This is likely to result in a collaborative work with Doubrov and Grant.

The following technical comments come directly from the participants.

Maciej Dunajski

BORIS DOUBROV.

- Links between geometry of ordinary differential equations, twistor theory and conformal structures.

  According to the work of Daniel Grossman systems of 2 second order ODE's with vanishing generalized Wunschmann (or Wilczynski) invariant have a remarkable property. Their solution space has a natural anti-self dual conformal structure of split signature. This has been discussed with Maciej Dunajski with a number of explicit examples constructed. There seem to be an ultimate relation between solutions of heavenly equation and ODE's with certain higher order invariants equal to 0.

  According to works of Ezra Newman, general conformal structures of signature (1,3) and (2,2) naturally appear on solution spaces for two second order PDE's on one function of two variables with some invariants equal to 0. There seem to be a direct link between these invariants and classical projective differential geometry of surfaces in RP^3. More research is required in this direction.

- Rational curves in parabolic geometries.

  We discussed with Ben McKay how rationality of certain curves in Cartan geometries implies vanishing of certain invariants and often the flatness of the connection. The role of sl_2 subalgebras in the model algebra and various technical computational aspects were discussed. One of the open questions is how McKay's results on parabolic geometries can be generalized to more general non-parabolic cases.

- Cartan connections associated with overdetermined systems of differential equations.

  There was a number of fruitful discussions with Rod Gover on the Cartan geometries associated with (over)determined systems of finite type differential equations (either ordinary or partial). There is a number of partial results in this direction, but no complete picture up to now. There are two natural parts of this problem: construction of the flat Cartan connection in case all compatibility conditions are satisfied, and interpreting compatibility conditions in terms of the curvature tensor of non-flat Cartan connection.

- Old and new version of Fefferman construction.

  An excellent presentation of Goerge Sparling on his vision of Fefferman constructions shows that there should be apparent generalizations of this construction for more general structures given by the identification of the tangent vector bundle with

  1. tensor product of two vector bundles (so-called almost Grassmanian structures)
  2. skew-symmetric square of a fixed vector bundle (so-called almost spinorial structures)

  According to George Sparling, the reduction of these structures by an infinitesimal symmetry has a striking similarity with the classical Fefferman construction. This has also been discussed with Maciej Dunajski in case of reductions of (2,2) conformal structures and their reconstruction from reduced data.

ROD GOVER. Differential equations and their solution spaces determine geometric structures. In some cases these structures completely characterise the equation and/or its solution space. One effective approach to extracting this data and invariants thereof is via Cartan connections or tractor connections. These are connections on prolonged differential systems, the differential system involved can come from the original equation itself or, in another direction, may be derived from the solution space of this. These techniques and comparisons of approaches was examined and also discussed with the participant Boris Doubrov. Another application looked at (and discussed with Maciej Dunajski) was the construction of invariants obstructing affine connections from being projectively related to a Levi-Civita metric connection. There is an established tractor connection related to this problem and this proved effective in recovering invariants in low dimensions. It remains to be seen if the vanishing of these invariants is sufficient to solve the problem. Further progress was made in a study of the solution space structure (and related issues) for linear operators which arise from a composition of commuting linear operators. This was subsequently publicly archived at arXiv:math/0701377.

JAMES GRANT.

- M. Dunajski and I completed the final version of our paper "Multidimensional integrable systems from deformations of Lie algebra homomorphisms" (joint with I.A.B. Strachan), which has now been submitted for publication.

- Based upon work that I presented in my talk at BIRS, I had conversations with M. Dunajski concerning the possibility of finding projective structures on hyperbolic monopoles moduli spaces. Such structures may allow an approach to the scattering theory of hyperbolic monopoles, and is a possible direction for future collaborative research.

- Based upon remarks of B. Doubrov after the talk of M. Dunajski, I became aware of the work of Grossman concerning the connection between anti-self-dual conformal structures of split signature on four-manifolds and equivalence classes of pairs of second order ordinary differential equations under point transformations. During the time at BIRS, I had several conversations with M. Dunajski and B. Doubrov concerning this correspondence, and carried out initial calculations investigating the ordinary differential equations that correspond to hyper-symplectic and Ricci-flat anti-self-dual four-manifolds. At this moment, I am investigating the symmetries of the equations governing hyper-symplectic structures, and the corresponding transformations that these symmetries generate on the space of second order differential equations modulo point transformations.

GEORGE SPARLING.

- Non-traditional applications of twistor theoretic ideas: integral geometry, projective structures, mini-twistor spaces, dispersion free equations, ODE's, perhaps leading in particular to a general theory of ODE's where one can trace the full structure through from the most elementary considerations: $F(x, y; a, b, c, ..) = 0 \to ....?$

- Applications of the same to such areas as the ODEs associated to null hypersurface twistor curves in space-time.
- Understanding of Fefferman-type constructions: conformal holonomy characterizations of Fefferman structures; generalized such structures used in connection with third order ODEs with non-vanishing Wunschmann invariant and in connection with the applications of the Xi-transform in general relativity. Setting of Fefferman-type theory in the context of parabolic geometry.
- That non-traditional signatures such as (2, 2) (self-dual Yang–Mills, gravity, Radon transform, etc), the Xi-transform (3,3) and (4, 4), F-theory (2, 10) or (3, 9) have a vital role to play.

LOUIS BOYA. The meeting was most interesting. I was prepared to discuss topics of two-times physics in relation to my own work on F-Theory. When preparing my talk, I kept in mind that F-theory is not very popular, and probably unknown to most mathematicians. In the talk I emphasis ed the current difficulties wit string theory with not many new results coming out of it, and how F theory may posssibly may provide new impact.

I interacted with Hao Xing whose research interests were similar to mine. The atmosphere was very congenial, and I have had ample discussions with most of the participants. I learned about the Xi transform, and how it lead to a model of six dimensions with split signature (3, 3) whereas in F-Theory we are dealing with 12 = (2, 10) signature. I especially enjoyed the talks by the organizers, G. Sparling and M. Dunajski, as well as some mathematicians , in particular Ben McKay.

Incidentally, I managed also to attend some talks on a parallel meeting on pure mathematics (low-dimensional topology). I do not have to talk about the surroundings, which were marvelous, with realistic possibilities of trekking and skiing. I think I was just too fortunate to be involved and being able to attend.

BEN MCKAY. During the Xi Transform workshop, I learned a lot about the geometry of ordinary differential equations, particularly from the talk of Boris Doubrov, who explained the relationship between the Wilczynski theory of linear ordinary differential equations and the work of Tresse, Wunschmann and Cartan on nonlinear ordinary differential equations. Doubrov also gave an outline of his approach to finding differential invariants for systems of higher order equations under arbitrary coordinate transformations, following a surprisingly strong analogy with the theory of second order equations and parabolic geometries. I thought some more about the problem of finding and classifying complex 2nd order ODEs whose generic solution is a closed Riemann surface. I realized during the workshop that this problem has some subtleties related to KAM theory. I also came to appreciate from talking to Maciej Dunajski and George Sparling that there are some nice relations between ordinary differential equations and split signature pseudo-Riemannian 4-manifolds, which I had not previously considered. Unfortunately, as I was just about to discover a major theorem, my head exploded from eating too much.

HAO XING. During this workshop, I discussed the possible generalization of Chern-Simons action to the F-theory context. Comparing to the Chern-Simon action in M theory, which is calculated by index of Dirac operator on 11 dimensional spacetime, we propose to define the Chern-Simon action in F-theory by a loop index on the loop space of a 12 dimensional spacetime. I discussed the loop index theory discovered by Witten in late 1980's. Based on this definition of Chern-Simons action on F-theory, I also discussed the connection among F-theory, type IIA, type IIB string theory and M theory using generalized T-duality. Moreover, I explained the difference and connection between our F-theory with the orginal F- theory proposed by Vafa, which is an elliptic fibration over 10 dimensional manifold. This talk I gave in this workshop is based on a joint work with my advisor Igor Kriz.

Moreover, I learned a lot from other people in this workshop. Luis taught me a lot on F-theory, and I told him a connection between loop index and monster group. I learned what is anti–self–dual metric from Maciej and a lot of conformal and Cartan geometry from other people.

List of Participants

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Chapter 51

Stochastic models of influenza dynamics (07frg109)

May 06 - May 13, 2007

Organizer(s): Jonathan Dushoff (McMaster University), David Earn (McMaster University, Department of Mathematics & Statistics), Joshua Plotkin (University of Pennsylvania)

The recent workshop at BIRS offered us a fantastic opportunity for collaboration and focused, productive research. The workshop exceeded our expectations in terms of the breadth of the academic subjects we explored, and the collaborations we established.

A subset of our group has been collaborating for several years. We have used mathematical models to study the spread and evolution of influenza viruses. The purpose of this workshop was to attempt reconciliation of our stochastic models with empirical data on influenza epidemics; to examine alternative methods of fitting model parameters; and to continue a collaboration with Marc Lipsitch from the Harvard School of Public Health. We have progress to report on all of these goals.

During our workshop at Banff, we completed revising a manuscript that uses empirical data about influenza deaths in the US over the past century to highlight a theoretical puzzle about influenza persistence after a pandemic. The most basic, longstanding mathematical model of disease transmission divides the population into three classes (Susceptibles, Infectious, and Recovered/Immune individuals) and describes flow between these classes with a system of three ordinary differential equations. Given this standard model of disease, and given the empirical influenza epidemic curve and infection rates observed in the United States in 1918, we have estimated that a very large proportion of the population was infected (and thereafter immune) to the Spanish Flu of 1918. According to these estimates, only a very small proportion of the population remained susceptible to influenza after the pandemic - too small to support the initiation of another epidemic the following season. But the empirical data indicate that another influenza epidemic did indeed occur in 1919, which raises a theoretical puzzle. Our manuscript describes this enigma and offers several hypotheses for its resolution: the virus may have evolved to such an extent in 1918 that it could re-infect individuals in 1919; or the virus could have persisted in 1919 due to heterogeneities in the host population and ”pockets” of remaining susceptibles; or (perhaps most intriguing) the virus may have evolved a greater ability to spread so that it could persist in 1919, despite the small number of susceptible hosts to support it. Our manuscript does not attempt to resolve this enigma, but rather to describe how the puzzle arises from the combination of standard mathematical models and empirical data from the 1918 influenza pandemic.

The second major topic we discussed in our workshop involves the non-parametric inference of model parameters from empirical time-series data. Recent developments due to Wallinga and Teunis (2004) have allowed for direct estimates of a pathogen s reproductive number based only on a time-series of incidence counts. Unlike parametric fitting procedures, this approach is both elegant and widely applicable over a range of models. However, for pathogens like influenza we rarely have a time-series of infection events, but rather have only a time-series of death events. We spent a large amount of our time at the workshop studying how to modify the method of Wallinga & Teunis to accommodate a death time-series, instead of an incidence time-series.

One approach to this question has involved modifying the method of Wallinga & Teunis to deal with the distribution of times between one death event and another related death event. We have discovered that this approach suffers from several technical and one major conceptual difficulty. The conceptual difficulty is that a death event on day t may been caused from a transmission event that eventually led to a death event on day $t > t$. As a result, this modified version of Wallinga & Teunis's approach involved summations over both past and future events, wheras the original Wallinga & Teunis method was one-sided. This causes difficulty in many practical settings, and the sense in which the reproductive number on day t is estimated does not agree with the original Wallinga-Teunis method.

An alternative approach is to deconvolve the observed death timeseries, using knowledge of the incidence-to-death transition kernel, to impute an underlying incidence time-series which can then be analyzed by the original Wallinga-Teunis technique. In fact, we had considered this problem at a previous FRG only to find that iteration techniques for deconvolution seemed poorly behaved. In this FRG, however, we made substantial progress in deconvoluting epidemiological data by applying the Lucy-Richardson technique, which uses an implicitly Bayesian approach that guarantees positivity at every iterate. This algorithm has not previously been applied to epidemiological data, and clearly has important epidemiological applications (recovering incidence from mortality time series, or infection from onset etc), but it has important limitations. Although the algorithm works perfectly when data are measured without noise, applications to noisy data are more complicated involving heuristic choices to balance the smoothness of the deconvolved timeseries against the accuracy of the convolution. We are currently writing a manuscript that describes deconvolution, its relationship to Wallinga & Teunis, and its applicability to questions in disease dynamic modeling.

List of Participants

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Chapter 52

Global attraction to solitary waves in nonlinear dispersive Hamiltonian systems (07frg137)

May 20 - May 30, 2007

Organizer(s): Vladimir Buslaev (St. Petersburg University), Andrew Comech (Texas A&M University), Alexander Komech (University of Vienna), Boris Vainberg (University of North Carolina, Charlotte)

Overview of the Field

The long time asymptotics for nonlinear wave equations have been the subject of intensive research. The modern history of nonlinear wave equations starts in nonlinear meson theories [Sch51a, Sch51b]. The well-posedness was addressed in early sixties by Jörgens [Jör61] and Segal [Seg63a]. Later Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and local attraction to zero were considered.

Global attraction (for large initial data) to zero may not hold if there are stationary or *quasistationary* solitary wave solutions of the form

$$\psi(x,t) = \phi(x) e^{-i\omega t}, \qquad with \quad \omega \in \mathbb{R}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{52.0.1}$$

We will call such solutions *solitary waves*. Other appropriate names are *nonlinear eigenfunctions* and *quantum stationary states* (the solution (52.0.1) is not exactly stationary, but certain observable quantities, such as the charge and current densities, are time-independent indeed).

According to “Derrick’s theorem” [Der64], time-independent soliton-like solutions to Hamiltonian systems, under rather general assumptions, are unstable. On the other hand, *quasistationary* solutions may be stable. This is caused by the additional conservation laws, which may prevent a slightly perturbed solitary wave from tumbling in the direction of lower energy states. This stimulated the study of the existence and stability properties of solitary waves in the Hamiltonian systems with symmetries.

Existence of solitary waves was addressed by Strauss in [Str77], and then the orbital stability of solitary waves in a general case was proved in [GSS87]. The asymptotic stability of solitary waves was considered by Soffer and Weinstein [SW90, SW92], Buslaev and Perelman [BP93, BP95], and then by others.

The existing results suggest that the set of orbitally stable solitary waves typically forms a *local attractor*, that is, attracts any finite energy solutions that were initially close to it. Moreover, a natural hypothesis is that the set of all solitary waves forms a *global attractor* of all finite energy solutions.

Current State of Things: Example

We are interested in the stability properties of solitary waves and long-time asymptotics of finite energy solutions. This field remains very active for the last thirty years, and yet many questions are not understood. Let us illustrate the state of things on the example of the Nonlinear Klein-Gordon Equation:

NLKG: $$\partial_t^2 u(x,t) = \partial_x^2 u + F(u), \qquad x \in \mathbb{R}, \quad t \geq 0, \quad u(x,t) \in \mathbf{C}.$$

This equation describes oscillations of a string with certain elasticity properties (represented by a smooth function $F(u)$, "the nonlinearity"). The string is infinite and stretched along the x-axis. Real and imaginary parts of $u(x,t)$ are the y- and z-coordinates of the piece of the string above the point x at the moment t. We assume that F is smooth and satisfies $F(e^{is}u) = e^{is}F(u)$, $s \in \mathbb{R}$, so that the equation is $\mathbf{U}(1)$-invariant. For a particular $F(u)$, one would like to know:

1. *Are there solutions of the form $\phi_\omega(x)e^{-i\omega t}$ with $\phi_\omega(x)$ localized ("solitary waves")?*
2. *Which solitary waves are orbitally stable (so that small perturbations do not grow)?*
3. *Which solitary waves are asymptotically stable (so that small perturbations disperse)?*
4. *As $t \to \infty$, does* any *finite energy solution look like outgoing solitary waves?*

These most natural questions in the context of nonlinear dispersive $\mathbf{U}(1)$-invariant Hamiltonian systems are the central questions of the PDE Theory. It is mainly due to the research of W. Strauss and his school (since the seventies) that **Questions 1, 2** are understood rather well. In particular, it is known that for the same nonlinearity some solitary waves could be stable while others could be unstable.

The asymptotic stability (**Question 3**) could be viewed as the *local attraction* to solitary waves. At present, not much is known about asymptotic stability, especially in the translation-invariant case; only the nonlinear Schrödinger equation has been studied in the works by Buslaev and Perelman; this was developed by Cuccagna and others. (No results for NLKG as of yet.) The asymptotic stability is proved under very strong assumptions on the spectrum and the order of vanishing of the nonlinearity. At the same time, one expects asymptotic stability for any orbitally stable solitary wave.

The last question (**Question 4**) is about the *global attraction*. Namely, we would like to know whether the system has a *finite-dimensional attracting set*, and whether it is formed by solitary waves. Properties of global attractors are well-understood for the *dissipative* systems (such as Navier-Stokes equation). Yet, essentially nothing is known even for NLKG in 1D.

Both the asymptotic stability and the global attraction are based on the "disersive damping": the nonlinearity increases the spectrum of the perturbation, creating the dispersive waves that carry the excess energy and charge away. This mechanism (called "friction by dispersion") is still not described rigorously.

Presentation Highlights

The core of the focussed research group was two intensive lecture courses by Professors Buslaev and Vainberg. Vladimir Buslaev presented a series of lectures on completely integrable systems, concentrating on the example of the completely integrable cubic Schrödinger equation. Such systems are the only example of dispersive models where the solitary asymptotics are known.

Boris Vainberg presented a series of lectures on quasiclassical asymptotics. Our expectation is that these methods could be applied to solitary asymptotics, giving a very detailed description of higher density – low energy electron waves in the magnetic field, when the nonlinear quantum effects become important and Vlasov-type classical particle description is no longer applicable.

Both these courses were exceptional in addressing very deep aspects of the theory and also giving the hands-on experience with the main technical methods.

Andrew Comech gave a minicourse on convergence to solitary waves in nonlinear nonlintegrable dispersive models based on the Klein-Gordon equation. The convergence to solitary waves in these models is the only existing result when the convergence of any finite energy solution to one of solitary waves is caused by dispersion.

Besides the lecture courses, Alexander Komech, Elena Kopylova, Galina Perelman, and David Stuart presented the most recent developments in the theory of local and global attraction to solitary waves.

Scientific Progress due to the Focused Research Group

The discussions during the workshop resulted in an active research in the following directions:

1. Global attraction to solitary waves. The results on global attraction have been proved for several systems:
   (a) For the 1D Klein-Gordon equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillators [KK07a, KK08b];
   (b) For the nD Klein-Gordon equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillator, with the mean field interaction [KK07b, KK08a].
2. Asymptotic stability of solitary waves. The convergence to solitary waves for the slightly perturbed initial data has been proved for the following systems:
   (a) For the wave equation coupled to a particle [IKV07];
   (b) For 1D Schrödinger equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillator [BKKS08] with no nonzero discrete spectrum;
   (c) For 1D Schrödinger equation coupled to nonlinear $\mathbf{U}(1)$-invariant oscillator [KKS08] with a nonzero discrete spectrum.
3. Dispersive decay in weighted energy norm. Such a decay has been proved for 2D discrete Schrödinger and Klein-Gordon equations [KKV08].
4. Discussions in the workshop encouraged David Stuart to start work on the asymptotic stability of Ginzburg-Landau vortices in the Chern-Simons-nonlinear Schrödinger system, using some ideas developed in the context of nonlinear Schrödinger equations. However, in the Chern-Simons context there is a richer dynamical structure including nontrivial vortex interactions, which can be described precisely in the self-dual limit.

The above results were reported during the Miniworkshop *"Global Attractors in Hyperbolic Hamiltonian Systems"* at the 5th European Congress of Mathematics in Amsterdam, July 2008.

Two more topics were actively discussed during the workshop, and are the subject of ongoing research:

1. Stability of solitary waves in nonlinear Dirac equation. It was suggested (and now numerically confirmed [CC08]) that the small amplitude solitary wave solutions to the nonlinear Dirac equation in 1D are (generically) spectrally stable.
2. Wave-particle dualism and the solitary asymptotics in the cathode rays. The participants found a way to advocate the possibility of explaining the electron diffraction results (Davisson – Germer experiment) in terms of solitary wave asymptotics for Maxwell-Dirac system.

List of Participants

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Research in Teams Reports

Chapter 53

Graph Colouring Problems Arising in Telecommunications (07rit015)

Mar 18 - Mar 25, 2007

Organizer(s): Bruce Reed (McGill University)

The channel assignment problem in radio or cellular phone networks is the following: we need to assign radio frequency bands to transmitters (each station gets one channel which corresponds to an integer). In order to avoid interference, if two stations are very close, then the separation of the channels assigned to them has to be large enough. Moreover, if two stations are close (but not very close), then they must also receive channels that are sufficiently apart.

Such problem may be modelled by $L(p, q)$-labellings of a graph G. The vertices of this graph correspond to the transmitters and two vertices are linked by an edge if they are very close. Two vertices are then considered close if they are at distance 2 in the graph. Let $dist(u, v)$ denote the distance between the two vertices u and v. An $L(p, q)$-*labelling of* G is an integer assignment f to the vertex set $V(G)$ such that:

- $|f(u) - f(v)| \geq p$, if $dist(u, v) = 1$, and
- $|f(u) - f(v)| \geq q$, if $dist(u, v) = 2$.

As the separation between channels assigned to vertices at distance 2 cannot be smaller than the separation between channels assigned to vertices at distance 1, it is often assumed that $p \geq q$.

The *span* of f is the difference between the largest and the smallest labels of f plus one. The $\lambda_{p,q}$-*number* of G, denoted by $\lambda_{p,q}(G)$, is the minimum span over all $L(p, q)$-labellings of G.

Moreover, very often, because of technical reasons or dynamicity, the set of channels available varies from transmitter to transmitter. Therefore one has to consider the list version of $L(p, q)$-labellings. A k-*list-assignment* L of a graph is a function which assigns to each vertex v of the graph a list $L(v)$ of k prescribed integers. Given a graph G, the *list* $\lambda_{p,q}$-*number*, denoted $\lambda^l_{p,q}(G)$ is the smallest integer k such that, for every k-list-assignment L of G, there exists an $L(p, q)$-labelling f such that $f(v) \in L(v)$ for every vertex v.

The problem of determining $\lambda_{p,q}(G)$ has been studied for some specific classes of graphs (see the survey of Yeh [22]). Generalizations of $L(p, q)$-labellings in which for each $i \geq 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i, have also been studied (see for example [18] or [16]). Surprisingly, list $L(p, q)$-labellings have been studied only very litte explicitely and appear only very recently in the literature [14]. However, some of the proofs for $L(p, q)$-labellings also work for list $L(p, q)$-labellings.

Note that $L(1, 0)$-labellings of G correspond to ordinary vertex colourings of G and $L(1, 1)$-labelling of G to the vertex colourings of the square of G. Hence the $\lambda_{1,0}$-number of a graph G equals its *chromatic number* $\chi(G)$, and its $\lambda^l_{1,0}$-number eqauls its *choice number* $ch(G)$. The *square* of a graph G, denoted G^2, is the graph with vertex set $V(G)$ such that two vertices u, v are linked by an edge in G^2 if and only if u and v are at distance at most 2 in G. Formally, $E(G^2) = \{ uv \mid dist_G(u, v) \leq 2 \}$. Obviously, $L(1, 1)$-labellings of G correspond to vertex colourings of G^2. So $\lambda_{1,1}(G) = \chi(G^2)$ and $\lambda^l_{1,1}(G) = ch(G^2)$

It is well known that $\omega(G) \leq \chi(G) \leq ch(G) \leq \Delta(G)+1$, where $\omega(G)$ denotes the *clique number* of G, i.e., the size of a maximum clique in G, and $\Delta(G)$ denotes the *maximum degree* of G. Similar easy inequalities may be obtained for $L(p,q)$-labellings: $q\,\omega(G^2)-q+1 \leq \lambda_{p,q}(G) \leq \lambda^l_{p,q}(G) \leq p\,\Delta(G^2)+1$. As $\omega(G^2) \geq \Delta(G)+1$, the previous inequality gives $\lambda_{p,q} \geq q\,\Delta+1$. However, a straightforward argument shows that $\lambda_{p,q} \geq q\,\Delta + p - q + 1$. In the same way, $\Delta(G^2) \leq \Delta^2(G)$ so $\lambda^l_{p,q}(G) \leq p\,\Delta^2(G)+1$ and the greedy algorithm shows $\lambda^l_{p,q}(G) \leq (2\,q-1)\,\Delta^2(G) + (2\,p-1)\,\Delta(G)+1$. Taking a $L(\lceil p/k \rceil, \lceil q/k \rceil)$-labelling and multiplying each label by k, we obtain a $L(p,q)$-labelling. This proves the following easy observation.

Proposition 53.0.1 *For all graphs G and positive integers k,p,q we have*

$$\lambda_{p,q}(G) \;\leq\; k\,(\lambda_{\lceil p/k \rceil, \lceil q/k \rceil}(G)-1)+1.$$

In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [7]. In their seminal paper, Griggs and Yeh [9] observed that a greedy algorithm yields $\lambda_{2,1}(G) \leq \Delta^2+2\Delta+1$, where Δ denotes the maximum degree of the graph G. Moreover, they conjectured that this upper bound can be decreased to Δ^2+1.

Conjecture 53.0.2 ([9]) *For every $\Delta \geq 2$ and every graph G of maximum degree Δ,*

$$\lambda_{2,1}(G) \leq \Delta^2+1.$$

This upper bound would be tight: there are graphs with degree Δ, diameter 2 and Δ^2+1 vertices, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. Thus, their square is a clique of order Δ^2+1, so the span of every $L(2,1)$-labelling is at least Δ^2+1.

However, such graphs exist only for $\Delta = 2,3,7$ and possibly 57, as shown by Hoffman and Singleton [11]. So one can ask how large may be the $\lambda_{2,1}$-number of a graph with large maximum degree. As it should be at least as large as the largest clique in its square, one can ask what is the largest clique number $\gamma(\Delta)$ of the square of a graph with maximum degree Δ. If Δ is a prime power plus 1, then $\gamma(\Delta) \geq \Delta^2-\Delta+1$. Indeed, in the projective plane of order $\Delta-1$, each point is in Δ lines, each line contains Δ points, each pair of distinct points is in a line and each pair of distinct lines has a common point. Consider the *incidence graph* of the projective plane: it is the bipartite graph with vertices the set of points and lines of the projective plane, and every line is linked to all the points it contains. The properties of the projective plane implies that the set of points and the set of lines form two cliques in the square of this graph, and there are $\Delta^2-\Delta+1$ vertices in each.

Jonas [13] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits a $(2,1)$-labelling with span at most $\Delta^2+2\Delta-3$. Subsequently, Chang and Kuo [5] provided the upper bound $\Delta^2+\Delta+1$ which remained the best general upper bound for about a decade. Král' and Škrekovski [17] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [5], Gonçalves [8] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2+\Delta-1$. Note that Conjecture 53.0.2 is true for planar graphs of maximum degree $\Delta \neq 3$. For $\Delta \geq 7$ it follows from a result of van den Heuvel and McGuiness [10], and Bella et al. [3] proved it for the remaining cases.

Combining results obtained at the workshop with earlier work, Havet, Reed, and Sereini have shown that Conjecture 53.0.2 holds for sufficiently large Δ. I.e. they prove:

Theorem 53.0.3 *There is a Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$,*

$$\lambda_{2,1}(G) \leq \Delta^2+1.$$

This is one of the two main outcomes of the workshop, we now describe the second.

Because the transmitters are laid out on earth, $L(p,q)$-labellings of planar graphs are of particular interest. There are planar graphs for which $\lambda_{p,q} \geq \frac{3}{2}\,q\,\Delta + c(p,q)$, where $c(p,q)$ is a constant depending on p and q. For example, consider a graph consisting of three vertices x, y and z together with $3\,k-1$ additonal vertices of degree two, such that z has k common neighbours with x and k common neighbours with y, x and y are connected and have $k-1$ common neighbours (see Figure 53.1).

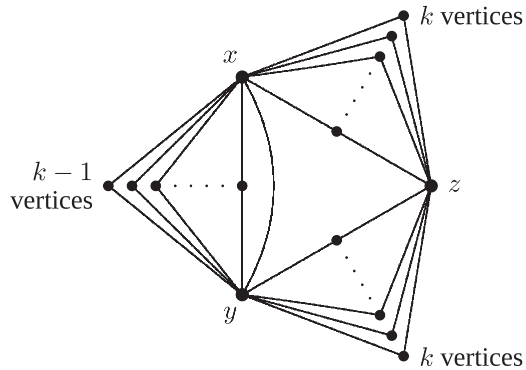

Figure 53.1: The planar graphs G_k.

This graph has maximum degree $2\,k$ and yet its square contains a clique with $3\,k+1$ vertices (all the vertices except z).

A first upper bound on $\lambda_{p,q}(G)$, for planar graphs G and positive integers $p \geq$ has been proved by Van den Heuvel and McGuinness [10] : $\lambda_{p,q}(G) \leq 2\,(2\,q-1)\,\Delta+10\,p+38\,q-24$. Molloy and Salavatipour [19] improved this bound by showing the following :

Theorem 53.0.4 (Molloy and Salavatipour [19]) *For a planar graph G and positive integers p, q,*

$$\lambda_{p,q}(G) \;\leq\; q\left\lceil \frac{5}{3}\,\Delta \right\rceil + 18\,p + 77\,q - 18.$$

Moreover, they described an $O(n^2)$ time algorithm for finding an $L(p,q)$-labelling whose span is at most the bound in their theorem.

The celebrated Four Colour Theorem by Appel and Haken [2] states that $\lambda_{1,0}(G) = \chi(G) \leq 4$ for planar graphs. Regarding the chromatic number of the square of a planar graph, Wegner [21] posed the following conjecture which is mentioned in Jensen and Toft [12, Section 2.18].

Conjecture 53.0.5 (Wegner [21]) *For a planar graph G of maximum degree Δ :*

$$\lambda_{1,1}(G) \;=\; \chi(G^2) \;\leq\; \begin{cases} 7, & \textit{if } \Delta = 3, \\ \Delta+5, & \textit{if } 4 \leq \Delta \leq 7, \\ \left\lceil \frac{3}{2}\,\Delta \right\rceil + 1, & \textit{if } \Delta \geq 8. \end{cases}$$

Wegner also gave examples showing that these bounds would be tight. For $\Delta \geq 8$, these are the same examples is in Figure 53.1.

Kotoschka and Woodall [15] conjectured that, for every square of a graph, the list-chromatic number equals the choose number. This conjecture and Wegner's one imply directly the following :

Conjecture 53.0.6 *For a planar graph G of maximum degree Δ :*

$$\lambda^l_{1,1}(G) \;=\; ch(G^2) \;\leq\; \begin{cases} 7, & \textit{if } \Delta = 3, \\ \Delta+5, & \textit{if } 4 \leq \Delta \leq 7, \\ \left\lceil \frac{3}{2}\,\Delta \right\rceil + 1, & \textit{if } \Delta \geq 8. \end{cases}$$

Wegner also showed that if G is a planar graph with $\Delta = 3$, then G^2 can be 8-coloured. Very recently, Thomassen [20] solved Wegner's conjecture for $\Delta = 3$ and Cranston and Kim [6] showed that the square of every connected graph (non necessarily planar) which is subcubic (i.e., with $\Delta \leq 3$) is 8-choosable, except for the Petersen graph. However, the 7-choosability of the square of subcubic planar graphs is still open. The first upper bound on $\chi(G^2)$ in terms of Δ was obtained by Jonas [13] who showed $\chi(G^2) \leq 8\,\Delta - 22$. This bound was later improved by Wong [**?**] to $\chi(G^2) \leq 3\,\Delta + 5$ and then by Van den Heuvel and McGuinness [10] to $\chi(G^2) \leq 2\,\Delta + 25$. Better bounds were then obtained for large values of Δ. It was shown that $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ for $\Delta \geq 749$ by Agnarsson and Halldórsson [1], and that $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ for $\Delta \geq 47$ by Borodin et al. [4]. Finally, the best known upper bound before the workshop was obtained by Molloy and Salavatipour [19] as a special case of Theorem 53.0.4 :

Theorem 53.0.7 (Molloy and Salavatipour [19]) *For a planar graph G,*

$$\lambda_{1,1}(G) \;=\; \chi(G^2) \;\leq\; \left\lceil \frac{5}{3}\,\Delta \right\rceil + 78.$$

As mentioned in [19], the constant 78 can be reduced for sufficiently large Δ. For example, it was improved to 24 when $\Delta \geq 241$.

Havet, McDiarmid, Reed, and Van Den Heuvel, combining results obtained at the workshop with earlier results, managed to prove:

Theorem 53.0.8 *The square of every planar graph G of maximum degree Δ has list chromatic number at most $(1 + o(1))\,\frac{3}{2}\,\Delta$. Moreover, given lists of this size, there is an acceptable colouring in which the colours on every pair of adjacent vertices of G differ by $\Delta^{1/4}$.*

As a corollary, for every planar graph G and any fixed p we get that $\lambda^l_{p,1}(G) \leq (1 + o(1))\,\frac{3}{2}\,\Delta(G)$.

Together with Proposition 53.0.1, this yields:

Corollary 53.0.9 *Let $p \geq q$ be two fixed integers. Then for any planar graph G we have $\lambda_{p,q}(G) \leq (1 + o(1))\,\frac{3}{2}\,q\,\Delta(G)$.*

Note that using exactly the same proof as for Theorem 53.0.8, one can

show that for any fixed $p \geq q$, for every planar graph G, $\lambda^l_{p,q}(G) \leq (1 + o(1))\,\frac{3}{2}\,(2\,q - 1)\,\Delta(G)$.

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Chapter 54

Bioeconomics of Invasive Species: Integrating Ecology, Economics and Management (07rit141)

May 15 - May 20, 2007

Organizer(s): Reuben Keller (University of Notre Dame), Mark Lewis (University of Alberta), David Lodge (University of Notre Dame), Jason Shogren (University of Wyoming)

Overview of the Field

Over the last seven years the National Science Foundation funded Integrated Systems for Invasive Species (ISIS) research group has worked to integrate economic, ecological and mathematical theory for responding to the risks from biological invasions. One of the products of this work will be an edited book, to be published by the Oxford University Press, titled "Bioeconomics of Invasive Species: integrating ecology, economics, policy and management". The four workshop attendees are the editors of this book, and the workshop at BIRS was to review and respond to draft chapters from other authors, and to write introductory and conclusion chapters. The workshop met each of these aims.

Background to project

Biological invasions occur when species are introduced to regions beyond their native range, establish large populations, and cause undesirable impacts to human health, the environment, or economic systems. Recent research reveals that biological invasions are one of the strongest drivers of global environmental change. Invasive species are now often in the public discourse. At the same time, economists have begun to take a real interest in determining how invasive species interact with economic systems, and how invaders should be controlled to optimize societal welfare. Although each of these bodies of work has greatly expanded our understanding of the impacts of invasions, little integration between economics and ecology has occurred that would allow managers and policy-makers to identify the optimal expenditures on, for example, prevention and control of invasive species. Because the level of effort expended on invasive species management is intricately linked to the costs and projected benefits of that management, there is an urgent need for greater synthesis between ecology and economics.

The ISIS group came together largely as a result of our common interest in the bioeconomics of invasive species in freshwater ecosystems. Many of us have worked in the Laurentian Great Lakes region, which is arguably the most studied ecosystem in the world with respect to the ways that economics and ecology combine to determine both the impacts of invasive species on society, and the responses of society to the risks and impacts from invaders. The first serious invader to enter the Great Lakes was the sea lamprey, which gained access after the Welland Canal was constructed. This new waterway was built in response to the economic imperative to allow oceangoing ships access to the lakes. Sea lamprey parasitize larger fish, and this led to the devastation of lake trout populations, which led in turn to declines in opportunities for recreational anglers, and the almost complete loss of the commercial fishery. Government agencies responded by searching for methods to limit the sea lamprey population, and poisons specific to sea lamprey have been developed. These are now applied to lamprey breeding streams at a cost of roughly $16million per year, paid by the U.S. and Canadian governments. In addition to sea lamprey, over 180 other species have invaded the Great Lakes, and their pathways of introduction can be linked to, for example, the aquarium, live food and nursery trades, ballast water of intercontinental ships and intentional release to enhance commercial and recreational fisheries. It is impossible to understand the invasion history of the Great Lakes without considering the economic history of the region. Indeed, the invasion history of all ecosystems is intricately linked to economic factors. Our book will be the first to present a unified bioeconomic approach for researching and addressing the problems of invasive species.

Our aim throughout the ISIS project has been to develop a synthetic ’bioeconomic’ research program for addressing the risks from invasive species. Mathematical biology has been at the heart of this effort for at least three reasons. First, mathematicians have conducted much of the most productive research on invasive species over the last several decades. For example, the development of integro-difference equation applications for predicting the spread of invasive species has contributed greatly to our understanding of species dynamics. These methods are thus essential to any genuinely synthetic approach. Second, although economists and ecologists routinely use complicated modeling approaches, members of either group are generally only familiar with the paradigms of their field. Hence, combining these models into a synthetic approach requires the broader perspective of mathematicians. Finally, both ecological and economic data contain large amounts of uncertainty, and there is a need for more advanced mathematical methods for explicitly dealing with this uncertainty.

We have a number of interconnected aims in preparing and publishing this book. First, we want to cement the role of bioeconomic research as the best approach to designing policy and management systems for invasive species. As society becomes more aware of the problems associated with global environmental change there is an increasing need for environmentally sound, and economically rational, responses. We present such a framework in this book for addressing the impacts of invasive species. No other book that we are aware of presents such an integrated framework for addressing an environmental problem. Hence, we believe that our book will inspire bioeconomic research on issues both within, and beyond, biological invasions.

Second, as well as providing the tools for bioeconomic research, we aim to show scientists how that research can be conducted so that it results in realistic and acceptable policy recommendations. Because the field of bioeconomics is relatively young, there has not yet been a great deal of development beyond theoretical applications. The authors of this book are an exception to this, and we see this book as an opportunity to expand our results, and those of others, more fully into a context that is useful to researchers and society.

Third, we recognize that the greatest potential for encouraging additional bioeconomic research is to make it accessible to junior graduate students and policy-makers. Our book will be written at such a level, and contains sufficient introductory material that ecologists, economists, mathematicians and interested policy-makers and managers will be able to see the merits of and adopt our approach.

Proposed Table of Contents for “Bioeconomics of Invasive Species” book

Chapter 1: Introduction to Biological Invasions: Biological, Economic, and Social Perspectives.
Chapter 2: Integrating Ecology and Economics to Guide Risk Management.
Chapter 3: Trait-Based Risk Assessment for Invasive Species.
Chapter 4: Forecasting Suitable Habitat for Invasive Species Using Ecological Niche Models and the Policy Implications of Forecasts.
Chapter 5: Propagule Pressure and Establishment.
Chapter 6: Estimating Dispersal and Predicting the Distribution of Invasive Species.
Chapter 7: Uncertain Invasions: A Biological Perspective.

Chapter 8: Uncertain Invasions: A Bioeconomic Perspective.
Chapter 9: Risk Perception and Communicating the Risk of Invasive Species.
Chapter 10: Modeling Integrated Decision-Making Responses to Invasive Species.
Chapter 11: Bioeconomics of Species Invasions in the Laurentian Great Lakes.
Chapter 12: A Case Study on Rusty Crayfish: Interactions Between Empiricists and Theoreticians.
Chapter 13: Advances in Ecological and Economical Analysis of Invasive Species: Zebra Mussel as a Case Study.
Chapter 14: Putting Bioeconomic Research Into Practice.

List of Participants

Keller, Reuben (University of Notre Dame)
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Chapter 55

String theory and inflationary cosmology (07rit135)

Jun 03 - Jun 10, 2007

Organizer(s): James Cline (McGill University)

Overview of the Field

The paradigm of inflation continues to be confirmed by experiments like WMAP, and will be probed more deeply by the Planck satellite in the near future. Because of the extremely high energies present in the early universe during inflation, string theorists have put great effort into making predictions for cosmology which might be the only way of verifying the theory experimentally. Inflation from string theory has developed rapidly during the last few years, with ref. [1] setting the trend for a new level of mathematical rigor in the subject. While the early models had to make numerous assumptions, for example concerning the stability of the extra dimensions of string theory, the current trend is to eliminate assumptions as much as possible in favor of rigorous calculations, so that the models become increasingly derived from the highly constrained mathematical framework of string theory, rather than merely inspired by it.

Recent Developments

One of the most highly studied proposals for inflation from string theory is that inflation is due to the relative motion of D-branes, higher dimensional objects intrinsic to string theory, within the compact internal dimensions which are equally essential. The main realizations involve D3- and anti-D3 branes (where 3 denotes the spatial dimension) and D3 and D7 branes. The major challenge is to find ways of making the potential between the branes sufficiently flat to support a long period of inflation, during which the size of the universe grew by a factor of at least e^{60}. This question cannot be fully addressed until stability of the size of the extra dimensions has been assured by some mechanism, since the dynamics of this compact space has a strong effect on the force between the branes. Warped compactifications in type IIB string theory have provided a mathematically rigorous framework for stabilizing the extra dimensions, and have thus made the problem of constructing a flat potential well-posed.

The accepted method for studying this problem is to derive a low energy effective lagrangian for the position of the inflationary D3 brane in the language of 4D supergravity. An essential ingredient is thus the superpotential, which leads to the potential. In ref. [1], it was recognized that there should exist corrections to the superpotential describing the force on a D3 brane in a warped throat which could potentially be tuned to give a flat potential [1], but these corrections had never been explicitly calculated, only parametrized. Thus the details of how flatness could be achieved in a specific string background remained to be elucidated. An important step in this direction was taken by the Princeton group, [3], which computed the string theory corrections to the superpotential. Our previous team effort at BIRS in 2006 was able to quickly capitalize on the results of this paper, by showing that inflation does not work when the D7 branes are embedded in the extra dimensions of the throat in a certain way (the Ouyang embedding), which we believed to be quite typical [2]. Our well-cited work influenced the Princeton group to consider a wider class of embeddings, and they found one (the Kuperstein embedding) which was compatible with inflation. This discovery, which we helped to inspire, led to two highly influential papers from them, ref. [4], which set the new standard for the state-of-the-art brane-antibrane inflation model derived from string theory. This was the status of the subject when we began our meeting in 2007.

Directions of research

We pursued two different research directions in response to the recent progress of ref. [4]. One was to start a more in-depth study of the inflationary dynamics in the framework developed in those papers, since their emphasis was on the construction of the model, with more of a preliminary analysis of the details of inflation and comparison to current experimental constraints. We quickly discovered a novel feature which the authors of [4] had not noticed: for some range of parameters, one could find a *minimum* in the number of e-foldings of inflation, N_e. This is quite different from the usual behavior, where N_e is a monotonic function of the parameters of the potential. Since it is usually difficult to get the experimentally required minimum number $N_e > 60$, this finding was rather striking.

Our second line of attack was to try to improve the method of lifting up the potential from negative values, typical of supergravity and which would lead to anti-de Sitter rather than approximately de Sitter solutions to Einstein's equations. The usual method of doing so, employed in [4], is to include anti-D3 branes in the throat, which explicitly break supersymmetry, and which therefore call into question the consistency of an approximately supersymmetric description of the dynamics. Prompted by recent studies of the α' (higher derivative) corrections to the low energy action from string theory, in particular corrections to the Kähler potential, as well as the effects of spontaneously breaking supersymmetry by turning on magnetic flux on the D7-branes [5], we attempted to get uplifting from the α' corrections, and to search for a new inflationary scenario in which the attraction of the D3 brane toward the bottom of the warped throat might be counteracted by its attraction to the D7 branes, due to the magnetic fluxes.

Outcome of the Meeting

The research we started resulted in two papers, one of which has been accepted for publication in JHEP [6], and the other recently submitted to Phys. Rev. D [7]. The first paper did not succeed in finding a new inflationary scenario as we had initially hoped, but it did obtain new results concerning the potential of the moduli in the modified setup (where magnetization of the D7 branes and α' corrections were included), in particular the minima of the angular directions in the warped throat, and it also succeeded in uplifting the negative vacuum energy without explicit supersymmetry breaking, which was one of the goals. The second paper used Monte Carlo Markov chain techniques to numerically explore the parameter space of the D3 brane inflation model as developed by ref. [3]. It succeeded in finding parameters where all experimental constraints (the spectral index and amplitude of power of CMB fluctuations) are satisfied, which had never been done in previous papers. Furthermore it found an optimal region in the parameter space where the fine tuning problem pointed out by the original authors is eliminated. Near this region, acceptable inflation models are found in which any given parameter need only be adjust to a part within a few. This represents an improvement by 8 orders of magnitude in the fine-tuning problem for the original choice of parameters by the authors of [3]. It is a striking result because usually at least a tuning of one part in 100 is needed to get 60 e-foldings of inflation.

Both of the papers which came out of our team meeting were realized with the collaboration of graduate students. The benefits of BIRS thus extended to these students, even though they were not present as official team members.

List of Participants

Burgess, Cliff (Conseil Européen pour la Recherche Nucléaire, McMaster University and Perimeter Institute)
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D. Cremades, M. P. Garcia del Moral, F. Quevedo and K. Suruliz, “Moduli stabilisation and de Sitter string vacua from magnetised D7 branes,” JHEP **0705**, 100 (2007) [arXiv:hep-th/0701154].

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[7] L. Hoi and J. M. Cline, “How Delicate is Brane-Antibrane Inflation?,” arXiv:0810.1303 [hep-th].

Chapter 56

Mapping Quantitative Traits in Humans (07rit143)

Jun 29 - Jul 08, 2007

Organizer(s): David Siegmund (Stanford University)

Overview of the Field

Genetic mapping of quantitative traits attempts to determine locations in the genome contributing to quantitative traits, e.g., blood pressure, serum cholesterol, serum insulin, and if possible any gene-gene and gene-environment interactions that may be involved. Because inheritance involves a random selection of genetic materials that is passed down to each succeeding generation, gene mapping necessary involves statistical analysis of genetic data. Controlled breeding experiments play a considerable role in simplifying the problem of gene mapping in plants and animals, but gene mapping in humans involves special complications because of one's inability to perform suitable breeding experiments.

There are two general approaches to gene mapping in humans: linkage, which is based on (close) relationships in families, and association, which is based on the (distant) relationships in populations.

Outcome of the Meeting

The main outcome of the meeting was a paper giving a unified foundation to the statistical problem of mapping quantitative traits by either family based or association based methods. This paper was published in *Proc Natl Acad Sci U S A*. A second manuscript that builds on the foundation in the first paper and provides relevant statistical software has now been submitted for publication.

List of Participants

Dupuis, Josee (Boston University School of Public Health)
Siegmund, David (Stanford University)
Yakir, Benjamin (Hebrew University Mount Scopus)

Bibliography

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Chapter 57

Recent Advances in Mathematical Relativity (07rit136)

Aug 05 - Aug 12, 2007

Organizer(s): Piotr Chrusciel (University of Tours & Oxford University), Greg Galloway (University of Miami), Daniel Pollack (University of Washington)

Goal of the Research In Teams week

The three co-organizers met at BIRS in August 2007 to commence work on a review of the most significant recent advances in the mathematical understanding of Einstein's theory of gravitation. The aim of this work was to produce an article for publication in the Bulletin of the American Mathematical Society, which would be accessible to a broad audience within the mathematical community, and which would highlight both the major breakthroughs which have recently been made, as well as the challenging open problems which remain.

Progression and use of facilities

At the start of the meeting we made decisions over the basic areas which we would focus on in the article. We decided to begin with a concise introduction to Lorentzian geometry and causal theory and then have significant sections on black holes, the Cauchy problem, initial data sets, global evolution and marginally trapped surfaces. Our time at BIRS was divided between group meetings where we discussed the topics to be included, individual time spent actually writing different sections, and again group meetings where we critiqued each others progress and worked together on detailed parts of the manuscript. The combination of excellent facilities where we could work both individually and as a group, and the very pleasant surroundings of the Banff Center, were crucial to the success of the weeks work. Informal, relaxed time was spent by the group during meals at the wonderful dining hall and over a couple of hikes in the area where progress on the project was also made in more ethereal ways.

Scientific Progress Made

Over the course of the week we made substantial progress on the manuscript. We completed rough drafts of the sections on Lorentzian geometry and causal theory, black holes, the Cauchy problem and initial data sets. We left BIRS with over 70 pages of TeXed manuscript. Given the difficulty in completing such a piece of work with three co-authors widely separated geographically, having a large block of time to work exclusively on this at BIRS was crucial to the eventual success of the project.

Over the subsequent year additional work was done on the manuscript individually by each of the co-authors, and during visits by Galloway to Oxford University, and by Chruściel and Pollack to the University of Miami. The final work on the manuscript was completed at the Institut Mittag-Leffler, in Djursholm, Sweden in Autumn 2008.

Outcome of the Meeting

The paper *Mathematical General Relativity: A Sampler* [1] has been completed and submitted to Bulletin of the American Mathematical Society. It will appear shortly as an official preprint in the Mittag-Leffler Preprint series. It is an 84 page document with 328 references which gives an up-to-date snapshot of the state of the art in the field of mathematical relativity. Included in this work is a list of 20 significant open problems which are discussed within the text and summarized at the end. We hope that this work will serve to draw new researchers into the field and raise the awareness within the general mathematical community of this fascinating area, with its unique confluence of analysis, geometry and physics

A copy of the preprint is being sent together with this report.

List of Participants

Chrusciel, Piotr (University of Tours & Oxford University)
Galloway, Greg (University of Miami)
Pollack, Daniel (University of Washington)

Bibliography

[1] P.T. Chruściel, G. Galloway and D. Pollack, *Mathematical General Relativity: A Sampler*, 1–84, Submitted for Publication to the Bulletin of the American Mathematical Society, October, 2008.

Chapter 58

Conformal and CR geometry: Spectral and nonlocal aspects (07rit132)

Aug 26 - Sep 09, 2007

Organizer(s): Andreas Cap (University of Vienna), A. Rod Gover (University of Auckland)

Overview of the Field

Conformal geometry has occupied an important position in mathematics and physics since early last century when Bateman, Dirac and others pointed out the conformal invariance of the fundamental equations governing electromagnetism and massless spin particles. More recently it has turned out to play an important role in many areas of mathematics and physics. Often these connections are rather subtle and unexpected. For example the smooth boundary of a domain in complex n-space inherits a so-called CR structure. As discovered by Ch. Fefferman, a CR structure is equivalent to a special conformal geometry. Thus there is an intimate relationship between conformal geometry and complex analysis. In the setting of elliptic PDEs and hard geometric analysis, certain core problems based on Riemannian geometry have yielded to conformal approaches. The touchstone in this area is the Yamabe problem of rescaling a metric to one with constant scalar curvature; this was finished by Schoen for structures of positive definite signature on compact manifolds in 1984, building on major earlier contributions of Aubin (1976), Trudinger (1968), and Yamabe (1960). The Yamabe problem is a semilinear PDE with a critical power nonlinearity. This may be viewed as a "nonlinear variant" of a certain conformally invariant linear equation. Many of the recent breakthroughs in Physics have involved conformal geometry in some essential way; the latest examples being string theory, and the AdS/CFT (anti-de Sitter/conformal field theory) correspondence.

The basic setup for conformal geometry is a (suitably) smooth manifold equipped with a conformal structure, that is an equivalence class of Riemannian metrics; two metrics are equivalent if they are related by multiplying by a positive smooth function. Thus, for example, on a conformal manifold one can measure angles between tangent vectors, but not the length of a tangent vector. This leads to immediate difficulties. Weyl's invariant theory classifies all Riemannian (or pseudo-Riemannian) invariants and invariant differential operators between natural bundles. However the corresponding problem for conformal geometry is much more difficult and is still not completely solved. The essential difference between these problems can be seen in the model spaces for the two geometries (Riemannian and conformal).

Each of the model spaces is a homogeneous space G/H. In the Riemannian case, H is the orthogonal group, so its representation theory is very well understood. In the conformal case the stabiliser H of a point is a parabolic subgroup of G=SO(p+1,q+1) (where (p,q) is the signature of the conformal structure). The finite-dimensional representations of H are not completely reducible in general and there is no complete classification of representations.

On the one hand the "parabolic" nature of conformal geometry certainly slowed its early development, but, on the other hand, it has encouraged a strong two-way interaction between the study of conformal structures and deep aspects of representation theory. For example conformally invariant operators (differential and pseudo-differential) give intertwinors between appropriate representation spaces. On the other hand the algebraic ideas of Kostant, Zuckermann and others have played a key role in the recent rapid progress in the construction and understanding of conformally invariant differential operators (as in for example Eastwood's "curved translation principle", pioneered in [12], and the "Bernstein-Gelfand-Gelfand [BGG] machinery" of Cap, Slovák, and Souček [10]).

On the geometric side a breakthrough in the modern theory was Fefferman's article "Parabolic invariant theory in complex analysis," [13]. This was concerned with CR geometry. The parallel development for conformal geometry (due to Fefferman and Graham and known as the "Fefferman-Graham ambient metric construction") came soon after, in 1985. In fact the article released at that point was a conference announcement which sketched the main ideas and potential applications. The detailed version of the article has been publicly archived in October of this year, see [14]; of course this necessarily links in the huge volume of work over the last two decades based around these constructions. The original works of Fefferman and Graham were motivated by invariant theory but established powerful geometric constructions that can be applied effectively to many problems related to these structures. A common theme is to use the constructions to turn geometric problems into well defined algebraic problems in parabolic representation theory. For the invariant theory Bailey, Eastwood, and Graham solved many of the key problems for the conformal and CR cases, in [2] . This was partly based on ideas from the analogous projective problem as solved by Gover in [16].

Linking several of the recent developments is a tool known as tractor calculus. For example this is the main geometric tool underlying the curved translation principle and the BGG machinery mentioned above. In the conformal setting the basic structures of tractor calculus were given by T.Y. Thomas and E. Cartan early-mid last century. (Indeed the basic tractor bundle and connection may be viewed as an induced structure for the Cartan principal bundle and connection.) However the calculus has been developed significantly recently by ourselves and our collaborators (especially Bailey, Eastwood and Graham). Through this, for example, we now have a direct link between the ambient constructions of Fefferman and Graham and the normal Cartan conformal and CR connections. Since the Cartan/tractor machinery fits well with representation theory, while the ambient metrics allow for considerable geometric simplification this link is a powerful tool which is being increasingly exploited.

On the geometric analysis side, an intriguing thread over the past decade or so is the problem of prescribing the Q-curvature. This is an invariant (due to Thomas Branson) which acts very much like the Gauss curvature in dimension 2 – the PDE describing its prescription is semilinear, and has an inhomogeneity and an exponential nonlinearity. For example, recent work of Djadli and Malchiodi shows that there is a Q-charge (basically the integral of Q) which, under pinching-type conformal changes that change the topology, "bubbles off" in a quantised way, much like the Yang-Mills topological charge. Furthermore, they prove existence of a metric with constant Q–curvature in the case where the charge is not an integral multiple of the bubbling charge. In other recent work, Graham and Hirachi have shown that the integral of Q has, as its total metric variation, the Fefferman-Graham obstruction tensor – the even dimensional obstruction to extending the Fefferman-Graham ambient metric construction to all orders. As such, it is an important object for the AdS/CFT correspondence. Recent work of Graham and Zworksi [21], Fefferman and Graham [14], Chang, Qing, and Yang [preprint], and Albin [1] makes the connection between the Q- curvature and AdS/CFT. At the same time, the Chang-Qing-Yang paper links this up with developments of about a decade ago, when it was established that a functional based on Q gives the main term in higher-dimensional Polyakov formulas, i.e. formulas for the quotient of functional determinants (of, for example, the Yamabe operator) at conformally related metrics. That is, this work provides a link to the role played by the Q-curvature in string theory.

Open Problems and Recent Developments

In recent work the applicants and collaborators have made several closely linked discoveries. These include: invariant operator non-existence results, [18], the existence of a new classes of conformally invariant elliptic differential complexes on conformally curved structures [3], new torsion quantities for these "detour complexes" which generalise Cheeger's 1/2-torsion [articles in progress]; subcomplexes in Bernstein–Gelfand–Gelfand sequences [8]; operators between differential forms which generalise the Q-curvature. An efficient version of tractor calculus to exploit the relation between conformal and CR geometry introduced by Fefferman [7].

The original aim of the meeting was to exploit these tools to construct and study global invariants for the global analysis and spectral theory on manifolds, however we were drawn in another direction due to an unexpected new development. Elliptic differential complexes and their generalisations provide a key tool for obtaining global invariants. For example on the locally flat structures there are the BGG complexes of [10]; these capture integrability conditions and their cohomologies give global invariants. However in the curved setting finding invariant differential sequences that yield useful complexes is non-trivial and the problem is for the most part completely open; some progress and interesting new directions have been mentioned already, the detour complexes of e.g. [3, 4, 19] and the subcomplexes of [8], which lead to deformation complexes for various parabolic geometries, see [5].

Investigations into the construction and application of invariant operators is often frustrated by the complexity of the operators involved and their constructions. Thus any steps toward a simplification and unification of the picture presents an important development. Very recently a new discovery was made which offers the possibility of a vast simplification in the construction and understanding of invariant operators and so our activity focussed on developing this new tool and and its applications. The central tool is the so-called *curved Casimir operator* of [9]. For G a semi-simple Lie group one has the quadratic Casimir that acts on induced G-representations. This is very well understood in the case of representations induced from a maximal compact subgroup, in which the Casimir essentially is a Laplacian. In [9] the authors show that in the parabolic case, the Casimir is an operator of order at most one, which admits a natural generalisation to to an invariant differential operator on any curved parabolic geometry. A parabolic geometry of type (G, P) is a manifold equipped with a P-principal bundle $\mathcal{G}$ (which via a soldering form type relation may be viewed as a generalisation of the idea of a higher order frame bundle) and on this a canonical Cartan connection which gives a suitably equivariant total parallelism, in a way that generalises the parallelism of G by the Maurer-Cartan form. By classical results of Cartan, Tanaka, and Chern–Moser, conformal and CR structures admit an equivalent description as parabolic geometries. It was shown in [6] that the exterior derivative on the total space of $\mathcal{G}$ descends to an invariant first order operator D, known as the fundamental derivative. This first order operator maps sections of an arbitrary natural vector bundle to sections of the tensor product of that bundle with the so–called adjoint tractor bundle. In terms of the fundamental derivative, the curved Casimir $\mathcal{C}$, acting on an arbitrary vector bundle associate to $\mathcal{G}$ is $(B \otimes id) \circ D^2$. Here B is the metric on the adjoint tractor bundle arising from the Killing form on $\mathrm{Lie}(G)$. Hence the first order operator $\mathcal{C}$ is a Laplacian-like contraction of the second power of D. It was shown in [9] that the naturality properties of the fundamental derivative carry over to $\mathcal{C}$.

Scientific Progress Made

The article [9] contains a systematic construction for a class of differential operators via curved Casimirs. This construction produces splitting operators, which form an essential ingredient in all constructions related to the curved translation principle and BGG sequences as described above. During the meeting we have extended these ideas to provide a systematic way for a direct (i.e. without going through iterated translations) construction of invariant linear differential operators acting between bundles associated to irreducible representation. This method also leads to non–standard operators and works in singular infinitesimal character, and hence produces operators which cannot be constructed using the BGG machinery. We have also developed a general machinery to compute the symbols of the resulting operators and hence in particular prove that they are non–zero (which is a major difficulty in the theory).

These basic tools work for arbitrary parabolic geometries. To check the power of our results we have developed some applications in conformal geometry; For this structure there are many existing results to compare to. Apart from several explicit examples, we have developed a general construction which produces almost all the formally self-adjoint operators between irreducible tensor bundles. (These are exactly the operators one does no obtain via the standard BGG machinery of [10, 11]). For each appropriately weighted irreducible tensor bundle E the construction gives for every dimension $n \geq 3$ a differential operator P :

$E[k-n/2] \to E[-k-n/2]$ and we prove that, for all but a finite set of dimensions the operator is of order $2k$ with non–zero symbol of that order (and in particular non-trivial). This family includes, for example, operators with leading term a power of the Laplacian acting on densities and so, as one of the simplest special cases, we obtain a new construction the conformal powers of the Laplacian along the lines of the Graham-Jenne-Mason-Sparling operators [20].

The striking feature of the construction is its simplicity of the construction and its uniformity. For all parabolic geometries these operators are given explicitly by polynomials of the form $L = (\mathcal{C} - \lambda_0)(\mathcal{C} - \lambda_1)\cdots(\mathcal{C} - \lambda_\ell)$ applied to the section space $\mathcal{V}$ of natural bundles V. The constants λ_i that have to be used in each case are Casimir eigenvalues of irreducible subquotients of V, and hence can be computed explicitly using finite dimensional representation theory. If there are coincidences between these eigenvalues (which can be forced by appropriately adapting the conformal weight) then L determines an operator between the corresponding pairs of irreducibles. (If all these eigenvalues are different, one obtains the splitting operators constructed already in [9]).

The more difficult side of the construction is to establish information on the operators constructed. An obvious minimal goal here is to calculate the symbol; in fact a knowledge of the symbol is also sufficient for many applications. Calculating this is complicated by the fact that starting with a representation inducing V, the number of factors in the composition defining L is in general much higher than the order of the resulting invariant operator. So although the symbol of a single curved Casimir well understood, this does not easily lead to information on the symbol of L. To compute the latter, we developed an algorithm to rewrite the (manifestly invariant) operator L as a composition of the order-of-L number of first order operators; each of the first order operators involved is not individually invariant but once one overall and simple choice is made then these are determined systematically by the operator and the algorithm. The symbols of these first order ingredients can be computed using purely algebraic tools, hence reducing the computation of the symbol to a combinatorial problem. Proving that the symbol of the operator L is nonzero in all but finitely many dimensions uses the information from the algorithm but also draws significantly on results from finite dimensional representation theory. Remarkably, both arguments based on compact groups, and arguments based on weights (and hence on complex groups and Lie algebras) are used to establish the result.

Given all these features it seems the constructions we developed during the RIT session, combined with the earlier observations in [9], will have a significant international impact on the approach researchers use to construct invariant operators and related tools such as differential complexes.

It is difficult to imagine that these developments could have been achieved without an ideal setting such as that provided by BIRS and the RIT programme. We were working intensively during the whole period of the programme, switching between the development of general ideas and computing explicit examples; the latter were often were quite involved. Without being able to completely concentrate on this topic for an extended period, similar progress would certainly not have been possible.

Outcome of the Meeting, a Summary

We feel the RIT meeting was tremendously successful. We developed a simple approach to constructing invariant differential operators which applies uniformly to the entire class of parabolic geometries. This approach was tested for conformal structures, where it leads to very strong results and also proves its computational efficacy. The general theory surrounding the curved Casimir we developed, and the specific results obtained, demonstrate that the curved Casimir plays a fundamental role in the invariant operator theory of parabolic geometries. For some directions it seems likely that this will lead to a paradigm shift in the approach to constructing and treating invariant operators. Based on the work carried out at the BIRS meeting we have two articles in progress, with one near completion.

List of Participants

Cap, Andreas (University of Vienna)
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Summer School Reports

Chapter 59

2007 Summer IMO Training Camp (07ss005)

Jul 10 - Jul 22, 2007

Organizer(s): Bill Sands (University of Calgary)

The 2007 IMO Training Camp started on Friday July 6 with the arrival in Calgary of five of the student Team members. (The sixth student and all three adult Team members this year were from Calgary.) Also arriving were two students from Vancouver, chosen, along with one student from Edmonton and three from Calgary, to participate in the Calgary portion of the Camp. Yufei Zhao from Cambridge, Ontario, and Lily Yen and Mogens Hansen from Vancouver, also arrived to help out at the first part of the Camp.

The participants were: Canadian Team members: Kent Huynh, Steven Karp, Cynthia (Yan) Li, Alex Remorov, Jonathan Schneider, Jarno Sun;

Mexican Team members: Isaac Buenrostro Morales, Aldo Pacchiano Camacho, Marco Antonio Avila Ponce de Leon, Cristian Manuel Oliva Aviles, Fernando Campos Garcia, Manuel Jesus Novelo Puc;

Adult trainers: Bill Sands (Leader), Adrian Tang (Deputy Leader), Minh-Lac Bui (Deputy Observer), Rogelio Valdez Delgado (Mexican Deputy Leader), Amir Amiraslani, Ed Doolittle, Alex Fink, Mogens Hansen, Felix Recio, David Rhee, Lily Yen, Yufei Zhao;

Local (Alberta and BC) students: Mariya Sardarli from Edmonton; Julian Sun and Jonathan Zhou from Vancouver; and Jaclyn Chang, Jonathan Love, Danny Shi and Hunter Spink from Calgary.

Of the local students, Mariya, Jaclyn, Jonathan Love, and Hunter were already attending Joseph Lings CMS Regional Math Camp, so they only started attending the IMO Camp on Sunday morning after the CMS Camp had ended. Of these four, all but Mariya stayed in their own homes at night and only attended the Camp during the day.

Of the adults, Amir, Minh-Lac and I stayed in our own homes at night. Everyone else was housed in Cascade Hall, an apartment-style Residence on campus, two students to a room. Meals were catered by the Students Union, at set times in a certain room.

In the first morning (Saturday July 7), all the Team members were taken to a nearby mall to purchase uniforms (pants for the males, skirts for the females). While this was happening, the remaining Camp participants attended that mornings scheduled lecture in Joseph Lings CMS Regional Camp. That afternoon and the next morning, the IMO Camp continued with lectures and problem sets by Mogens Hansen and Lily Yen, and all Camp participants in attendance. On Sunday afternoon, all Camp participants were taken to the Calgary Stampede.

On Monday were lectures and problem sets by Yufei Zhao and Amir Amiraslani. Meanwhile, Lily and Mogens left the Camp to go back to Vancouver. Graham Wright arrived in Calgary in the late morning to prepare for the Media Event to take place the next day. That evening he treated the adult Team members to supper at a nearby restaurant.

In the morning of Tuesday July 10, after some fun math activity in the morning, the Media Event took place from noon till 1:00 PM. At 3:00 the Calgary portion of the Camp ended with the departure of Graham Wright and all the local students. The Team members and trainers were then driven to the BIRS facility in Banff for the remainder of the Training Camp.

BIRS of course was again an unmatchable setting for the Camp, with the facilities, the food, and the scenery all superb. Training continued, with an increased emphasis on mock contests. There was a lot of coming and going during the Camp. Felix Recio arrived on July 12 and stayed until July 16. Yufei Zhao left on July 14, and Ed Doolittle and Alex Fink arrived on July 16. I left on July 17 to travel to Vietnam, and David Rhee arrived later that day.

The most notable innovation at this years Camp was the inclusion of the Mexican IMO Team in the training at BIRS. Due to their airline arrangements, the Mexicans could not attend the entire Camp, but they arrived in the morning of July 14 and stayed until July 19. The Mexican Team consisted of six students and the Deputy Leader, Rogelio Valdez Delgado. Rogelio gave two lectures on geometry which were very well received.

Besides the concentrated training that took place at BIRS, the Canadian and Mexican Teams were taken on two excursions. On July 14 we walked up and down Sulphur Mountain, and then had the opportunity for a welcome dip in the Hot Springs. And on July 18 both Teams were driven to the Columbia Icefields.

One slight medical incident occurred one evening. We learned that Steven Karp had a small cut in his ear that resulted in a small amount of blood gathering there, perhaps as a result of an insect bite. It was bothering him, so Minh-Lac and I took him to the emergency room of the Banff hospital, which miraculously was completely empty of customers when we arrived. The nurse on duty looked at Stevens ear and recommended that we simply clean his ear and apply some antiseptic. So we bought the items at the drug store, and Minh-Lac expertly took care of his ear. For the rest of the Camp she kept watch on his ear, but the problem cleared up very quickly.

The Canadian Team left Banff on July 21 and returned to Calgary, from where they flew to Vietnam.

Many thanks to:

- The staff and management at BIRS, especially Brenda Shakotko, the BIRS StationManager, who made our stay there so memorable; also Nassif Ghoussoub, the BIRS head, who moved mountains on short notice to make it possible for the Mexican IMO Team to train with us at BIRS.
- The following Trainers during the IMO Camp: Lily Yen of Capilano College and Mogens Hansen of Vancouver; Yufei Zhao, Alex Fink and David Rhee, former IMO Team members and current university students; Felix Recio of the University of Toronto; Amir Amiraslani of the University of Calgary; and Ed Doolittle of the University of Regina.
- Alexis Roper and Betty Teare, administrative staff in the Department of Mathematics and Statistics of the University of Calgary, who helped to arrange the site of the Media Event, booked the food, and took the pictures at the Media Event.
- Jag Kaur Samra and Grady Semmens of Media Relations, and Dave Wood of the Teaching and Learning Centre, for assistance in setting up and running our successful and enjoyable Media Event in the Learning Commons on campus.
- Deputy Leader Adrian Tang, Deputy Observer Minh-Lac Bui, and trainer Ed Doolittle, who drove the Teams and Trainers to and from Calgary and to and from the Columbia Icefields.
- University of Calgary graduate student Peter Papez, who drove Vancouver student Jonathan Zhou to the Calgary airport on July 10.

List of Participants

Amiraslani, Amir (University of Calgary)
Antonio Avila Ponce De Leon, Marco (IMO 2007)
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