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BIRS

Contents

Five-day Workshop Reports

Chapter 1

Permutation Groups (09w5130)

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Overview

This 5-day workshop finds string theory at a crossroads of its development. For the past twenty five years it has been perhaps the most vigorously investigated subject in theoretical physics and has spilled over into adjacent fields of theoretical cosmology and to mathematics, particularly the fields of topology and algebraic geometry. At this time, it retains a significant amount of the momentum which most recently received impetus from the string duality revolution of the 1990's. The fruits of that revolution are still finding their way to the kitchen table of the working theoretical physicist and will continue to do so for some time.

After undergoing considerable formal development, string theory is now poised on another frontier, of finding applications to the description of physical phenomena. These will be found in three main places, cosmology, elementary particle physics and the description of strongly interacting quantum systems using the duality between gauge fields and strings.

The duality revolution of the 1990's was a new understanding of string theory as a dynamical system. All known mathematically consistent string theories were recognized to simply be corners of the moduli space of a bigger theory called M-theory. M-theory also has a limit where it is 11-dimensional supergravity. This made it clear that, as a dynamical system, string theory is remarkably rich. Its many limits contain quantum mechanical and classical dynamical systems which are already familiar to physicists and mathematicians as well as a host of new structures.

A byproduct of the duality revolution was the realization that, in the context of string theory, there is a simple concrete example of the long expected duality between quantized Yang-Mills gauge field theories and string theories in the form of the "Maldacena conjecture" – the conjectured exact duality between maximally supersymmetric $\mathcal{N} = 8$ Yang-Mills theory defined on flat four-dimensional Minkowski spacetime and the type IIB supersymmetric string theory on an $AdS_5 \times S^5$ background space-time. Although this duality is still a conjecture, there are a large number of nontrivial quantitative checks of it and it agrees in every instance.

On the face of it, this correspondence between a gauge theory and a string theory is remarkable. It is a one-to-one mapping of all of the quantum states and all of the observables from a non-trivial four dimensional quantum field theory, a close relative of the field theories which are used to describe elementary particle physics, and a ten dimensional string theory. It literally states that one set of observers, who are equipped with particle accelerators for instance, would see the world as four space-time dimensional and being composed of elementary building, particles, which are the quanta of the Yang-Mills theory, propagating scalar, spinor and vector fields. Another set of observers, equipped differently, would see the same world as being ten space-time dimensional with the elementary building blocks being fundamental strings. What is more, the duality is informative and useful in a technical sense as it relates the weakly coupled limit of one theory to the strongly coupled limit of the other theory, and mixes the classical and quantum limits of both theories. For example, it allows one to relate solutions of the classical nonlinear partial differential equations of classical type IIB supergravity – ordinary classical multivariate functions – to the very strong coupling limit of Yang-Mills theory, which is also the deep quantum limit where all variables are random variables and quantum fluctuations rule the day. That one could learn anything about this limit of a quantum field theory is remarkable by any measure. Moreover, in many instances, the AdS/CFT correspondence is the only way to obtain strong coupling information about one or the other of the theories.

The gauge-theory - string theory duality also can be viewed as giving a precise definition of the type IIB string theory. In spite of some decades of intense effort, string theory as a dynamical system still has a less precise formulation than classical or quantum field theory. Given the duality, the string theory can be given a technical definition which is at least as sound as the gauge theory.

One development in the AdS/CFT correspondence is the observation, originally due to Minahan and Zarembo (who were both speakers at this workshop) that the problem of finding the spectrum of $\mathcal{N} = 4$ supersymmetric Yang Mills theory is equivalent to that of solving a quantum integrable model, which is usually done using the Bethe Ansatz. This gives the as yet unrealized hope that the entire spectrum could be constructed and spawned a large amount of activity.

At the same time as it is a crossroads of string theory, this is the golden age of cosmology. Once regarded as a science that was starved for data, cosmology has burgeoned as ground and space-based astronomical observations supply a wealth of precise cosmological measurements. In the present epoch, after many years on the speculative side of the scientific spectrum, cosmological theory is now being confronted by cosmological fact. Questions that were until recently the stuff of speculation can now be analyzed in the context of rigorous, predictive theoretical frameworks whose viability is determined by observational data. The most surprising and exciting feature of cosmology's entrance into the realm of data-driven science is its deep reliance on theoretical developments in elementary particle physics. At the energy scales characteristic of the universe's earliest moments, one can no longer approximate matter and energy using an ideal gas formulation; instead, one must use quantum field theory, and at the highest of energies, one must invoke a theory of quantum gravity, such as string theory. Cosmology is thus the pre-eminent arena in which our theories of the ultra-small will flex their muscles as we trace their role in the evolution of the universe. As such, it give perhaps the most promising approach to confirming whether string theory is the correct theoretical model for planck scale physics. One avenue to doing this which has seen much discussion recently is through string models of inflation, the phase of rapid, exponential expansion of the early universe, whose existence seems to be confirmed by current data. The idea is that inflation stretches distance scales so drastically that even physics which occurs on distance scales as minute as the size of strings gets blown up to astronomical size and there is the hope that the physics which determines the structure at that scale is blown up with it and carries hints about its origin. This exciting possibility as well as some others are what occupy the present day string cosmologists.

The relationship between the highly mathematical subject of string theory and cosmology, which has traditionally been more phenomenological, is rapidly evolving. One of the main motivations for string theory is to find a framework capable of dealing with the singularities which arise in classical general relativity, most notably the cosmological singularity of the big bang. Moreover, according to present cosmological models, physics on scales close to the singularity, i.e., close to the Planck energy scale) is responsible for producing the structure we observe today in the Universe. Thus, the largely unexplored interface between cosmology and string theory is enormously rich and promising terrain.

Years of research have shown that cosmology requires input from new developments in fundamental physics in order to make significant progress. For example, such input is needed to provide a consistent basis for inflationary cosmology (or for alternatives such as the the pre-big-bang or ekpyrotic scenarios), to provide insight into the formation of structure, to provide possible solutions to the dark matter problem, to provide a mechanism for the apparent acceleration of the Universe, and to explain other observational facts which are still a mystery in the current models of cosmology. On the other hand, since it is unlikely that string theory will ever be directly tested through accelerator experiments, cosmological observations may well be the most promising way of confirming this approach to quantum gravity. Thus, sharpening string cosmology is both crucial for further breakthroughs in cosmology and to provide a means to one day test string theory itself.

This time of great progress in cosmology coincides with another important event, the startup of the Large Hadron Collider particle physics experiment. This is the first particle physics experiment in almost three decades where there is a reasonable expectation of seeing truly new physics, and could well be the most important one in the life-time of currently active particle physicists. Some of this physics is related to cosmology. For example, supersymmetric extensions of the standard model – which could very well be the new physics – have particles which are candidates for dark matter. Finally, mathematics has made significant progress in unraveling the nature of string theory. It is clear that its understanding at the most fundamental level will require sophisticated, most likely new mathematics. Some of this new mathematics is already there and bringing it into contact with physics is particularly timely.

Over the years there have been many fruitful interactions between string theory and various fields of mathematics. Subjects like algebraic geometry and representation theory have been stimulated by new concepts such as mirror symmetry, quantum cohomology and conformal field theory.

String cosmology

There were six speakers in the string cosmology section covering a spectrum from the phenomenology of inflation to the appearance of cosmological structures in solvable string models of toy cosmologies.

- Alexander Westphal discussed his recent work on monodromy in the cosmic microwave background. He presented a simple mechanism for obtaining large-field inflation, and hence a gravitational wave signature, from string theory compactified on twisted tori. For Nil manifolds, he obtained a leading inflationary potential proportional to $\phi^{2/3}$ in terms of the canonically normalized field phi, yielding predictions for the tilt of the power spectrum and the tensor-to-scalar ratio, $n_s \approx 0.98$ and $r \approx 0.04$ with 60 e-foldings of inflation; he noted the possibility of a variant with a candidate inflaton potential proportional to $\phi^{2/5}$. The basic mechanism involved in extending the field range – monodromy in D-branes as they move in circles on the manifold – arises in a more general class of compactifications, though his methods for controlling the corrections to the slow-roll parameters require additional symmetries.

- Robert Brandenberger reviewed the current status of string gas cosmology. String gas cosmology is a string theory-based approach to early universe cosmology which is based on making use of robust features of string theory such as the existence of new states and new symmetries. A first goal of string gas cosmology is to understand how string theory can effect the earliest moments of cosmology before the effective field theory approach which underlies standard and inflationary cosmology becomes valid. String gas cosmology may also provide an alternative to the current standard paradigm of cosmology, the inflationary universe scenario.

- Nemanja Kaloper discussed the relationship between quintessence and the string landscape. He argued that quintessence may reside in certain corners of the string landscape. It could arise as a linear combination of internal space components of higher rank forms, which are axion-like at low energies, and may mix with 4-forms after compactification of the Chern-Simons terms to four dimensions due to internal space fluxes. The mixing induces an effective mass term, with an action which preserves the axion shift symmetry. The symmetry is then broken spontaneously by background selection. With several axions, several 4-forms, and a low string scale, as in one of the setups already invoked for dynamically explaining a tiny residual vacuum energy in string theory, the 4D mass matrix generated by random fluxes may have ultra-light eigen-modes over the landscape, which are quintessence. He illustrated how this works in simplest cases, and outlined how to get the lightest mass to be comparable to the Hubble scale now, $H_0 \sim 10^{-33}\mathrm{eV}$. The shift symmetry protects the smallest mass from perturbative corrections in field theory. If the ultra-light eigen-mode does not couple directly to any sector strongly coupled at a high scale, the non-perturbative field theory corrections to its potential will also be suppressed. Finally, if the compactification length is larger than the string length by more than an order of magnitude, the gravitational corrections may remain small too, even when the field value approaches M_{Pl}.

- Matt Kleban spoke about his recent work entitles “Watching Worlds Collide”. He showed how to extend their previous work on the cosmology of Coleman-de Luccia bubble collisions. Within a set of approximations he showed how to calculate the effects on the cosmic microwave background (CMB) as seen from inside a bubble which has undergone such a collision. He showed that that the effects are always qualitatively similar–an anisotropy that depends only on the angle to the collision direction–but can produce a cold or hot spot of varying size, as well as power asymmetries along the axis determined by the collision. With other parameters held fixed the effects weaken as the amount of inflation which took place inside our bubble grows, but generically survive order 10 efolds past what is required to solve the horizon and flatness problems. In some regions of parameter space the effects can survive arbitrarily long inflation.

- Joanna Karczmarek talked about her work on matrix model cosmology. She reviewed the idea that the leading classical low-energy effective actions for two-dimensional string theories have solutions describing the gravitational collapse of shells of matter into a black hole. She reviewed the argument that string loop corrections can be made arbitrarily small up to the horizon, but α' corrections cannot. She used the matrix model to show that typical collapsing shells do not form black holes in the full string theory. Rather, they backscatter out to infinity just before the horizon forms. The matrix model was also used to show that the naively expected particle production induced by the collapsing shell vanishes to leading order. This agrees with the string theory computation. From the point of view of the effective low energy field theory this result is surprising and involves a delicate cancellation between various terms.

- Washington Taylor spoke about his recent work on inflationary constraints on type IIA string theory. He discussed how to prove that inflation is forbidden in the most well understood class of semi-realistic type IIA string compactifications: Calabi-Yau compactifications with only standard NS-NS 3-form flux, R-R fluxes, D6-branes and O6-planes at large volume and small string coupling. With these ingredients, the first slow-roll parameter satisfies epsilon ¿= 27/13 whenever V ¿ 0, ruling out both inflation (including brane/anti-brane inflation) and de Sitter vacua in this limit. His proof is based on the dependence of the 4-dimensional potential on the volume and dilaton moduli in the presence of fluxes and branes. He also described broader classes of IIA models which may include cosmologies with inflation and/or de Sitter vacua. The inclusion of extra ingredients, such as NS 5-branes and geometric or non-geometric NS-NS fluxes, evades the assumptions used in deriving the no-go theorem. He focused on NS 5-branes and outlined how such ingredients may prove fruitful for cosmology.

Mathematical String Theory

The presentations which could be classified as Mathematical string theory. They were centered around issues in supersymmetry and duality.

- Simeon Hellerman discussed his recent proof that every unitary two-dimensional conformal field theory (with no extended chiral algebra, and with central charges c_L, c_R ¿ 1) contains a primary operator with dimension Delta$_1$ that satisfies 0 ¡ Delta$_1$ ¡ (c_L + c_R)/12 + 0.473695. Translated into gravitational language using the AdS$_3$ /CFT$_2$ dictionary, this result proves rigorously that the lightest massive excitation in any theory of 3D gravity with cosmological constant Lambda ¡ 0 can be no heavier than 1/(4 G_N) + o(—Lambda—ˆ(1/2)). In the flat-space approximation, this limiting mass is twice that of the lightest BTZ black hole. The derivation of the bound applies at finite central charge for the CFT, and does not rely on an asymptotic expansion at large central charge. Neither does the proof rely on any special property of the CFT such as supersymmetry or holomorphic factorization, nor on any bulk interpretation in terms of string theory or semiclassical gravity. The only assumptions are unitarity and modular invariance of the dual CFT. The proof demonstrates for the first time that there exists a universal center-of-mass energy beyond which a theory of ”pure” quantum gravity can never consistently be extended.

- Charles Doran spoke about the recent work on the classification scheme of so-called adinkraic off-shell supermultiplets of N-extended worldline supersymmetry without central charges. He showed how, with collaborators, he was recently able to complete the constructive proof that all of these trillions or more of supermultiplets have a superfield representation. While different as superfields and supermultiplets, these are still super-differentially related to a much more modest number of minimal supermultiplets. He discussed how they were constructed.

- Keshav Dasgupta spoke about he derivation of a novel deformation of the warped resolved conifold background with supersymmetry breaking ISD (1,2) fluxes by adding D7-branes to this type IIB theory. He showed that they allow spontaneous supersymmetry breaking without generating a bulk cosmological constant. In the compactified form, the background will no longer be a Calabi-Yau manifold as it allows a non-vanishing first Chern class. In the presence of D7-branes the (1,2) fluxes can give rise to non-trivial D-terms. He reviewed the study the Ouyang embedding of D7-branes in detail and showed that in this case the D-terms are indeed non-zero. He also showed that, in the limit approaching the singular conifold, the D-terms vanish for Ouyang's embedding, although supersymmetry appears to be broken. He also discussed constructing the F-theory lift of their background and demonstrated how these IIB (1,2) fluxes lift to non-primitive (2,2) flux on the fourfold. The seven branes correspond to normalisable harmonic forms. he briefly sketched a possible way to attain an inflaton potential in this background once extra D3-branes are introduced and point out some possibilities of restoring supersymmetry in our background that could in principle be used as the end point of the inflationary set-up.
- Sunil Mukhi spoke about how to obtain the complete set of constraints on the moduli of N=4 superstring compactifications that permit ”rare” marginal decays of 1/4-BPS dyons to take place. The constraints are analysed in some special cases. The analysis extends in a straightforward way to multi-particle decays. He discussed the possible relation between general multi-particle decays and multi-centred black holes.

String theory Gauge theory duality

Gauge theory - string theory duality was the largest category of presentations. The topics centered around the formal structure of duality, including the conjectured integrability of the planar limit of the gauge theory and the classical limit of string theory as well as applications to the study of strongly interacting gauge theories and to gravity.

- Mark van Raamsdonk spoke about his work which attempts to provide some insights into the structure of non-perturbative descriptions of quantum gravity using known examples of gauge-theory / gravity duality. He argued that in familiar examples, a quantum description of space-time can be associated with a manifold-like structure in which particular patches of spacetime are associated with states or density matrices in specific quantum systems. He also argued that quantum entanglement between microscopic degrees of freedom plays an essential role in the emergence of a dual spacetime from the nonperturbative degrees of freedom. In particular, in at least some cases, classically connected spacetimes may be understood as particular quantum superpositions of disconnected spacetimes.
- Joe Minahan, Konstantin Zarembo, These speakers reviewed the status of integrability of the planar limit of $\mathcal{N}=4$ supersymmetric Yang-Mills theory and its dual, the classical limit of IIB superstring theory. They also discussed new results about the two-loop anomalous dimensions for fermionic operators in the ABJM model and the ABJ model. They discussed the appropriate Hamiltonian and reviewed the argument that it is consistent with a previously predicted Bethe ansatz for the ABJM model. The difference between the ABJ and ABJM models is invisible at the two-loop level by cancelation of parity violating diagrams. They showed how to construct a Hamiltonian for the full two-loop OSp(6—4) spin chain by first constructing the Hamiltonian for an SL(2—1) subgroup, and then showed the lift to OSp(6—4). They showed that this Hamiltonian is consistent with the Hamiltonian found for the fermionic operators.
- Alex Buchel discussed the use of the AdS/CFT correspondence to study first-order relativistic viscous magneto-hydrodynamics of (2+1) dimensional conformal magnetic fluids. He showed that the first

order magneto-hydrodynamics constructed following Landau and Lifshitz from the positivity of the entropy production is inconsistent. He proposed additional contributions to the entropy motivated dissipative current and, correspondingly, new dissipative transport coefficients. He used the strongly coupled M2-brane plasma in external magnetic field to show that the new magneto-hydrodynamics leads to self-consistent results in the shear and sound wave channels.

- Troels Harmark, Gianluca Grignani, Marta Orselli each gave one of a series of talks which reviewed aspects of their recent interesting work on the string dual of the recently constructed $N = 6$ super-conformal Chern-Simons theory of Aharony, Bergman, Jafferis and Maldacena (ABJM theory). They focused in particular on the $SU(2) \times SU(2)$ sector. They showed how to find a sigma-model limit in which the resulting sigma-model is two Landau-Lifshitz models added together. They considerd a Penrose limit for which they can approach the $SU(2) \times SU(2)$ sector. Finally, they showed how to find a new Giant Magnon solution in the $SU(2) \times SU(2)$ sector corresponding to one magnon in each $SU(2)$. Putting these results together, they found the full magnon dispersion relation and they compared this to recently found results for ABJM theory at weak coupling.

- Xi Yin reviewed study spin chain operators in the N=6 Chern-Simons-matter theory recently proposed by Aharony, Bergman, Jafferis and Maldacena to be dual to type IIA string theory in AdS4xCP3. He discussed the two-loop dilatation operator in the gauge theory, and compared to the Penrose limit on the string theory side.

- Pallab Basu, Anindya Mukerjee, each gave one of a series of two seminars where they reviewed their work on the large N SU(N) gauge theories on a compact manifold S^3 X R (with possible inclusion of adjoint matter) which is known to show first order deconfinement transition at the deconfinement temperature. This includes the familiar example of pure YM theory and N=4 SYM theory. They discussed the effect of introduction of N_f fundamental matter fields in the phase diagram of the above mentioned gauge theories at small coupling and in the limit of large N and finite N_f/N. They found some interesting features like the termination of the line of first order deconfinement phase transition at a critical point as the ratio N_f/N is increased and absence of deconfinement transition thereafter (there is only a smooth crossover). The results have implications for QCD, which unlike a pure gauge theory does not show a first order deconfinement transition and only displays a smooth crossover at the transition temperature.

- Hong Liu reviewed recent work where they showed that, for a class of conformal field theories (CFT) with Gauss-Bonnet gravity dual, the shear viscosity to entropy density ratio, η/s, could violate the conjectured Kovtun-Starinets-Son viscosity bound, $\eta/s \geq 1/4\pi$. He argued, in the context of the same model, that tuning η/s below $(16/25)(1/4\pi)$ induces microcausality violation in the CFT, rendering the theory inconsistent. This is a concrete example in which inconsistency of a theory and a lower bound on viscosity are correlated, supporting the idea of a possible universal lower bound on η/s for all consistent theories.

- David Kutasov reviewed his work on Seiberg duality in the context of AdS/ cft duality. He argued that N=2 supersymmetric Chern-Simons theories exhibit a strong-weak coupling Seiberg-type duality. He also discussed supersymmetry breaking in these theories.

- Jaume Gomis discussed his recent work on maximally supersymmetric 2+1-dimensional gauge theory. He discussed adding a supersymmetric Faddeev-Popov ghost sector to the recently constructed Bagger-Lambert theory based on a Lorentzian three algebra and obtained an action with a BRST symmetry that can be used to demonstrate the absence of negative norm states in the physical Hilbert space. He showed that the combined theory, expanded about its trivial vacuum, is BRST equivalent to a trivial theory, while the theory with a vev for one of the scalars associated with a null direction in the three-algebra is equivalent to a reformulation of maximally supersymmetric 2+1 dimensional Yang-Mills theory in which there a formal SO(8) superconformal invariance.

- Martin Kruczenski reviewed a computation of the 1-loop correction to the effective action for the string solution in AdS_5 x S^5 dual to the circular Wilson loop. More generically, the method he used can be applied whenever the two dimensional spectral problem factorizes, to regularize and define the fluctuation determinants in terms of solutions of one-dimensional differential equations. A such it can be applied to non-homogeneous solutions both for open and closed strings and to various boundary conditions. In the case of the circular Wilson loop, he obtained, for the 1-loop partition function a result which up to a factor of two matches the expectation from the exact gauge theory computation. The discrepancy can be attributed to an overall constant in the string partition function coming from the normalization of zero modes, which have not been fixed.

- Takuya Okuda and Diego Trancanelli each gave talks which discussed Wilson loop correlators at strong coupling. They reviewed the computation at strong coupling the large N correlation functions of supersymmetric Wilson loops in large representations of the gauge group with local operators of N=4 super Yang-Mills. The gauge theory computation of these correlators is performed using matrix model techniques. They showed that the strong coupling correlator of the Wilson loop with the stress tensor computed using the matrix model exactly matches the semiclassical computation of the correlator of the ’t Hooft loop with the stress tensor, providing a non-trivial quantitative test of electric-magnetic duality of N=4 super Yang-Mills. They then perform these calculations using the dual bulk gravitational picture, where the Wilson loop is described by a ”bubbling” geometry. By applying holographic methods to these backgrounds they calculate the Wilson loop correlation functions, finding perfect agreement with our gauge theory results.

String theory and particle physics

- Savdeep Sethi and Katrin Becker each gave seminars discussing their work on torsional heterotic geometries. They discussed the construction of new examples of torsional heterotic backgrounds using duality with orientifold flux compactifications. They explained how duality provides a perturbative solution to the type I/heterotic string Bianchi identity. The choice of connection used in the Bianchi identity plays an important role in the construction. They proposed the existence of a much larger landscape of compact torsional geometries using string duality. They also presented some quantum exact metrics that correspond to NS5-branes placed on an elliptic space. These metrics describe how torus isometries are broken by NS flux.

- Melanie Becker spoke about her work on new heterotic non-Kahler geometries. New heterotic torsional geometries are constructed as orbifolds of T^2 bundles over K3. The discrete symmetries considered can be freely-acting or have fixed points and/or fixed curves. She gave explicit constructions when the base K3 is Kummer or algebraic. The orbifold geometries can preserve N=1,2 supersymmetry in four dimensions or be non-supersymmetric.

- Herman Verlinde discussed his work on a holographic perspective on D-brane model building for elementary particle physics. He spoke about geometric aspects of extensions of the supersymmetric standard model that exhibit a periodic duality cascade. In the spirit of the holographic correspondence, the growth of the gauge group rank towards the UV is interpreted as a gradual decompactification transition. He showed that this class of models typically develop a duality wall in the UV, and presented an efficient method for estimating the hierarchy between the on-set of the cascade and the formation of the wall. As an illustrative example, he studied the model introduced by Cascales, Saad and Uranga which has an known geometric realization in terms of D-branes on an SPP/Z_3 singularity.

- Arkady Vainshtein spoke about his work on Dyon dynamics near marginal stability and non-BPS states. He showed how to derive the general form of the moduli-space effective action for the long-range interaction of two BPS dyons in N=2 gauge theories. This action determines the bound state structure of various BPS and non-BPS states near marginal stability curves, and he utilized it to compute the leading correction to the BPS-mass of zero-torsion non-BPS bound states close to marginal stability.

List of Participants

Aschbacher, Michael (California Institute of Technology)
Baginski, Paul (University of California, Berkeley)
Bamberg, John (Ghent University)
Blackburn, Simon (Royal Holloway, University of London)
Burness, Tim (University of Southampton)
Cameron, Peter (Queen Mary University of London)
Damian, Erika (University of East Anglia (UK))
De Medts, Tom (Ghent University)
Diaconis, Persi (Stanford University)
Giudici, Michael (University of Western Australia)
Guest, Simon (Baylor)
Guralnick, Robert (Univeristy of Southern California)
Kantor, William (University of Oregon)
Korchagina Capdeboscq, Inna (Warwick University)
Liebeck, Martin (Imperial College)
Lubotzky, Alex (Hebrew University of Jerusalem)
Lucchini, Andrea (University of Padova)
Lyons, Richard (Rutgers University)
Magaard, Kay (Birmingham)
Malle, Gunter (Universitaet Kaiserslautern)
Muehlherr, Bernhard (Université Libre de Bruxelles (ULB))
Neumann, Peter M (The Queen's College)
Parker, Christopher (University of Birmingham)
Praeger, Cheryl (University of Western Australia)
Pyber, Laci (Renyi Institute of Mathematics Budapest)
Reichstein, Zinovy (University of British Columbia)
Saxl, Jan (University of Cambridge)
Schneider, Csaba (University of Lisbon)
Segev, Yoav (Ben Gurion University)
Seitz, Gary (University of Oregon)
Seress, Akos (The Ohio State University)
Shareshian, John (Washington University)
Steinberg, Benjamin (Carleton University)
Tent, Katrin (Universität MÚnster)
Testerman, Donna (Ecole Polytechnique Federale de Lausanne)
Tiep, Pham Huu (University of Arizona)
Van Bon, John (Universita della Calabria)
Weiss, Richard (Tufts University)
Wilson, Rob (Queen Mary London)
Ziegler, Martin (Mathematisches Institut Freiburg)

Chapter 2

Stability Theoretic Methods in Unstable Theories (09w5113)

Feb 08 - Feb 15, 2009

Organizer(s): Thomas Scanlon (University of California at Berkeley), Saharon Shelah (Einstein Institute of Mathematics), Bradd Hart (McMaster University), Madhur Anand (Univeristy of Guelph), Frank Wagner (Université Lyon I), Alf Onshuus (Universidad de los Andes), Anand Pillay (University of Leeds, UK), Alf Onshuus (Universidad de los Andes)

Overview of the Field, Recent Developments and Open Problems.

The idea of applying methods and results from stability theory to unstable theories has been an important theme over the past 25 years, with o-minimality, smoothly approximable structures, and simple theories being key examples.

But there have been some key recent developments which bring new ideas and techniques to the table. One of these is the investigation of abstract notions of independence, leading for example to the notions of thorn forking and rosiness. Another is the discovery that forking, weight, and related notions from stability are meaningful in dependent theories. Another is the formulation of notions of stable, compact, or more general domination, coming from the analysis of theories such as algebraically closed valued fields and o-minimal theories.

The level of different approaches and techniques which end up overlapping was the reason we decided it would be a perfect time for a research meeting where the most prominent researchers would come together and discuss the ideas, results and goals that were showing up in different contexts. The dominant subjects of the meeting were the following.

Dominating local structure.

In the talks given by Hrushovski and Macpherson about stable domination and by Hasson about stable types in theories interpretable in o-minimal structures, there were clear indications towards the possible applications of understanding and using the stable-like "pieces" (types or sorts) within a particular theory. This is a completely new approach towards traditional stability theory (and general classification theory) where the main idea is not anymore to find dividing lines between different theories but instead one tries to find well behaved parts within a particular model. It is clear by results from Haskell, Hrushovski, Pillay and Macpherson that whenever the stable sorts "dominate" (the precise definition of this can be found in [HHM08]) all the types in the structure, many of the stability theory results can be applied to understand the global structure. In [HP09] these ideas of finding particularly well behaved sorts that could give global information about the whole structure proved to be particularly promising when one expanded the well behaved part from stable to compact in order to apply the results to structures which were groups interpretable in theories with an underlying order.

There are two possible lines of research that stem out from the results described here. On the one hand, there should be an ongoing research in stably dominated and compactly dominated structures. There are quite a few interesting examples in this areas and the results so far have been very impressive and promising. The other venue that should be explored is to find other possible definitions that can work as well (as part of the structures in a “local” manner) as local stability and local compactness have worked so far. Some possible ideas would be to find definitions for local dependence that are good enough to be significant (in the sense of finding examples) and strong enough to prove for example some of Shelah’s theorems concerning types in dependent theories.

Measures.

In 1987 Keisler ([Kei87]) wrote a paper where he showed that many aspects of forking could be understood and generalized when one studied measures on the definable sets as an extension of types (types would just be a 0-1 measure on the space, where the measure of a formula is 1 if and only if the formula is in the type). Work by Hrushovski, Peterzil and Pillay and lately by Simon shows that this analysis can provide a deeper understanding of many theories which had an underlying order (bounded o-minimal theories) and actually conclude many interesting and far reaching results in the definable groups of such theories.

It is quite possible that a deeper understanding of invariant Keisler measures will shed more light upon how forking can affect the type definable subsets in particular theories.

Pregeometries.

There were two aspects in which pregeometries (matroids) have been studied lately. The first is the possible equivalences to linearity. Linearity is a concept which is model theoretically hard to define, to prove and to use. In (forking-)minimal theories there are concept like one-basedness and local modularity which are equivalent to linearity and much easier to use. Using pairs of structures one can define notions which are equivalent to linearity and the study of applications of these new equivalences seems very promising.

Matroids have played a role in model theory since Zilber started studying the different possible matroids that were defined by algebraic closure in strongly minimal sets. However, in many ways when one tried to use this sort of arguments in higher dimensions one had to use quite sophisticated model theoretic tools such as weight and regular types in order to apply them. In combinatorics, a generalization of matroids (greedoids) has been developed during the last couple of decades. Even though there is no good definition for infinite dimensional greedoids, a definition which would be essential if one wanted to apply greedoids to model theory, the development of these definitions and the possible applications are quite promising ideas for the near future.

Dependent Theories.

Dependent (also called “NIP”) theories have been studied intensively during the last decade. Original work of Shelah has had many applications and has been very influential in oncoming work. Some of the ongoing questions include.

- Number of non forking extensions. It is known that one can prove that if for any set A of cardinality κ the number of non forking extensions of any type over A is at most 2^κ then the theory is dependent. It is also known that in a theory T the number of non forking extensions of a type p as described above is 2^{2^κ}. But there is a big gap between the two and the study of this gap should reveal the properties of non forking in dependent theories.

- Counting types. Shelah conjectured that any type in a dependent theory had boundedly many coheir extensions (where the bound depended on the model, not on the size of the parameter set of the different types). Using this he was able to prove existence of indiscernible subsequences of large enough sequences, but Kaplan and Shelah disproved this conjecture. However, it is possible that one can have a bound in the number of definable heirs or other significant extensions which are very important to the study of the best known unstable dependent theories (o-minimal theories).

- Generic pair conjecture. The generic pair conjecture in dependent theories was proved in [Sh08-2], but it has been studied in the more general context of abstract elementary classes (which is a categorical approach to model theory) and this may provide a way to define dependent theories in this more abstract setting.

- Weight, strong dependence and dp-minimality. Strong dependence and dp-minimal theories have been some of the most successful strengthenings of dependent theories. Both of this notions can be defined using the appropiate definition of weight.

- Other notions of independence. Studying NTP_2 structures (see the item below) Chernikov, Kaplan and Usvyatsov came up with the notion of strong non forking. How this notion relates to non forking, to non th-forking and to the notion of non splintering defined by Grossberg, Van Dieren and Villaveces for abstract elementary classes seems to be a meaningful area of research that can help us understand the extensions of types. Studying this within dependent theories and even restricting oneself to generically stable types seems to be particularly promising.

Other divding lines in classification theory; ideas from Shelah's "Classification Theory".

Much of stability theory has been developed based on ideas from Shelah's book [She78] and even notions like dependent theories described above have the origins in this extremely influential book. However, there are still many unexplored ideas from this book, some of which have resurfaced lately and seem to be quite promising.

$\mathbf{NTP_2}$.

Kaplan and Chernikov have been able to extend many of the results which are true for both simple and dependent theories. The study of coheirs, heirs, and non forking seems to be quite meaningful in this context. Also, viewing things in this level of generality may help understanding and defining interesting concepts in dependent thories. The new definition of strong forking by Kaplan and Usvyatsov seems to be part of this. One should also try to understand the relation between the notion of non splintering (defined by Grossberg, Van Dieren and Villaveces) and strong non forking, even in the context of dependent theories.

$\mathbf{NTP_1}$

Kim gave a very nice talk where he tried to analyse NTP_1 theories. The best example of NTP_1 theories are the ω-free PAC fields studied by Chatzidakis in [Cha02]. Once again, studying in this level of generality is also helpful to find significant independence notions which may coincide with forking in stable theories and therefore be hidden by it. In his talk, Kim suggested studying the independence notion that comes from defining that a formula $\phi(x, a)$ "strong divides" over A (not to be confused with strong dividing defined for th-forking, although it would seem that in many cases they would coincide) if given any Morley sequence $\langle a_i \rangle$ of a over A the conjunction

$$\bigwedge_i \phi(x, a_i)$$

is k-inconsistent.

The main idea was that if this notion has local character in NTP_1 theories (which it appears to have) one can develop in NTP_1 theories an internal analysis of this independence notion.

Amalgamation. There have been many approaches, particularly in Abstract Elementary Classes, about amalgamation problems (finding a common realization to a family of types). The principal impetus in this area was Hrushovski's paper [Hr06] where he studied, in stable theories, the relation between the existence problem for the amalgamation, the uniqueness, and the existence of certain definable groupoids. He also proved that for any structure M of a stable theory T one could find an expansion M^* of M by new sorts such that in M^* one had existence and uniqueness for all the amalgamation problems, further suggesting that this notion is related to something close to higher dimensional groupoids.

Perhaps the best way to describe the possible impact of this is to say that in stable theories, the concepts of 1-uniqueness and 2-existence are related to the very important notion of imaginaries. In this same direction, Hrushovski proved that 2-uniqueness and 3-existence was related to the definition of groupoids which can be seen as a generalization of imaginaries. It is therefore quite likely that studying the n-existence and the n-uniqueness problems can point out to objects which are inherent to the structure but which we have yet to analyse as independent sorts.

Other topics discussed in the meeting.

Other subjects discussed during the meeting which are quite interesting and where one can find model theoretic results but which are not quite as well structured yet include

(i) Other dividing lines in the classification of first order theories include rosiness, strict order property, SOP_n.

(ii) Use of stability-style techniques in some of the "good" classes of theories from (i) thorn rank, meta-analysability, definable types,

Model theory and group dynamics. This talk, given by Prof. Newelski was one of the most interesting talks of the meeting, but since the approach is quite new (even for this very new branch of model theory) it was hard to place among the previous subject headings. The main idea is to interpret the basic notions of group dynamics in the model theoretic setting and use these to understand extensions of types. In his talk, Newelski used some of the notions of group dynamics to define weak generics and understand how co-heir extensions can also be studied under this approach; but it is quite likely that this approach can be used in order to define new types of interesting extensions of types or give a deeper understand of the existing ones.

Presentation Highlights

Before commenting on the highlights of the meeting, we should describe the structure of it. The most innovative aspect of the meeting was the scheduling of some "structured working time" where the classrooms would be preassigned in order to have people work on and discuss different questions or interesting subjects and each participant would choose which of the "working groups" he would like to join. During most days we would have talks in the morning, and one talk after both lunch and dinner after each of which we would get together and then break up into working groups. Both the attendance of the "gathering" talks and of the working groups was quite impressive and all around the meeting was a great success.

The greatest achievement of the meeting (more of which we will talk about in the following section) was the understanding of the structure of a new branch of model theory which is starting to take shape after many researchers from diverse overlapping subjects got together for this meeting. It became clear that this new approach to model theory has two main subbranches. The first consists to generalize the results from stability and simplicity theory to even broader contexts like dependent, NTP$_1$ and NTP$_2$ theories, bringing the traditional "dividing lines" from classification theory even further and being able to conclude quite interesting results in even broader contexts. The second one is to use this dividing lines within a single theory. By this we mean that understanding the "stable parts" of a particular structure can have amazing consequences that in many cases can even say quite important things about the general structure.

We asked the participants what they thought were the highlights of the meeting. Hrushovski's and Simon's talks about invariant measures and stably dominated types and Chernikov's talk about NTP$_2$ structures were the three talks that most people considered highlights of the meeting, whereas canonical bases in NTP$_2$ structures and trying to define a linear order in an unstable dependent theory (or find a counterexample) were the working sessions people most frequently mentioned.

Scientific Progress Made and Outcome of the Meeting

As mentioned before, the biggest scientific progress made and outcome of the meeting was the awareness that was built around this fast growing subject, and the beginnings of the understanding of the structure of the subject and how the different approaches interact with each other. This feeling was reinforced when we asked the participants what they thought was the outcome of the meeting and we got replies such as "...awareness that a lot is happening very fast around generalisations of stability.", "...this meting managed to describe the state of the art in NIP and related areas, a currently rapidly evolving subject." and "It was a great success, many interesting projects were begun and it will be very interesting to reconvene in Banff in the future to hear the outcome of these investigations!" among many other similar replies. We all have a strong feeling that this meeting will provide a very important first step in a very promising area of model theory, and many of the participants mentioned that organizing a follow-up meeting in few years would be worthwhile.

As for scientific progress per se, many of the participants (Dzamonja, Gismatullin, Hart, Hrushovski, Kim, Macpherson, Malliaris, Onshuus, Usvyatsov) explicitly claimed to have made progress in research they had either started before the meeting or started working during the meeting, and it is quite likely that there will be many upcoming publications and results that were made (or significantly advanced) during the meeting. We should include the following partial reports.

- Hrushovski mentioned achieving a better understanding of the metastability in dependent theories and how much more went through that he thought at first.
- Gismatullin solved one of the questions he asked about groups without proper subgroups of finite index.
- Hart included the following report:

  "Suppose that p is a depth zero, regular type over a model M and N is dominated by a realizations of p over M. Moreover, assume that $M \subseteq_{na} N$. This situation arises on the leaves of a decomposition tree for any model of a countable, classifiable theory. In the calculation of the uncountable spectrum for countable theories, for cardinal arithmetic reasons, it was unnecessary to understand the exact structure of N over M. Hrushovski has shown that in the case where p is not locally modular, N is prime over M and any realization of p in N. There are examples to show that this is not true when p is locally modular. During the Banff meeting, Bouscaren, Hart and Laskowski worked to finalize the details of suggestions of Hrushovski's that in the case where p is non-trivial and locally modular then N is "controlled" over M by the generic of a definable group non-orthogonal to p. Progress was made and a paper should be forthcoming."

List of Participants

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Boxall, Gareth (University of Leeds)
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Dolich, Alfred (Chicago State University)
Dzamonja, Mirna (University of East Anglia)
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Gismatullin, Jakub (University of Wroclaw)
Goodrick, John (University of Maryland)
Hart, Bradd (McMaster University)
Haskell, Deirdre (McMaster University)
Hasson, Assaf (Oxford University)
Hils, Martin (Université Paris Diderot Paris 7)
Hrushovski, Ehud (Yale University)
Kamensky, Moshe (University of Waterloo)
Kaplan, Itay (Hebrew University of Jerusalem)

Kikyo, Hirotaka (Kobe University)
Kim, Byunghan (Yonsei University)
Krupinski, Krzysztof (Wroclaw University)
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Chapter 3

Data Analysis using Computational Topology and Geometric Statistics (09w5112)

Mar 08 - Mar 13, 2009

Organizer(s): Peter Bubenik (Cleveland State University), Gunnar Carlsson (Stanford University), Peter Kim (University of Guelph)

Overview of the Field

Mathematical scientists of diverse backgrounds are being asked to apply the techniques of their specialty to data which is greater in both size and complexity than that which has been studied previously. Large, high-dimensional data sets, for which traditional linear methods are inadequate, pose challenges in representation, visualization, interpretation and analysis. A common finding is that these massive data sets require the development of new theory and that these advances are dependent on increasing technical sophistication. Two such data-analytic techniques that have recently developed independently of each other have come to the fore, namely, Geometric Statistics and Computational Topology. Although the former uses geometric arguments, while the latter uses algebraic-topological arguments, and hence they appear disparate, there is substantial commonality and overlap just as in the more traditional overlap between geometry and topology. Thus the purpose of this workshop is to bring together these two research directions and explore their overlap, particularly in the service of statistical data analysis.

A standard paradigm assumes that the data comes from some underlying geometric structure, such as a curved submanifold or a singular algebraic variety. The observed data is obtained as a random sample from this space, and the objective is to statistically recover features of the underlying space and/or the distribution that generated the sample.

In Geometric Statistics one uses the underlying Riemannian structure to recover quantitative information concerning the probability distribution and/or functionals thereof. The idea is to extend statistical estimation techniques to functions over Riemannian manifolds, utilizing spectral methods adapted to the Riemannian structure.

One then considers the magnitude of the statistical accuracy of these estimators. Considerable progress has been achieved in terms of optimal estimation in the minimax sense. These ideas have far reaching implications in the analysis of high-dimensional data such as, for example, in astronomy, biomechanics, medical imaging, microwave engineering and texture analysis.

In Computational Topology, one attempts to recover more qualitative global features of the underlying data instead, such as connectedness, or the number of holes, or the existence of obstructions to certain constructions, based upon the random sample. In other words, one hopes to recover the underlying topology. An advantage of topology is that it is stable under deformations and thus insensitive to errors introduced in the sampling.

A combinatorial construction such as the alpha complex or the Čech complex converts the discrete data into an object for which it is possible to compute the topology. However, it is quickly apparent that such a construction and its calculated topology depend on the scale at which one considers the data. A multiscale solution to this problem is the technique of persistent homology. It quantifies the persistence of topological features as the scale changes. Persistent homology is useful for visualization, feature detection and object recognition. It has been successfully applied to analyze natural images, neurological data, gene-chip data, protein binding and sensor networks.

Although Geometric Statistics and Computational Topology have a disparate appearance and seem to have different objectives, it has recently been noticed that they share a commonality through statistical sampling. In particular it has been noticed that the metric distance of persistent homology in Computational Topology, is intimately related to the sup-norm metric between the underlying density that generates a random sample on a Riemannian manifold, and its statistical estimator. Consequently, the qualitative and quantitative data analyses are intimately linked, which is not surprising because of the close connection between geometry and topology traditionally.

Recent Developments and Open Problems

The use of geometric and topological methods for statistical data analysis is currently being pursued in the three allied fields of computer science, mathematics and statistics. Although each field has their own particular approach and questions of interest, the amount of similarity is striking and this workshop was able to synthesize all three fields together. The open problems that were considered was the development of computational and statistical algorithms and methods using aspects of geometry and topology when data over the geometric object was only available.

We can summarize the type of investigations as it pertains to the aforementioned three fields. A more detailed description is provided in the following section:

- In computer science the pursuit naturally focused on efficient algorithms and visualization. Some specific items discussed included algorithms for the discrete approximation of the Laplacian, algorithms for approximating cut-locus, data reduction techniques, and recovery from noisy data;

- In mathematics the interest focused on certain constructions. Here such topics included zigzag persistence, Hodge theory, and recovering the topology over a random field;

- In statistics parameter estimation was the main interest and topics included bootstrapping and MCMC on manifolds, geodesic PCA, asymptotic minimaxity, conditional independence, statistical multiscale analysis and analysis over the Euclidean motion group.

Additionally, some physical applications were also discussed such as brain mapping, network analysis and biomechanics of osteoarthritis.

Presentation Highlights (in alphabetical order)

Dominique Attali (CNRS, Grenoble)
Persistence-sensitive simplification of functions on surfaces in linear time.
Let f be a real-valued function defined on a triangulated surface S. The persistence diagram of f encodes the homological variations in the sequence of sublevel sets $S_t = f^{-1}(-\infty, t]$. A point (x, y) in the persistence diagram of f corresponds to a homological class which appears in S_x and disappears in S_y. The distance $y - x$ of the point (x, y) to the diagonal represents the importance of the associated homological class: the further away a point is from the diagonal, the more important the associated feature. An ε-simplification of f is a map g on S whose persistence diagram consists only of those points in the diagram of f that are more

than ε away from the diagonal. The speaker gave an algorithm for constructing an ε-simplification of f which is also ε-close to f. This was a joint work with M. Glisse, S. Hornus, F. Lazarus and D. Morozov.

Peter Bubenik (Cleveland State University)
Persistent homology and nonparametric regression.
The talk focused on estimating the persistent homology of sublevel sets of a function on a compact Riemannian manifold, from a finite noisy sample. The Stability Theorem of Cohen-Steiner, Edelsbrunner and Harer bounds the distance between the persistent homologies of the sublevel sets of two functions by the supremum norm of the difference between the two functions. Using this result, the above topological problem was translated to the statistical nonparametric regression problem on a compact manifold under the sup-norm loss. The main result was a calculation the sharp asymptotic minimax bound. Furthermore, the construction of the estimator in the proof is well-suited to calculations of the persistent homology of its sublevel sets. These techniques were applied to brain image data. Initial results indicated the possibility of distinguishing autistic and control subjects by the topology of their brains. This was joint work with Gunnar Carlsson, Moo Chung, Peter Kim, and Zhiming Luo.

Gunnar Carlsson (Stanford University)
Generalized Persistence, Noise, and Statistical Significance
Persistent homology has been shown to be a useful way to detect qualitative structure in various kinds of data sets. The speaker showed that a generalized form of persistence, called “zig-zag persistence”, can be useful both in removing noise in certain geometric problems as well as in understanding statistical significance of qualitative geometric invariants. This was joint work with V. de Silva and D. Morozov.

Fred Chazal (INRIA)
Geometric inference for probability distributions
Data often comes in the form of a point cloud sampled from an unknown compact subset of Euclidean space. The general goal of geometric inference is then to recover geometric and topological features (Betti numbers, curvatures,. . .) of this subset from the approximating point cloud data. In recent years, it has appeared that the study of distance functions allows one to address many of these questions successfully. However, one of the main limitations of this framework is that it does not cope well with outliers nor with background noise. The speaker showed how to extend the framework of distance functions to overcome this problem. Replacing compact subsets by measures, he introduced a notion of distance function to a probability distribution in $\mathbb{R}^n$. These functions share many properties with classical distance functions, which makes them suitable for inference purposes. In particular, by considering appropriate level sets of these distance functions, it is possible to associate in a robust way topological and geometric features to a probability measure. This was joint work with David Cohen-Steiner.

Moo Chung (University of Wisconsin-Madison)
Eigenfunctions of Laplace-Beltrami operator in cortical manifolds
In quantifying cortical and subcortical anatomy of the human brain, various differential geometric methods have been proposed. Many such successful methods are inherently implicit and without explicit parametric forms. Although there are a few parametric approaches such as spherical harmonic descriptors, their application has been limited to simple subcortical structures. The reason for the lack of more explicit parametric approaches is that it is difficult to construct an orthonormal basis for an arbitrary cortical manifold. The speaker proposed to use the eigenfunctions of the Laplace-Beltrami operator, which are computed numerically using the cotan formula. The eigenfunctions are then used in setting up a regression in the cortical manifold. In the heat kernel smoothing framework, smoothing is done by expanding the heat kernel using the eigenfunctions. The eigenfunction approach offers far more flexibility in setting up a statistical model than implicit approaches.

Vin de Silva (Pomona)
Zigzag persistence
Zigzag persistence is a new methodology for studying persistence of topological features across a family of spaces or point-cloud data sets. Building on classical results about quiver representations, zigzag persistence

generalises the highly successful theory of persistent homology and addresses several situations which are not covered by that theory. The speaker presented theoretical and algorithmic foundations with a view towards applications in topological statistics. As an important example, he discussed a particular zigzag sequence derived from the level sets of a real-valued function on a topological space. A powerful structure theorem, called the Pyramid Theorem, establishes a connection between this ”levelset zigzag persistence” and the extended persistence of Cohen-Steiner, Edelsbrunner and Harer. This theorem resolves an open question concerning the symmetry of extended persistence. Moreover, the interval persistence of Dey and Wenger can be understood in this context; in some sense it carries three-quarters of the information produced by the other two theories. This was joint work with Gunnar Carlsson and Dmitriy Morozov.

Tamal Dey (Ohio State University)
Topology by approximating cut locus from point data
A cut locus of a point p in a compact Riemannian manifold M is defined as the set of points where *minimizing* geodesics issued from p stop being minimizing. It is known that a cut locus contains most of the topological information of M. One can try to utilize this property of the cut loci to decipher the topology of M from a point sample. Recently it has been shown that Rips complexes can be built from a point sample P of M systematically to compute the Betti numbers, the rank of the homology groups of M. Rips complexes can be computed easily. However, the sizes of the Rips complexes tend to be large. Since the dimension of a cut locus is lower than that of the manifold M, a sub-sample of P approximating the cut locus is usually much smaller in size and hence admits a relatively smaller Rips complex. The speaker explored the above approach for point data sampled from surfaces embedded in any high dimensional Euclidean space. He presented an algorithm that computes a sub-sample P' of a sample P of a 2-manifold where P' approximates a cut locus. Empirical results show that the first Betti number of M can be computed from the Rips complexes built on these sub-samples.

Leo Guibas (Stanford)
Analysis of Scalar Fields over Point Cloud Data
Given a real-valued function f defined over some metric space X, is it possible to recover some structural information about f from the sole information of its values at a finite subset L of sample points, whose pairwise distances in X are given? The speaker provided a positive answer to this question. More precisely, taking advantage of recent advances on the front of stability for persistence diagrams, he introduced a novel algebraic construction, based on a pair of nested families of simplicial complexes built on top of the point cloud L, from which the persistence diagram of f can be faithfully approximated. He then derived from this construction a series of algorithms for the analysis of scalar fields from point cloud data. These algorithms are simple and easy to implement, have reasonable complexities, and come with theoretical guarantees. This was joint work with F. Chazal, S. Y. Oudot, and P. Skraba.

Susan Holmes (Stanford)
How to sample from a manifold: Applications to validation of Computational Topology and its algorithms
The speaker surveyed the classical methods of parametric bootstrapping and MCMC for generating samples from non uniform distributions. Then she presented work on how to draw samples from a manifold and show how this can be used to compute confidence statements for results from various outputs from computational topology algorithms such as JPlex. This was joint work with Persi Diaconis and Mehrdad Shahshahani.

Stephan Huckeman (Goettingen)
Intrinsic Statistics on Riemannian Manifolds
One goal in image analysis consists in describing statistical distributions of characteristic patterns, e.g. shapes of random physical objects. Typically such shapes live on non-Euclidean manifolds, possibly with unbound curvature at singularities (e.g. Kendall’s 3D shape space). While over the last decades statisticians have used Euclidean approximations to these manifolds thus making tools of classical multivariate analysis available for “sufficiently concentrated data”, this talk aimed at intrinsic generalizations of PCA and MANOVA thus broadening the scope of statistical image analysis.

Matt Kahle (Stanford University)
Moduli spaces of hard disks in a box

The speaker discussed a family of moduli spaces which generalize classical configuration spaces for points in the plane. However, the methods used for computing homology of configuration spaces are not easily applicable to these spaces, and even the number of components seems to be a fairly subtle question. So computational / applied methods were used to better understand this pure math problem. A combination of techniques, including simulated annealing and the nudged elastic band method, were used to compute the most basic topological features of these spaces. There was also a brief discussion of the statistical physics setting that motivates the problem, suggested by Persi Diaconis. This was ongoing joint work with Gunnar Carlsson and Jackson Gorham.

Andre Lieutier (Dassault Systemes)
A stable notion of curvature on point clouds
The speaker addressed the problem of curvature estimation from sampled compact sets. The main contribution was a stability result: the gaussian, mean or anisotropic curvature measures of the offset of a compact set K with positive μ-reach can be estimated by the same curvature measures of the offset of a compact set K' close to K in the Hausdorff sense. He showed how these curvature measures can be computed for finite unions of balls. The curvature measures of the offset of a compact set with positive μ-reach can thus be approximated by the curvature measures of the offset of a point-cloud sample. These results can also be interpreted as a framework for an effective and robust notion of curvature. This was joint work with Frederic Chazal, David Cohen-Steiner and Boris Thibert.

Zhiming Luo (University of Guelph)
Asymptotic minimax regression estimate under super-norm loss on Riemannian manifold
Relating to Peter Bubenik's talk "Persistent homology and nonparametric regression", the speaker gave more details on the minimax nonparametric regression estimator and the exact constant of the sharp asymptotic minimax bound on a compact Riemannian manifold. This was joint work with Peter Bubenik, Gunnar Carlsson, Moo Chung, and Peter Kim.

Facundo Memoli (Stanford)
A Metric Geometry approach to Object Matching
The problem of object matching under invariances can be studied using certain tools from Metric Geometry. The main idea is to regard objects as metric spaces (or measure metric spaces). The type of invariance one wishes to have in the matching is encoded in the choice of the metrics with which one endows the objects. The standard example is matching objects in Euclidean space under rigid isometries: in this situation one would endow the objects with the Euclidean metric. More general scenarios are possible in which the desired invariance cannot be reflected by the preservation of an ambient space metric. Several ideas due to M. Gromov are useful for approaching this problem. The speaker discussed different adaptations of these, and in particular he constructed an L^p version of the Gromov-Hausdorff distance using mass transportation ideas.

Yuriy Mileyko (Duke University)
Defining hierarchical order within reticular networks
While the Strahler Stream Order is a standard method for computing the hierarchical order within non-reticular networks, it cannot handle networks with loops. The speaker presented a new algorithm which can perform such a task for planar networks. This algorithm is based on ideas from persistent homology and may be regarded as a generalization of the Strahler Stream Order. From a topological point of view, the latter method defines a filtration of a network (based on tributaries) and updates the order of the edges at critical events, that is, when two connected components merge. Such an event can be regarded as a change in 0-dimensional homology. Therefore, he defined critical events for networks with loops as changes in 1-dimensional homology. Taking advantage of the planarity of a network, one can trace a sequence of such critical events and update the order of network edges. This work was motivated by the problem of analyzing the structure of leaf networks, and the speaker presented a few preliminary results of such an analysis. He also discussed possible generalizations of the new method to arbitrary networks.

Konstantin Mischaikow (Rutgers University)
Topology Guided Sampling of Nonhomogeneous Random Fields

Topological measurements are increasingly being accepted as an important tool for quantifying complex structures. In many applications these structures can be expressed as nodal domains of real-valued functions and are obtained only through experimental observation or numerical simulations. In both cases, the data on which the topological measurements are based are derived via some form of finite sampling or discretization. The speaker presented a probabilistic approach to quantifying the number of components of generalized nodal domains of non-homogeneous random fields in one space dimension via finite discretizations, i.e., he considered excursion sets of a random field relative to a non-constant deterministic threshold function. He gave explicit probabilistic a-priori bounds for the suitability of certain discretization sizes and also provided information for the choice of location of the sampling points in order to minimize the error probability. He illustrated the results for a variety of random fields, demonstrated how they can be used to sample the classical nodal domains of deterministic functions perturbed by additive noise, and discussed their relation to the density of zeros.

Sayan Mukherjee (Duke University)
Conditional Independence Models via Filtrations
The speaker presented a novel approach to infer conditional independence models or Markov structure of a multivariate distribution. Specifically, the objective is to place informative prior distributions over decomposable graphs and sample efficiently from the induced posterior distribution. The key idea is a parametrization of decomposable hypergraphs using the geometry of points in $\mathbb{R}^m$. This allows for the specification of informative priors on decomposable graphs by priors on a finite set of points. The constructions used have been well studied in the fields of computational topology and random geometric graphs. The framework underlying this idea was developed and its efficacy was illustrated using simulations.

Axel Munk (Goettingen)
Statistical Multiscale Analysis - From Jump detection to Image Analysis
The speaker discussed how to use statistical multiscale analysis (SMA) techniques in order extract jumps from noisy signals in various signal detection problems. This was applied to reconstruct the open states in ion channel experiments for biomembranes. In the second part SMA was extended to image analysis, i.e. to 2D and 3D. The resulting method is locally adaptive, i.e. it automatically adjusts locally any regularisation method to locally varying features, such as edges. This was illustrated with examples from biophotonic imaging.

Vic Patrangenaru (Florida State University)
Asymptotic Statistics and Nonparametric Bootstrap on Manifolds and Applications
Asymptotic statistical analysis and nonparametric bootstrap on smooth geometric objects, or manifolds, is an exciting and challenging field of research, extending multivariate limit theorems to the nonlinear case, where statistical theory and differential geometry are inextricably intertwined, and implementation requires innovative algorithms and high speed computation. This presentation dealt with recent developments in this young area of nonparametric statistics, which must also resolve associated geometric issues and problems of implementation. Asymptotic statistics on manifolds have a wide range of applications in many areas of science including geology, meteorology, biology, medical imaging, bioinformatics and machine vision. This was joint work with R. N. Bhattacharya, F. H. Ruymgaart and other collaborators.

Michael Pierrynowski (McMaster University)
Differential geometry reveals differences in the knee motion of elders with osteoarthritis
Knee motion, force and moment have been used by biomechanists to identify elders with and without knee osteoarthritis (OA). The knee adduction moment has received the most attention since it is associated with the severity and prognosis of OA which then informs clinicians to prescribe effective intervention. However, measuring the knee adduction moment clinically is problematic since it requires synchronized kinematic data acquisition and ground reaction force measures. For potential clinical use the speaker proposed a differential geometry analysis of the easier measured knee kinematics [SE(3)] that shows promise to detect the presence or absence of mild to moderate knee OA. This technique sums over repetitive gait cycles the curvatures and torsions from the translation component of SE(3) which are then geometrically interpreted using Frechel's Theorem. In a similar vein, he examined the length of the paths transcribed on a sphere (S^2) by the three

columns (orthonormal vectors) of the SO(3) orientation component. He reported that during repetitive normal overground gait, the sum of the curvatures and the path length of the third SO(3) vector are smaller in 52 elders with knee osteoarthritis compared to 47 elders with healthy knees. He discussed this finding in relation to OA knees having decreased non-linear motion paths and less tibia rotation during gait. This was joint work with Peter T. Kim.

Louis-Paul Rivest (Université Laval)
Some statistical models for SE(3) data
This presentation began by reviewing the occurrence, in the biomechanical literature, of data sets whose elements belong to SE(3), the 6-dimensional Lie group of 3D rigid body displacements. The construction of some probability models on SE(3) using distance measures was presented. These models were used to describe the dispersion of an observed SE(3) displacement around its "true value". They were used to construct loss functions for the estimation of the parameters of a statistical model for SE(3) data. The SE(3) model used to estimate the directions of the two rotation axes of the ankle was then be presented. Some statistical challenges associated with the estimation of the parameters of this model were reviewed, with some of the solutions that have been put forward. Statistical analyses carried out with the R-package Kinematics for the statistical modeling of SE(3) data were be used to illustrate the theory.

Stephen Smale (Toyota Technological Institute at Chicago)
Hodge Theory
The speaker discussed results on extensions of Hodge theory to metric spaces and the relations to the subject "Topology, Geometry and Data". This was joint work with Nat Smale.

Mikael Vejdemo-Johansson (Stanford University)
Persistent Cohomology and Circular Coordinates
An inherent assumption in algorithms for linear or non-linear dimension reduction (NLDR) is that the data will be representable faithfully and efficiently using real-valued coordinates. However, there are examples that challenge this assumption: the circle, for instance, being inherently one-dimensional, but using two real coordinates for a faithful representation. The speaker presented a strategy for constructing circle-valued functions on a statistical data set. He developed a machinery of persistent cohomology to identify candidates for significant circle-structures in the data, and used harmonic smoothing and integration to obtain circle-valued coordinate functions from representative cocycles of the cohomology classes recovered. He suggested that the enriched class of either real- or circle-valued coordinate functions permits a precise NLDR analysis of a broader range of realistic data sets.

Yusu Wang (Ohio State University)
Approximating Laplace-Beltrami Operator, Integrals and Gradients in Non-statistical Discrete Settings
The Laplace-Beltrami operator of a given manifold (e.g, a surface) is a fundamental object encoding the intrinsic geometry of the underlying manifold. It has many properties useful for practical applications from areas such as graphics and machine learning. For example, its relation to the heat diffusion makes it a primary tool for surface smoothing in graphics. However, many a time, the underlying manifold is only accessible through a discrete approximation, either as a mesh or simply as a set of points. The important question is then how to approximate the Laplace operator and other geometric invariants from such discrete setting. Previously, much work has been done on approximating Laplace operator from points sampled from some probabilistic distribution. The speaker described her recent results on approximating the Laplace operator from either piecewise linear manifolds (e.g, meshes) or simply general point cloud data. She then gave several applications of the constructed discrete Laplace operator, including estimating the gradient, critical points, and the integral of an input function from point cloud data.

Scientific Progress Made

The lectures were organized so that there was approximately equal representation from each of the fields of computer science, mathematics and statistics. Throughout the week, each lecture was well attended with much discussion and enthusiasm displayed by the audience. Questions and inquiries were made from scientists within each field, but also from the other fields as well. The scientific progress that was most prominent therefore can be summarized in terms of the cross-fertilization between the above three fields, and the attempt to bridge the “language gap” between the three fields. The benefits of this synergy was carried on much beyond the lectures, and what was particularly fascinating was that in some fields, what they had been struggling with, was something fairly well known in others.

As mentioned in the introduction, an important goal was broadening the horizons of statistical data analysis. It terms of this it is evident that the statisticians have considerable advantages. What is missing from statisticians is the plethora of geometrical and topological techniques as neither is typically carried out in graduate statistical training. In particular, development of statistical techniques usually take place over Euclidean space. On the other hand, computer scientists and mathematicians have the geometric and topological training, as well as experience in development of algorithms. Nevertheless, statistical techniques needed in non-Euclidean settings are limited consequently, statistical expertise for these types of structures are not readily available as compared to that available for Euclidean space. Consequently, significant scientific progress was made in communicating ideas of each others fields in the pursuit of using geometric and topological methods for statistical data analysis.

Outcome of the Meeting

Throughout the meeting, the organizers informally and continually canvassed the participants as to their thoughts on the progress of the workshop thus far. A common approval by participants was the sentiment. On the final day of the workshop, an informal discussion was held to discuss the outcome of the meeting as well as possible future like events. A very enthusiastic approval was relayed by all participants along with a very sincere desire going forward, to have more of such meetings either as a BIRS workshop, or other formats and venues. It was expressed that an equal representation from computer science, mathematics and statistics is the ideal mixture. Furthermore, in future meetings, in addition to the methodological advances made, a greater quantity of physical applications using geometric and topological statistical data analysis techniques was desired.

List of Participants

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Cohen-Steiner, David (INRIA Sophia Antipolis)
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Chapter 4

Invariants of Incidence Matrices (09w5071)

Mar 29 - Apr 03, 2009

Organizer(s): Qing Xiang (University of Delaware), Chris Godsil (University of Waterloo), Peter Sin (University of Florida)

Overview of the Field

Incidence matrices arise whenever one attempts to find invariants of a relation between two (usually finite) sets. Researchers in design theory, coding theory, algebraic graph theory, representation theory, and finite geometry all encounter problems about modular ranks and Smith normal forms (SNF) of incidence matrices. For example, the work by Hamada [9] on the dimension of the code generated by r-flats in a projective geometry was motivated by problems in coding theory (Reed-Muller codes) and finite geometry; the work of Wilson [18] on the diagonal forms of subset-inclusion matrices was motivated by questions on existence of designs; and the papers [2] and [5] on p-ranks and Smith normal forms of subspace-inclusion matrices have their roots in representation theory and finite geometry.

An impression of the current directions in research can be gained by considering our level of understanding of some fundamental examples.

Incidence of subsets of a finite set

Let X_r denote the set of subsets of size r in a finite set X. We can consider various incidence relations between X_r and X_s, such as inclusion, empty intersection or, more generally, intersection of fixed size t. These incidence systems are of central importance in the theory of designs, where they play a key role in Wilson's fundamental work on existence theorems. They also appear in the theory of association schemes.

Incidence of subspaces of a finite vector space

This class of incidence systems is the exact q-analogue of the class of subset incidences. The possible incidence relations are inclusion or, more generally, intersection in subspace of fixed dimension. These examples have been studied for their relation to questions in representation theory of the general linear group. In some cases, they have been applied to finite geometry and to construct error-correcting codes.

Incidence of distinguished subspaces of a vector space

In the presence of a quadratic, Hermtian or symplectic form, we may refine the above incidence systems by considering distinguished subspaces such as totally isotropic or nonsingular ones. The corresponding classical group acts and there are connections to its representation theory and to the geometry of the associated polar spaces.

General Problem: Computation of invariants

Incidence matrices have invariants at several levels of rigidity. If we consider the matrices as representing linear maps over fields then we wish to compute its eigenvalues and rank in every characteristic p (p-rank for short). Since there is usually a group G acting which preserves the incidence relation, the linear map becomes a homomorphism of G-modules, raising deeper questions about the G-module structure of the domain, codomain, image and kernel of the map.

The incidence matrix is integral, and can also be regarded as the matrix of a homomorphism of free abelian groups. Thus, the invariant factors (or Smith normal form) of the matrix form a stronger set of invariants than the p-ranks, which can be deduced immediately from the former. This time the group action raises questions about representations over the integers and over p-adic rings.

Finally, incidence matrices have been used as parity check or generator matrices of codes. Then the relevant invariants are those which are preserved by automorphisms of the code, such as the minimum weight of a codeword or, more generally the weight enumerator.

Thus it can be seen that there is a multitude of natural problems, depending on the choice of incidence system and the choice of invariant. These problems share many common features but their origins and the reasons for studying some of them are very diverse, so that published work on these questions is scattered across the literature of the subdisciplines. It is no easy task just to keep track of which ones have been answered!

Presentation Highlights and Scientific Progress Made

Design theory

Let v, k, t and λ be integers with $v \geq k \geq t \geq 0$ and $\lambda \geq 1$. A *t-design* on v points with block size k and index λ is an incidence structure $\mathcal{D} = (X, \mathcal{B})$ with:

1. $|X| = v$,
2. each $B \in \mathcal{B}$ is a k-element subset of X,
3. for any set $T \subset X$ of t points, there are exactly λ blocks containing all points in T.

Next we define a class of subset-inclusion matrices. Let X be a v-set. Let W_{tk} denote the $\binom{v}{t}$ by $\binom{v}{k}$ matrix whose rows are indexed by the t-subsets of X, whose columns are indexed by the k-subsets of X, and where the entry in row T and column K is

$$W_{tk}(T, K) = \begin{cases} 1, & \text{if } T \subseteq K, \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

With the definition of W_{tk}, it becomes clear that a t-design is nothing but a $\binom{v}{k}$ by 1 vector x with nonnegative integer entries such that

$$W_{tk}x = \lambda \mathbf{j}, \tag{4.2}$$

where $\mathbf{j}$ is the all-one $\binom{v}{t}$ by 1 vector. Therefore investigating the Smith normal form of W_{tk} is important for the study of t-designs. Rick Wilson in his talk spoke of his work [18] on a diagonal form of W_{tk} and various applications of this result, including application to a zero-sum Ramsey-type problem. It should be noted that the p-ranks of W_{tk} and the p'-case of the subspace-inclusion matrices were treated by representation theoretic methods in the work of Frumkin and Yakir [8]. Rick Wilson also noted that empty intersection relation between subsets and the inclusion relation are essential the same in the set case. This seemingly trivial point was made by at least three of the speakers. Navi Singhi talked about his work on tags on subsets [16], and G. B. Khosrovshahi described his work with his collaborators on special bases of the null space of W_{tk}. These treatments seek to impose orderings on the object, in other words to break their symmetry. Both talks have a strong algorithmic flavor. Vladimir Tonchev talked about his recent work with Dieter Jungnickel on counterexamples to the Hamada conjecture, which states that the geometric designs $\mathrm{PG}_d(n, q)$ and $\mathrm{AG}_d(n, q)$ are characterized as the designs of minimum p-ranks among all designs with the given parameters. It should be noted that Hamada's conjecture implies that for any prime p, the only projective plane of order p is $\mathrm{PG}(2, p)$. Previously, only a few counterexamples (with concrete parameters) to Hamada's conjecture were known. Recently Jungnickel and Tonchev [10] constructed an infinite family of counterexamples. However it should be noted that Hamada's conjecture for symmetric designs with classical parameters is neither proved nor disproved.

Related to Singhi's talk, we mention that so far attempts to define a theory of tags for vector spaces have not been fruitful. But in work of Paul Li [13] solving the conjecture of Brouwer on the 2-rank of the symplectic dual polar spaces over $\mathrm{GF}(2)$, one can clearly see similar ideas about ordering (i.e., breaking symmetry) applied to good effect.

Strongly regular graphs

A *strongly regular graph srg* (v, k, λ, μ) is a graph with v vertices that is regular of valency k and that has the following properties:

1. For any two adjacent vertices x, y, there are exactly λ vertices adjacent to both x and y.
2. For any two nonadjacent vertices x, y, there are exactly μ vertices adjacent to both x and y.

It is well known that strongly regular graphs are equivalent to two-class association schemes. Many of the problems above had already been considered in the context of strongly regular graphs. This area has been a rich source of examples and interesting problems on invariants of incidence matrices (more appropriately, adjacency matrices). There are theorems which characterize graphs by the invariants and interesting examples of nonisomorphic graphs with the same invariants. In his talk, Andries Brouwer surveyed the results from his work with Van Eijl [4]. For various graphs the SNF of the adjacency matrix is given. Kneser graphs are defined (graphs on flags of a building of spherical type, adjacent when far apart) and it is shown by examples that in the thin case these Kneser graphs often have the property that the SNF of the adjacency matrix A equals the SNF of the diagonal matrix with the spectrum of A on the diagonal, while in the thick case the SNF has only powers of p. Quite a few very interesting remarks were made in the talk. For example, Brouwer commented that it is generally easier to consider the relations of "far apart" rather than "close together". Examples of this include Kneser graphs and their q-analogues. This philosophy is borne out in the vector space setting where we know the p-ranks for r-dimensional subspaces versus s-dimensional subspaces for all r and s when the relation is zero intersection, but we know the p-ranks only when either r or s is equal to 1 if we consider inclusion. A nice open problem along these lines is to compute the integral invariants for zero intersection of r-subspaces and s-subspaces in projective space. After showing an old proof by Brouwer and Van Eijl for the p-ranks of Paley graphs and large submatrices, Brouwer commented that the "and large submatrices" part is interesting: representation-theoretic methods usually give the p-rank of the full matrix but do not give information on submatrices. Willem Haemers talked about the work of his former student Rene Peeters [14] on p-ranks and SNF of distance-regular graphs.

Representation theory

In recent years representation theory has proven itself to be an extremely powerful tool for the exact calculation of p-ranks.

New results announced at the workshop included the solution by P. Sin of the p-rank problem for point-hyperplane incidences in orthogonal geometries, originally raised in a 1995 paper of Blokhuis and Moorhouse [3], and the solution of the analogous question for hermitian geometries by P. Sin and O. Arslan. This work applies fairly sophisticated techniques of representation theory of algebraic groups in characteristic p, such as the Jantzen Sum Formula and the theory of good filtrations. The p-ranks in question turn out to be the (previously unknown) dimensions of irreducible representations and the above theory reduced the problem to some complicated but tractable combinatorics.

Representation theory is an important tool in the work Bardoe-Sin [2] describing the permutation module for $\mathrm{GL}(n,q)$ on the points of projective space. This work yields the p-rank of many incidence systems on which $\mathrm{GL}(n,q)$ acts, including the famous Hamada formula for the p-rank of points versus subspaces of a fixed dimension. Later the p-rank for the incidence relation of zero intersection between r-subspaces and s-subspaces for any r and s was determined from detailed knowledge of this module.

The work of Chandler, Sin and Xiang [6, 7] on the p-ranks for symplectic spaces also depends heavily on representation theory. A special case of their computations which is of interest to other areas include the p-ranks for the symplectic generalized quadrangles in characteristic p, which are consequently all known now.

The success of representation theory in the above problems suggests applying the representation theory of the symmetric group to incidences of subsets of a set. In a way the elegant matrix method of Wilson serves the same purpose as representation theory. Nevertheless, it may still be an instructive exercise to recast this body of work in the language of symmetric group representations, Young tableaux, Specht modules etc. This might also throw some light on open problems such as the incidence of subsets with prescribed intersection size.

Coding theory

Recently, electrical engineers have been interested in low-density parity-check (LDPC) codes defined by incidence matrices of generalized polygons. The dimensions of the codes are 2-ranks, which are known, having been computed by eigenvalue methods and representation theory. However, one can also ask questions about weight enumerators, for which such methods cannot be used. The earliest work in this direction was by Bagchi and Sastry [1] but the subject has been dormant until the recent interest. L. Storme, J-L. Kim, K. Mellinger and others have brought new life to the subject. In this talk, Leo Storme survey his recent results with his collaborators on codewords of small weights in codes arising from projective planes, on codewords of small or large weights in codes arising from the classical generalized quadrangles. The methods used here are mainly from finite geometry. See [12, 11] for more details.

Computation

Incidence matrix problems can easily stretch computers to the limit. For example, in the process of computing the SNF of a $(0,1)$-incidence matrix, the entries of the matrices arising from intermediate steps can get extremely large even though the entries in the original matrix are very small (here the entries are 0 or 1). Saunders gave an overview of the LinBox package, which contains efficient algorithms for computing SNF of integral matrices. In particular, Saunders explained the importance and effectiveness of probabilistic algorithms. Brouwer explained the need for ways to parallelize computations.

Open Problems

Many open problems and conjectures were proposed in the talks of the workshop or during informal discussions. Some were already mentioned in previous sections. Here we collect a few of them.

1. Let V be an $(n+1)$-dimensional vector space over GF(q), where $q=p^t$. For any i, $1\le i\le n$, we use $\mathcal{L}_i$ to denote the set of all i-dimensional subspaces of V. For integers r,s, $1\le s\le r\le n$, let $A_{r,s}(q)$ denote the (0,1)-incidence matrix with rows indexed by elements Y of $\mathcal{L}_r$ and columns indexed by elements Z of $\mathcal{L}_s$, and with (Y,Z)-entry equal to 1 if and only if $Z\subseteq Y$. The p-rank of $A_{r,s}(q)$ is known when $s=1$. What is the p-rank of $A_{r,s}(q)$ when $1<s<n$?

2. Using the notation in Problem 1, for integers r,s, $1\le s\le r\le n$, let $B_{r,s}(q)$ denote the (0,1)-incidence matrix with rows indexed by elements Y of $\mathcal{L}_r$ and columns indexed by elements Z of $\mathcal{L}_s$, and with

(Y, Z)-entry equal to 1 if and only if $Z \cap Y = \{0\}$. The p-rank of $B_{r,s}(q)$ is known from the work of P. Sin [15]. What is the SNF of $B_{r,s}(q)$?

3. In [6, 7], the symplectic analogues of Hamada's formula were given. How about orthogonal and Hermitian analogues of Hamada's formula?

4. How to define tags on subspaces of a finite dimensional vector space so that we can use them to solve p-rank and SNF problems for incidence relations between subspaces?

5. In [17], it was shown that every commutative semifield of order congruent to 1 modulo 4 gives rise to a strongly regular graph with Paley parameters (or, a pseudo-Paley graph, for short). Assume that q is an odd prime power. Let j be a nonsquare in $K = \mathrm{GF}(q)$, and let $1 \neq \sigma \in Aut(K)$. The Dickson semifield $(K^2, +, *)$ is defined by

$$(a, b) * (c, d) = (ac + jb^\sigma d^\sigma, ad + bc).$$

Let

$$D(q, \sigma) = \{(x^2 + jy^{2\sigma}, 2xy) \mid (x, y) \in K^2,\ (x, y) \neq (0, 0)\}, \tag{4.3}$$

i.e., D is the set of nonzero "squares" of the Dickson semifield. Then the Cayley graph $X(K^2, D(q, \sigma))$ with vertex set K^2 and connecting set $D(q, \sigma)$ is a pseudo-Paley graph. Let $q = 3^t$, let A be the adjacency matrix of $X(K^2, D(3^t, \sigma))$, and let $r_t = \mathrm{rank}_3(A)$ (i.e., the rank of A over $\mathrm{GF}(3)$). The first few terms of the sequence $(r_t)_{t\geq 1}$ were computed by David Saunders and Guobiao Weng. For example, $r_1 = 4, r_2 = 20, r_3 = 85, r_4 = 376, r_5 = 1654, r_6 = 7283, r_7 = 32064$. Based on the above data, David Saunders conjectured that

$$r_t = 4r_{t-1} + 2r_{t-2} - r_{t-3},$$

for all $t \geq 4$. The significance of the conjecture lies in that its validity immediately implies that the pseudo-Paley graph constructed from the Dickson semifield (where $q = 3^t$) is not isomorphic to the Paley graph with the same parameters.

Conclusion

Invariants of incidence matrices have been studied by researchers in algebraic graph theory, representation theory, design theory and coding theory. Bringing together people working on distinct but overlapping and strongly analogous problems has helped to create a clear view of what is known and what are important open questions. The lectures and informal discussions have gone a long way to clarifying the relationships between the different theories, the role of different technical approaches and how the main problems fit together into a unified scheme. The territory has been charted clearly. We can now see the holes in our knowledge which can soon be filled in by existing methods and also the cutting edge problems where new results will mark significant progress. Experimental methods have also been examined in depth, with clear evidence of the value of computer work for producing conjectures and expert discussion of the precise limitations of computers in handling incidence problems.

A good example of work exhibiting influences from many sources was the talk of David Chandler. In his work with P. Sin and Q. Xiang on the integral invariants for incidence of points and subspaces of a fixed dimension in a finite projective space, combination of character sums and p-adic methods such as Stickelberger's theorem and Wan's theorem with representation theory of the general linear group. The character sums can be considered a technique imported from the theory of difference sets. He also showed how these ideas could be applied to solve a classical problem in Galois geometry on the size of intersections of unitals.

In organizing this workshop our goals were to provide a general context for a broad range of analogous problems which previously may have appeared isolated, and to publicize certain methods, approaches and problems from the component subdisciplines which were either unknown or had never been tried by researchers with other backgrounds. Our scientific aims will have been achieved if the workshop has accelerated the adoption of new methods, provoked interest in open problems and provided a framework for future collaborative work between different subdisciplines in which invariants of incidence matrices are important.

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Chapter 5

Semiparametric and Nonparametric Methods in Econometrics (09w5032)

Apr 05 - Apr 10, 2009

Organizer(s): Oliver Linton (London School of Economics), Joel L. Horowitz (Northwestern University), Enno Mammen (University of Mannheim), Yacine Ait-Sahalia (Princeton University)

Introduction by The Organizers

The main objective of this workshop was to bring together mathematical statisticians and econometricians who work in the field of nonparametric and semiparametric statistical methods. Nonparametric and semiparametric methods are active fields of research in econometric theory and are becoming increasingly important in applied econometrics. This is because the flexibility of non- and semiparametric modelling provides important new ways to investigate problems in substantive economics. Many of the most important developments in semi- and nonparametric statistical theory now take place in econometrics. Moreover, the development of non- and semiparametric methods that are suitable to the needs of economics presents a variety of mathematical challenges. Econometric research aims at achieving an understanding of the economic processes that generate observed data. This is different from fitting data that may be useful for prediction but that do not capture underlying causes. A large part of economic theory consists of models of equilibria of competing processes. Statistical data are a snapshot of the equilibrium but, by themselves, do not reveal the processes that led to the equilibrium. Consequently a reduced form model (e.g.\a conditional mean function) does not suffice for much economic research. Achieving an understanding of the economic processes requires a careful combining of economic theory and statistical considerations. This often requires the development of statistical tools that are specific to the problems that arise in economics and are unfamiliar in other statistical specialties. For example, econometric research has focused on developing methods to deal with endogenous covariates (that is, covariates that are correlated with a model's error terms), time series models that fit equilibria as stationary submodels (cointegration), and time series models for volatility processes (conditional variances) in finance. Semi- and nonparametric methods are being used increasingly frequently in applied econometrics. The models are not necessarily of the simple form of classical regression, "response = signal plus independent noise," where the signal can be recovered by nonparametric smoothing of the responses. Rather, the nonparametric functions enter the model in a much more complicated way. Mathematically this has led to challenging problems. Identifiability of a model is much more involved in nonparametric model specifications. In particular, this is the case for nonseparable models where the error terms do not enter additively into the model. Some nonparametric inference problems with endogenous covariates lead to statistical inverse problems and require the study of estimates and solutions of noisy integral equations. The mathematical analysis of nonparametric time-series models and of nonparametric diffusion models is strongly related to research in stochastic processes, Markov processes, stochastic analysis and financial mathematics. Empirical process theory is an essential tool for the understanding of uniform performance and of convergence rates of nonparametric estimates and for efficiency considerations in semiparametric models. All these problems were topics of talks and discussions at the workshop. The mathematical development in econometrics is complimentary to recent statistical applications in biology. There, the focus tends to be on dimension reduction for the statistical analysis of high-dimensional data. The intellectual charm of mathematical research in modern econometrics comes from the interplay between statistical and economic theory.

Abstracts

Identifying the Returns to Lying When the Truth is Unobserved, Arthur Lewbel, Boston College

Consider an outcome Y, an observed binary regressor D, and an unobserved binary D*. This paper considers nonparametric identification and estimation of the effect of D on Y, conditioning on D*. Suppose Y is wages, unobserved D* indicates college experience, and D indicates claiming to have been to college. This paper identifies the 'returns to lying' difference in wages, about 6% to 20%, between those who falsely claim college versus those who tell the truth about not having college.

Identification is obtained either by observing a variable V roughly analogous to an instrument, or by imposing restrictions on model error moments.

Testing Conditional Factor Models, Dennis Kristensen, Columbia University

We develop a new methodology for estimating time-varying factor loadings and conditional alphas based on nonparametric techniques. We test whether long-run alphas, or averages of conditional alphas over the sample, are equal to zero and derive test statistics for the constancy of factor loadings. The tests can be performed for a single asset or jointly across portfolios. The traditional Gibbons, Ross and Shanken (1989) test arises as a special case when there is no time variation in the factor loadings. As applications of the methodology, we estimate conditional CAPM and Fama and French (1993) models. We reject the null that long-run alphas on book-to-market and momentum decile portfolios are equal to zero even though there is substantial variation in the conditional factor loadings of these portfolios.

Semiparametric modeling and estimation of the dispersion function in regression, Ingrid Van Keilegom, Universite catholique de Louvain

Modeling heteroscedasticity in semiparametric regression can improve the efficiency of the estimator of the parametric component in the regression function, and is important for inference problems such as plug-in bandwidth selection and the construction of confidence intervals. However, the literature on exploring heteroscedasticity in a semiparametric setting is rather limited. Existing work is mostly restricted to the partially linear mean regression model with a fully nonparametric variance structure. The nonparametric modeling of heteroscedasticity is hampered by the curse of dimensionality in practice. Moreover, the approaches used in existing work need to assume smooth objective functions, therefore exclude the emerging important class of semiparametric quantile regression models. To overcome these drawbacks, we propose a general semiparametric location-dispersion regression framework, which enriches the currently available semiparametric regression models. With our general framework, we do not need to impose a special semiparametric form for the location or dispersion function. Rather, we provide easy to check sufficient conditions such that the asymptotic normality theory we establish is valid for many commonly used semiparametric structures, for instance, the partially linear structure and single-index structure. Our theory permits non-smooth location or dispersion functions, thus allows for semiparametric quantile heteroscedastic regression. We demonstrate the proposed method via simulations and the analysis of a real data set. (This is joint work with Lan Wang).

Asymptotic Theory for Nonparametric and Semiparametric Estimation with Spatial Data, Peter Robinson, London School of Economics

We develop conditions for asymptotic statistical theory for estimates of nonparametric and semiparametric models when the data are spatial, or spatio-temporal. We attempt to cover data that are regularly or irregularly-spaced, as well as ones where only pairwise distances are available, and cross-sectional data in which even this information is lacking but dependence is feared. The stress is on allowing for a broad range of spatial dependence, including long-range dependence, and heterogeneity, including conditional and unconditional heteroscedasticitya

On the Regularization Power of the Prior Distribution in Linear ill-Posed Inverse Problems, Anna Simoni, Toulouse School of Economics

We consider a functional equation of type ˆY = Kx + U in an Hilbert space. We wish to recover the functional parameter of interest x after observing ˆY . This problem is ill-posed because the operator K is assumed to be compact. We consider a class of models where the prior distribution on x is able to correct the ill-posedness even for an infinite dimensional problem. The prior distribution must be of the g-prior type and depends on the regularization parameter and on the degree of penalization. We prove that, under some conditions, the posterior distribution is consistent in the sampling sense. In particular, the prior-to-posterior transformation can be interpreted as a Tikhonov regularization in the Hilbert scale induced by the prior covariance operator. Finally, the regularization parameter may be treated as an hyperparameter and may be estimated using its posterior distribution or integrated out.

Efficient Estimation in ICA and ICA-Like Models, P.J. Bickel, UC Berkeley

The ICA (Independent Component Analysis) generalization of the multivariable Gaussian model corresponds to observing n iid observations of the form:

$$X = AZ,$$

where X, Z are p=1 and the components of Z are independent. It is well known that if Z has at most one Gaussian component A is identifiable up to permutation and scaling of the rows. Chen and Bickel (2006) Ann.Statist. constructed efficient estimates of A, ie ones which achieved the information bound if the components of Z had distributions known up to scale.

We'll discuss estimation in a generalization and a related model. The generalization is to the case where:

$$X = (X_1, \ldots, X_n) \qquad\qquad X_i = AZ_i$$

the X_i are p dimensional, the Z have dimension $q < p$, p is unknown, the Zi components are independent but differ in i through scale only and the rows of A are sparse. A second model discussed is due to Blanchard et al (2006), JMLR. We indicate its identity to the "sliced inverse regression" model of D.Cook(2007)Statistical Science and discuss efficient estimation in that model as well.

Inference Based on Conditional Moment Inequalities, Donald W. K. Andrews, Yale University

In this paper, we propose an instrumental variable approach to construct confidence sets for the true parameter in models defined by conditional moment inequalities/ equalities. We show that by properly choosing instrument functions, one can transform conditional moment inequalities/equalities into unconditional ones without losing identification power. Based on the unconditional moment inequalities/equalities, we construct confidence sets by inverting Cramér-von Mises-type tests. Critical values are chosen using generalized moment selection (GMS), plug-in asymptotic, and subsampling procedures. We show that the proposed confidence sets have correct uniform asymptotic coverage probabilities. New methods are required to establish these results because an infinite-dimensional nuisance parameter affects the asymptotic distributions. We show that the tests considered are consistent against all fixed alternatives and have power against a broad array of n-1/2-local alternatives, though not all such alternatives. We extend the results to allow for an infinite number of conditional or unconditional moment inequalities/equalities.

Identification and Estimation of Marginal Effects in Nonlinear Panel Models, Whitney Newey, MIT

This paper gives identification and estimation results for marginal effects in nonlinear panel models. We find that linear fixed effects estimators are not consistent, due in part to marginal effects not being identified. We derive bounds for marginal effects and show that they can tighten rapidly as the number of time series observations grows. We also show in numerical calculations that the bounds may be very tight for small numbers of observations, suggesting they may be useful in practice. We propose two novel inference methods for parameters defined as solutions to linear and nonlinear programs such as marginal effects in multinomial choice models. We show that these methods produce uniformly valid confidence regions in large samples. We give an empirical illustration.

Gaussian process priors in nonparametric and semiparametric estimation, Aad van der Vaart VU University Amsterdam

We discuss the use of Gaussian processes as priors for an unknown function in a Bayesian analysis. We study the posterior distribution of the function or a parameter in the ”frequentist” set-up, where the data are generated according to a fixed ”true distribution”. We explain how the rate of contraction of the posterior to the true parameter depends on the properties of the Gaussian process (reproducing kernel Hilbert space and small ball probability), how (random) scaling influences this rate, and how this changes if we are interested in a (semiparametric) functional. The first part of the talk is based on joint work with Harry van Zanten; the second part is mainly based on work of Ismael Castillo, both from VU University Amsterdam.

Nonparametric partial-frontier estimation: robustness and efficiency, Irene Gijbels, Katholieke Universiteit Leuven, Belgium,

One of the major aims in recent nonparametric frontier modeling is to estimate a partial frontier well inside the sample of production units but near the optimal boundary. Two concepts of partial boundaries of the production set have been proposed: an expected maximum output frontier of order m = 1, 2, . . . and a conditional quantile-type frontier of order 2]0, 1]. In this talk, we answer the important question of how the two families of partial production frontiers are linked. For each order m, we specify the order for which both partial boundaries can be compared. A discussion on the breakdown as well as on the efficiency of the nonparametric order-m frontiers and the order- frontiers is provided. Some asymptotic results are discussed. The theoretical findings are illustrated through some simulations and data analysis. This talk is based on joint work with Abdelaati Daouia.

Estimation of nonparametric models with simultaneity, Rosa L. Matzkin, University of California, Los Angeles

We introduce new estimators for nonparametric, nonadditive models with simultaneity. The estimators are computed as simple functionals of nonparametric estimators of the distribution of the observable variables, using constructive methods for identification. They are shown to be consistent and asymptotically normal. It is shown that when each structural equation possesses an exclusive observable exogenous variable, then under some restrictions on the derivatives with respect to those variables, one can estimate all the remaining derivatives by matrix inversion and multiplication, analogous to Least Squares. The paper analyzes in detail the identification and estimation of a single equation model when simultaneity is present and an instrument is available.

Identification in Accelerated Failure Time Competing Risks Models, Sokbae Lee University College London

We provide new conditions for identification in accelerated failure time competing risks models. In our model, we specify unknown regression functions and the joint survivor function of latent disturbance terms nonparametrically. We show that the model can be identified with covariates that are independent of latent errors, provided that certain rank conditions are satisfied. We present a simple example in which our rank conditions for identification are verified. Our identification strategy does not have the problem of identification at near zero".

On Robust Estimation of Moment Condition Models with Dependent Data, Y. Kitamura, Yale University

Moment condition models are frequently used in dynamic econometric analysis. They are particularly useful when one wishes to avoid fully parameterizing the dynamics in the data. Even with such flexibility, however, it is often highly desirable to use an estimation method that is robust against small deviations from the model assumptions. For example, measurement errors, which cause vexing problems in time series analysis, can contaminate observations and thereby leading to such deviations.

Though GMM is generally considered to be a robust estimator, this paper demonstrates that an alternative estimator, which is termed the blockwise minimum Hellinger distance estimator (the blockwise MHDE), possesses desirable optimal properties in terms of robustness. Simulations confirm these theoretical results, and GMM is found to be quite sensitive to deviations of data from the model specification. Previous results obtained by Kitamura, Otsu, and Evdokimov (2008) are extended in several aspects.

Semiparametric estimation of markov decision processes with continuous state space, S.T. Srisuma, London School of Economics

We provide two-step root T consistent estimators for the structural parameters for a class of semiparametric Markovian discrete choice models. Such models are popular in applied work, in particular with labor and industrial organization. We extend the simple methodology of Pesendorfer and Schmidt-Dengler (2008) to allow for continuous observable state space. This extension is non-trivial as the value functions, to be estimated nonparametrically in the first stage, are defined recursively in a non-linear functional equation. Utilizing structural assumptions, we show how to consistently estimate these infinite dimensional parameters as a solution to an integral equation of type 2, see Linton and Mammen (2005), the solving of which is a well-posed problem. We employ the method of kernel smoothing in the first stage and also provide the distribution theory for the value functions.

Functional Linear Instrumental Regression, Sébastien Van Bellegem, Toulouse School of Economics

This talk is devoted to the functional linear model $Y = \langle Z, \varphi \rangle + U$ where Z is a random element in a Hilbert space $\mathcal{H}$, φ is an unknown function of $\mathcal{H}$, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{H}$ and U is a random error that is orthogonal to some functional instrumental variable W. A particular case is given when $\mathcal{H} = \mathbb{R}^p$, in which case we recover the standard linear instrumental regression. We show that solving this problem is a linear ill-posed inverse problem, with a known but data-dependent operator. Our goal is to analyse the rate of convergence of the Tikhonov-regularized estimator, when we premultiply the problem by an instrument-dependent operator B. This extends the Generalized Method of Moments to functional GMM. We then discuss the optimal choice of B and propose an extension of the notion of "weak instrument" to this nonparametric framework.

Nonparametric estimation of Exact consumer surplus with endogeneity in price, Anne Vanhems, Toulouse School of Economics

This paper deals with nonparametric estimation of variation of exact consumer surplus with endogenous prices. The variation of exact consumer surplus is linked with the demand function via a non linear differential equation and the demand is estimated by nonparametric instrumental regression. We analyze two inverse problems: smoothing the data set with endogenous variables and solving a differential equation depending on this data set. We provide some nonparametric estimator, present results on consistency and optimal choice of smoothing parameters, and compare the asymptotic properties to some previous works.

Identification and Estimation of a Nonparametric Transformation Model, Hidehiko Ichimura, University of Tokyo

A nonparametric version of the Box-Cox transformation model is considered. Namely, the dependent variable, Y, and a vector of explanatory variables, X , are observed, and (Y,X) is being generated by the model $\Lambda(Y) = m(X) + U$, where Λ is a strictly increasing unknown function and m$(\cdot)$ is an unknown function, U is an unobserved random variable that is independent of X with the cumulative distribution function F.

Sufficient conditions under which identification of Λ, m(.), and F are achieved are discussed, estimators of these parameters developed, and their consistency and asymptotic distribution theory established.

Nonparametric Identification in Generalized Competing Risks Models with Applications to Second-Price Auctions, T. Komarova, London School of Economics

This paper proposes an approach to proving nonparametric identification for distributions of bidders' values in asymmetric second-price auctions. I consider the case where bidders have independent private values, and the only available data pertain to the winner's identity and to the transaction price. I provide conditions on observable data sufficient to guarantee point identification. My identification proof is constructive and based on establishing the existence and uniqueness of a solution to the system of non-linear differential equations that describes the relationships between unknown distribution functions and observable functions. It comprises two logical steps: proving the existence and uniqueness of a local solution, and then extending that solution to the whole support.

In addition to the main result, I demonstrate how this approach can obtain identification in more general auction settings, such as those with a stochastic number of bidders, or with weaker support conditions. I also show that my results can be extended to generalized competing risks models. Moreover, contrary to classical competing risks (Roy model) results, I describe how generalized models can yield implications that can help check for model misspecification. Finally, I provide a sieve minimum distance estimator and show that it consistently estimates the underlying valuation distribution of interest.

Sparse non-Gaussain component analysis, V.Spokoiny, Humboldt University

Non-gaussian component analysis (NGCA) introduced in Blanchard et al (2006) offered a method for high dimensional data analysis allowing for identifying a low-dimensional non-Gaussian component of the whole distribution in an iterative and structure adaptive way. An important step of the NGCA procedure is identification of the non-Gaussian subspace using Principle Component Analysis (PCA) method. This article proposes a new approach to NGCA called \sparse NGCA which replaces the PCA-based procedure with a new the algorithm we refer to as \convex projection.

On PSMD Plug-in Estimation of Functionals of Semi/nonparametric Conditional and unconditional Moment Models, Xiaohong Chen, Yale University

In this paper, we consider estimation of functionals of unknown parameters that are identified via the "plug-in" semi/nonparametric conditional and unconditional moment models, in which the generalized residual functions may be non-pointwise smooth with respect to the unknown functions of endogenous variables. We establish the asymptotic normality of the penalized sieve minimum distance estimator (PSMD) of any functionals, which may or may not be root-n estimable. For functionals that are root-n estimable, our PSMD estimator achieves the semiparametric efficiency bound of Ai and Chen (2005). Regardless whether the functionals are root-n estimable or not, we show that the profile optimally weighted criterion function is chi-square distributed. We provide two example applications: (1) root-n efficient estimation of weighted average derivative of nonparametric quantile instrumental variables (IV) regression; (2) pointwise asymptotic normality of nonparametric quantile IV regression.

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Chapter 6

Causal Inference in Statistics and the Quantitative Sciences (09w5043)

May 03 - May 08, 2009

Organizer(s): Erica E M Moodie (McGill University), David A Stephens (McGill University)

A Short Overview of the Field

Causal inference attempts to uncover the structure of the data and eliminate all non-causative explanations for an observed association. The goal of most, if not all, statistical inference is to uncover causal relationships. However it is not in general possible to conclude causality from a standard statistical inference procedure, it is merely possible to conclude that the observed association between two variables is not due to chance. Statistical inference procedures do not provide any information about which variable causes the other, or whether the apparent relationship between the two variables is due to another, confounding variable. An explicit introduction of the philosophy of and approaches to causation was first brought into the statistical sciences in 1986 by Paul Holland [1], although references to causal approaches exist in the literature up to 60 years prior (see, for example, [2, 3]). Since then, there has been an explosion of research into the area in a variety of disciplines including statistics (particularly biostatistics), computer science, and economics.

Causal inference in statistics is a broad area of research, under which many topics fall. In particular, the following themes were considered:

1. Inference and asymptotic theory

   Causal inference provides a natural testbed for classical asymptotic theory, in particular, semi-parametric inference. Consistent estimation of causal parameters is guaranteed under certain standard regularity and sampling conditions, by the standard asymptotic theory of estimating equations. More interesting, however, is first, the issue of semiparametric efficiency - optimal asymptotic variances can be deduced by appealing to semiparametric arguments, but estimation is complicated by the presence of nuisance parameters in different components of the model - and secondly, the issue of "double robustness" [4], where consistent estimates of causal parameters follows even under misspecification of the mean model, provided a nuisance model such as the intervention or treatment model or a missingness mechanism model is correctly specified. In addition, non-standard asymptotic theory is required for certain non-regular problems, for instance, those that arise for non-differentiable estimating functions in the study of dynamic regimes [5].

2. Balancing scores and inverse weighting: advances in biostatistics

The fundamental objective of causal inference is to balance the treatment groups so that the treated and untreated subjects are comparable with respect to confounding variables. There are two common approaches to achieving this balancing that are frequently employed in biostatistics. The first relies on modeling the probability of receiving treatment, so that comparisons between treatment groups may be made within strata of subjects who have similar profiles with respect to their likelihood of treatment exposure. Adjustment for the probability of receiving treatment is typically accomplished by weighted regression (Marginal Structural Models [6, 7]), adjustment in a regression model or matching (propensity scores [8]).

The second approach to causal comparisons is most relevant in the context of clinical trials. This approach aims to identify subjects who have complied with their randomly assigned treatment allocation and to compare response between treated and untreated subjects within strata of subjects who have similar profiles with respect to their likelihood of complying with the assigned treatment [9].

3. Instrumental variables and structural equation models: connecting statistics and econometrics

   As in the health sciences, economists are typically interested in causal relationships, such as determining whether a particular training program increases income. In an instrumental variables analysis, the key assumption of most causal methods – that all confounding variables have been recorded – is dropped; in its place, the analyst requires an instrument, i.e. a variable which predicts exposure but does not affect outcome via any other pathway [10]. Many important methods of causal inference including the instrumental variables approach to analysis and the Generalized Propensity Score – an extension of the traditional propensity score that facilitates the estimation of dose-response relationships – were developed by economists. These methods are particularly useful and are generally under-used by statisticians.

4. Adaptive treatment regimes

   Estimating the best sequence of treatment regime for a chronic illness such as hypertension or cancer presents many statistical challenges. In many such diseases, the potential for microbial resistance, toxic side-effects, and compliance with treatment over time can complicate the ability to decide when and how to recommend treatment changes. Typically, the individual tailoring of treatments has been done at the clinical level on an ad-hoc or experience-driven basis at the physician's discretion, and is not based on statistical evidence. The area of dynamic (or adaptive) treatment regimes, pioneered in the statistics literature by Dr. Susan Murphy [11] and Dr. James Robins [5], attempts to formalize the estimation of optimal decision rules for treatment over time, specific to time-varying patient characteristics. Sequential decision making problems such as the estimation of optimal adaptive treatment regimes have also been considered in the computer sciences, through methods in artificial intelligence, reinforcement learning, and control theory.

5. Bayesian causal inference

   There has been relatively little attention given to causal approaches such as marginal structural models in the Bayesian communities, although there are of course exceptions (e.g. [12]). Many of the methods of causal inference including regression on propensity scores, marginal structural models, and instrumental variables require two-step approaches in which a number of nuisance parameters much be estimated. A Bayesian approach would allow for cohesive propagation of the uncertainty in the models.

Presentation Highlights

The purpose of this inter-disciplinary workshop was threefold: to review recent advances in the causal inferences in statistic; to bring together researchers from related fields, in particular Economics, Computer Sciences, and Epidemiology, who work on causal inference methodology so that approaches and ideas may be shared; and finally, to increase the profile of causal inference amongst statisticians in Canada.

The workshop opened on the evening of May 3 with a social hour and posters presented by Bibhas Chakraborty (University of Michigan), Ashkan Ertefaie (McGill University), Sara Geneletti (Imperial College London), Jay Kaufman (McGill University), Benjamin Rich (McGill University), Susan Shortreed (McGill University), Elizabeth Stuart (Johns Hopkins University), and Yongling Xiao (McGill University).

The keynote speaker was Judea Pearl, a computer scientist and philosopher who formalized the topic of causal reasoning in his seminal book, "Causality: Models, Reasoning, and Inference" [13]. Dr. Pearl provided the workshop participants with a two-hour overview of causes and counterfacturals, introducing principles based on non-parametric structural equation models that are sufficient for solving many problems involving causal relationships.

Monday May 4, 2009

Pearl, Judea (University of California, Los Angeles) Discussed principles, based on non-parametric structural equation models enriched with ideas from logic and graph theory, that give rise to a formal calculus of counterfactuals and unify existing approaches to causation.

van der Laan, Mark (University of California, Berkeley) Presented an approach to causal effect estimation that uses cross-validation to select optimal combinations of many model fits, and subsequent targeted maximum likelihood estimation to target the fit towards the causal effect of interest. This approach takes away the need for specifying regression models, while still providing maximum likelihood based estimators and inference.

Geneletti, Sara (Imperial College London) Gave an overview of the decision theoretic framework for causal inference and discussed the pros and cons of this approach compared to one based on countrefactuals, arguing that DT provides a more concise, economical and justifiable approach to inference of treatment effects.

Neufeld, Eric (University of Saskatchewan) Presented visualizations offering an interesting pedagogical tool for explaining the ideas of causation, intervention, and confounding.

Tuesday May 5, 2009

Small, Dylan (University of Pennsylvania) Introduced the malaria attributable fraction (MAF), talked about difficulties in estimating this quantity, and presented a potential outcomes framework for defining and estimating the MAF, as well as a sensitivity analysis that assesses the sensitivity of inferences to departures from the assumption of random assignment of parasite densities.

VanderWeele, Tyler (University of Chicago) Demonstrated the use of marginal structural models, which can also be applied in the presence of time-dependent confounding, to test for sufficient cause interactions. He showed that lower bounds on the prevalence of such sufficient cause interactions could be determined.

Schaubel, Douglas (University of Michigan) Developed semiparametric methods to estimate the effect on restricted mean lifetime of a time-dependent treatment with application to data from a national organ transplant registry. The method involves weighting results from stratified proportional hazards models fitted using a generalization of case-cohort sampling. The evaluation of asymptotic and finite-sample properties of the proposed estimator was presented.

Henderson, Robin (University of Newcastle) Proposed a modelling and estimation strategy for optimal dynamic treatment regimes which incorporates the regret functions into a regression model for observed responses. This addresses problems of model building, checking and comparison that have had little or no attention so far.

Noorbaloochi, Siamak (Center for Chronic Disease Outcome Research) Showed how sufficiency and ancillarity concepts can be used to understand and construct methods to reduce bias due to imbalance in baseline predictors. Gave as an example the effective dimension reduction summaries provided by regression graphics as an alternative to propensity based analysis.

Wednesday May 6, 2009

Strumpf, Erin (McGill University) Provided an overview of two methods, instrumental variables and regression discontinuity, used by economists to identify causal effects of interest. Discussed how these methods have been used to address health-related questions.

Robins, James (Harvard School of Public Health) Showed that in certain special cases, a complete causal DAG can be discovered from data. This is an unusual result because the causal DAG is normally considered to be structural information necessarily external to the data. Under the assumption of faithfulness, conditional independence relationships among the observed variables impose constraints on possible DAGs for the data generating process—in the example considered, however, no such conditional relationships were present. Through manipulations of the joint density that correspond to specific operations on the associated (unknown) causal DAG, Robins showed that an exhaustive search could uncover conditional independence relationships that would have been present had one intervened on certain variables and that, in the special case considered, identify the causal DAG uniquely.

Goetghebeur, Els (Ghent University) Reviewed instrumental-variable-based methods of estimation for the causal odds ratio when outcomes are dichotomous. Comparisons made both formally and via simulation were presented.

Hirano, Keisuke (University of Arizona) Presented results on estimation and inference for partially identified models specified through moment inequalities that are of great interest in economics, but also closely related to problems in dynamic optimal treatment regimes.

Thursday May 7, 2009

Robins, James (Harvard School of Public Health) Presentation given by Dr. Robins in lieu of Andrea Rotnitzky. Proposed novel methods for using the data obtained from an observational database in one health care system to determine the optimal treatment regime for biologically similar subjects in a second health care system when the two health care systems differ in the frequency of, and reasons for, both laboratory tests and physician visits. Also proposed a novel method for estimating the optimal timing of expensive and/or painful diagnostic or prognostic tests.

Arjas, Elja (University of Helsinki and the National Institute for Health and Welfare) Presented a nonparametric Bayesian modeling/predictive inference approach to estimation of optimal dynamic treatment regimes. The proposed methods was illustrated using the Multicenter AIDS Cohort Study (MACS) data set.

Hernán, Miguel (Harvard School of Public Health) Presented an application of a *dynamic* marginal structural model. The application considered was determination of the optimal threshold in CD4 count for HAART initiation in HIV patients.

Joffe, Marshall (University of Pennsylvania) Outlined selective ignorability assumptions which can be used to derive valid causal inference in conjunction with structural nested models, illustrated on erythropoetin use and mortality among hemodialysis patients. Discussed the connection between selective ignorability assumptions and G-estimation with instrumental variables assumptions and estimation.

Gustafson, Paul (University of British Columbia) Considered the case of nonidentified models from a Bayesian perspective, with an emphasis on the example of instrumental variables analysis. Argued that in this context the posterior distribution on a parameter of interest may no longer concentrate to a single point as the sample size grows and it is therefore important to study the width of large-sample limiting posteriors, as well as their sensitivity to the choice of prior distribution.

Richardson, Thomas (University of Washington) Considered the problem of non-identifiability of the instrumental variable potential outcomes model in which the instrument, treatment and response are all binary using a Bayesian approach, also treated by Pearl in his book. After demonstrating sensitivity to the prior specification, went on to characterize the 15-dimensional parameter space for this problem in terms of 6 observed and 9 unobserved dimensions after re-parametrization.

Abadie, Alberto (Harvard University) Reviewed the topic of matching estimators and their distribution. Presented newly developed methods for the calculation of the asymptotic distribution of a broad class of matching estimators. In particular, the important case of the two-stage estimator obtained by matching on the propensity score were discussed.

Rosenblum, Michael (University of California, San Francisco) The targeted maximum likelihood estimator for the causal parameters of a marginal structural model was presented and advantageous properties of the estimator were discussed. A related method for diagnosing bias due to violation of experimental treatment assignment has been proposed. These methods were then applied to estimate the effect of medication adherence on viral suppression in a cohort of HIV positive, homeless individuals in San Francisco.

Friday May 8, 2009

Sekhon, Jasjeet (University of California, Berkeley) Concerning the problem of using matching to obtain good balance on observed covariates, proposed a non-parametric matching method based on an evolutionary search algorithm. Discussed advantages of this method over propensity score matching—in the example of Pulmonary Artery Catheterization considered, the proposed method is able to replicate the experimental results using observational data while propensity score matching does not. Also talked about difficulties and assumptions that apply to the use of matching in general.

McCandless, Lawrence (Simon Fraser University) Considered Bayesian techniques to adjust for unmeasured confounding. A novel methods for observational studies with binary covariates that models the confounding effects of measured and unmeasured confounders as exchangeable within a Bayesian framework was proposed, and its properties discussed.

Glynn, Adam (Harvard University) Demonstrated that the observation of a post-treatment variable can improve our knowledge about total causal effects when treatment assignment is non-ignorable and the assumptions necessary for the front-door technique do not hold. Presented a Bayesian model that provides a framework for sensitivity analysis when the treatment is unobserved, but a post-treatment proxy is.

Outcome of the Meeting

The workshop successfully brought together researchers from Statistics, School of Government, Economics, Computer Science, and Epidemiology with a common interest in quantitative methods for causal inference. Participants came from Canada, the United States, England, Belgium, and Finland and represented a great range of career stages. The seminars presented were of an exceptional quality, and participants took advantage of the unscheduled time to exchange ideas informally.

List of Participants

Abadie, Alberto (Harvard University)
Arjas, Elja (University of Helsinki)
Chakraborty, Bibhas (University of Michigan)
Ertefaie, Ashkan (McGill University)
Geneletti, Sara (Imperial College London)
Glynn, Adam (Harvard University)
Goetghebeur, Els (University of Ghent)
Graham, Dan (Imperial College London)
Gustafson, Paul (University of British Columbia)
Henderson, Robin (University of Newcastle)
Hernan, Miguel (Harvard School of Public Health)
Hirano, Keisuke (University of Arizona)

Joffe, Marshall (University of Pennsylvania)
Kaufman, Jay (McGill University)
Lefebvre, Genevieve (Université du Québec à Montréal)
McCandless, Lawrence (Simon Fraser University)
Neufeld, Eric (University of Saskatchewan)
Noorbaloochi, Siamak (Center for Chronic Disease Outcome Research)
Pearl, Judea (UCLA)
Platt, Robert (McGill University)
Redman, Mary (Fred Hutchinson Cancer Research Center)
Rich, Benjamin (McGill University)
Richardson, Sylvia (Imperial College)
Richardson, Thomas (University of Washington)
Robins, James (Harvard University)
Rosenblum, Michael (UC San Francisco)
Schaubel, Doug (University of Michigan)
Schnitzer, Mireille (McGill University)
Sekhon, Jasjeet (UC Berkeley)
Shortreed, Susan (Monash University)
Small, Dylan (University of Pennsylvania)
Strumpf, Erin (McGill University)
Stuart, Elizabeth (Johns Hopkins Bloomberg School of Public Health)
van der Laan, Mark (University of California Berkeley)
VanderWeele, Tyler (University of Chicago)
Xiao, Yongling (McGill University)

A special issue of the *International Journal of Biostatistics* is forthcoming which will be devoted to publishing research material presented at the workshop or stemming from discussions which took place at the workshop.

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Participant Feedback

Michael Rosenblum:

> I wanted to express my many thanks for your having organized such a wonderful conference at Banff! It was amazing–I learned so much and met such amazing researchers in causal inference. The goal of connecting statisticians, computer scientists, and economists was certainly achieved.
>
> In particular, the presentations of Judea Pearl, Jamie Robins, and Alberto Abadie were outstanding. The time built in to allow for discussions was great as well.

Thomas Richardson:

> The causal conference was very good. A good mix of people.

Benjamin Rich:

> I wanted to tell you what a great time I had at the workshop. It was an amazing week that I won't forget. I feel very lucky to have participated and I wanted to thank you again for giving me that opportunity. I think the poster was well received, and I learned a great deal from the many excellent talks I heard throughout the week.

Marshall Joffe:

> Thank you very much again for including me in the program at Banff last week. This was one of the best meetings I have ever attended: there were many interesting talks, and there was ample opportunity to converse with and meet many interesting and bright people working in the field, and to learn about many new developments. Additionally, the setting was unbeatable.

Dylan Small:

> I had a great time at the causal conference. The talks were great and it was great having chance to spend time with friends and meet new friends.

Tyler VanderWeele

> I wanted to write to thank you again for organizing the Banff workshop. I really enjoyed my time there. You did a very nice job of bringing together people with different backgrounds. The sessions Monday through Wednesday were quite interesting and, although I then had to leave after that, I imagine Thursday and Friday went just as well. Thank you again for all of your work in putting the conference together. Banff is lovely!

Chapter 7

Mathematical Immunology of Infectious Diseases (09w5054)

May 17 - May 22, 2009

Organizer(s): Jane Heffernan (York University), Beni Sahai (Cadham Provincial Laboratory), Robert Smith? (The University of Ottawa)

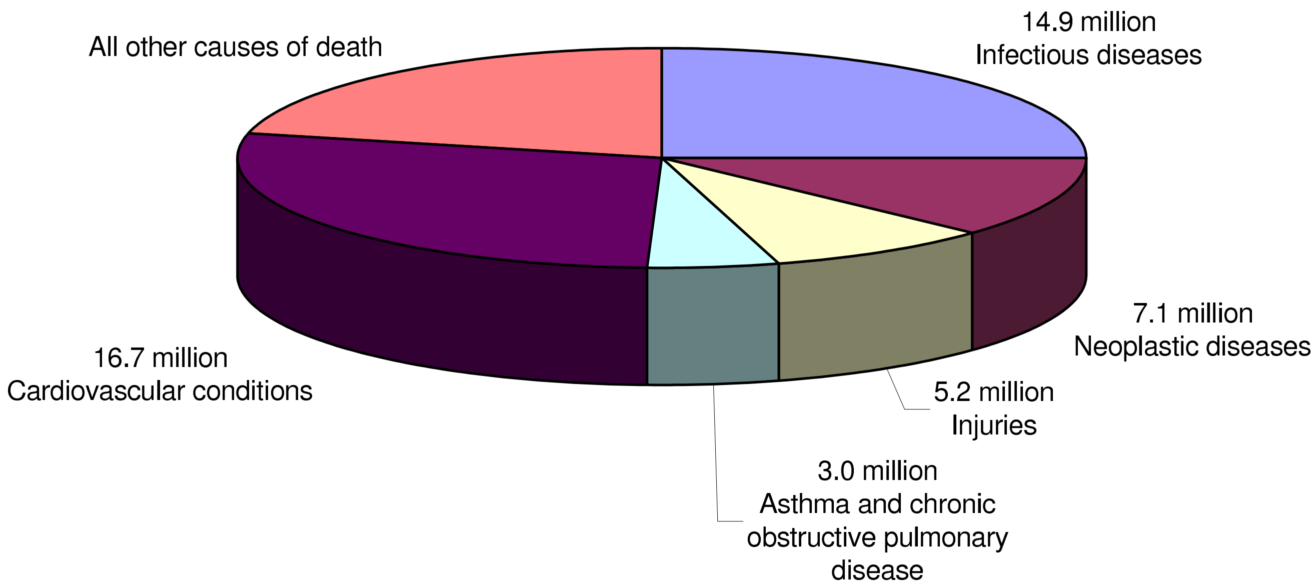
Overview of the Field

Infectious diseases are the second leading cause of death among humans worldwide, but number one cause of death in developing countries [22]. In addition to established infections that have coexisted with humans for thousands of years, the course of human history has been dotted with numerous emerging and re-emerging infectious diseases [22]. From the ages of recurring plagues when patients with leprosy and sexually-transmitted infections were severely stigmatized and dreaded, to mortifying epidemics of cholera, smallpox and other childhood infections, influenza pandemics of the preceding centuries, and more recent emergence of HIV, HCV, SARS, WNV, vCJD, drug-resistant microbes and the threats of bioterrorism, infectious diseases have inflicted incalculable suffering and socioeconomic devastation. With discovery of vaccines, antibiotics and other antimicrobial agents, major advances in prevention and management of many infectious diseases have been realized, yet significant misery continues from infection-related morbidities, deaths and staggering costs. World is currently in grips of two concurrent pandemics: HIV and H1N1 influenza.

The immune system offers a sophisticated, natural and arguably the most reliable defense mechanism against many infectious diseases. It is nonetheless unable to defend against many other infections, especially those by pathogens that adversely affect its operation. The outcome of infection in an individual impacts the population by contributing either to herd immunity or the spread of infection, depending on the ability of immune system to overcome the infection. The limitations of immune defenses become obvious by fatal or chronic infection and during major epidemics. Our ability to overcome these limitations to mitigate the burden of infectious diseases rests on better insights into the functioning of immune system, especially the nature of its interaction with specific pathogens.

An overview of the general principles of immunological control of infectious diseases

The immune response to infection affects an individual and the population as a whole.



Infectious diseases	**Annual deaths (million)**
Respiratory infections	3.96
HIV/AIDS	2.77
Diarrhoeal diseases	1.80
Tuberculosis	1.56
Vaccine-preventable childhood diseases	1.12
Malaria	1.27
STDs (other than HIV)	0.18
Meningitis	0.17
Hepatitis B and C	0.16
Tropical parasitic disease	0.13
Dengue	0.02
Other infectious diseases	1.76

Figure 7.1: Infectious diseases worldwide. About 25% of annual deaths worldwide are the direct result of infectious diseases [27]. This does not include millions who die as a consequence of past infections such as streptococcal rheumatic heart disease, cervical neoplasia after HPV infection, liver failure or hepatocellular carcinoma following chronic hepatitis B or C infection [18]

In an individual:

Once contracted, a pathogen must replicate in an individual in order to cause disease, evolve further and be recognized by the immune cells. The process leading to the onset of a disease is determined by a dynamic interplay between the pathogen and immune system. Depending on the pathogen, some fail to overcome the initial barriers presented by the nonspecific components of the innate immunity and do not replicate adequately to cause disease, whereas others readily overcome these barriers and replicate in sufficient numbers to produce disease in infected individuals. This interaction between the pathogen and the components of innate immunity triggers generation of an adaptive immune response that usually consists of pathogen-specific oligoclonal cytotoxic T cells (CTLs) and antibody molecules. The combined action of CTLs and antibodies is usually required to suppress pathogen replication and, in turn, reverse disease symptoms. CTLs suppress pathogen replication by destroying the infected cells that support its replication, whereas antibodies neutralize the pathogen by binding to an epitope on its surface that is required for pathogen to enter into a host cell or subject pathogen to destruction through opsonization by phagocytic cells or lyses by the complement system. The individuals so retain an immunological memory of the pathogen in form of long-lived pathogen-specific memory CTLs and B cells capable of producing antibodies against the pathogen in case of re-infection. The examples of infections typically controlled by the adaptive immune response include mumps, chickenpox, measles, rubella and influenza. Such infections are often vaccine-preventable and there are successful vaccination programs to prevent them.

However, when the replication cycle of a pathogen interrupts any step of the formation of CTLs or antibodies, by direct interaction or by assuming latency, the adaptive immune response does not develop or develops inadequately. Such responses fail to halt pathogen replication or reverse disease symptoms. Their examples include HIV, HCV, HSV-1, tuberculosis and malaria infections. Empirical vaccines directed against these pathogens have not been successful. These infections are therefore not vaccine-preventable.

In a population:

The spread of a disease from a single infected individual to others in a population of wholly susceptible individuals is described by its reproductive number, R_0, which is a combined function of the pathogen dynamics in the infected individual and the efficiency of pathogen transmission (see [11] for a review of R_0). The latter is affected by the stability of the pathogen and the mode and route of its transmission. The value of R0 can be greatly affected by (i) the contact structure which is different, for example, for school, home, healthcare facility, factory or office; (ii) the herd immunity which is affected by prior exposure to the same pathogen or vaccination; (iii) the control measures such as chemoprophylaxis, isolation or quarantine; and (iv) the immunodemography of the population concerned. The values of $R_0 > 1$ signify the spread of infection, whereas those < 1 are indicative of the control of spread and even the potential for eradication of the disease.

There are at least three distinct areas where R_0 is regulated immunologically; that is when immune system impacts the dynamics of infection in a population: first, by regulating the pathogen dynamics in vivo; second, by affecting herd immunity, and third, by an altered immunodemography through change in the immune status due to age, disease or therapy. There may yet be additional avenues of its impact such as emergence of drug resistance.

Importance of mathematical modeling in immunology and infectious diseases

Immunology is a science of dynamic interactions amongst a highly specialized array of lymphoid and myeloid leukocytes (white blood cells) and soluble cytokines released by them (Fig. 2). The latter trigger or affect the migration and interactions between various leukocytes and their responses directed to protect the integrity of the organism concerned.

Major advances in our knowledge of the organization and function of the immune system are symbolized by 30 Nobel Prizes in Physiology or Medicine to individuals for works in immunology and related fields during the last century. Yet, significant gaps in knowledge continue to exist. While experimental research has made outstanding contributions by identification and physical characterization of the key components of immune system and their mutual interactions, an understanding of their dynamical behavior as these relate to suppression or persistence of an infection remains limited. The underlying immunologic criteria vis--vis

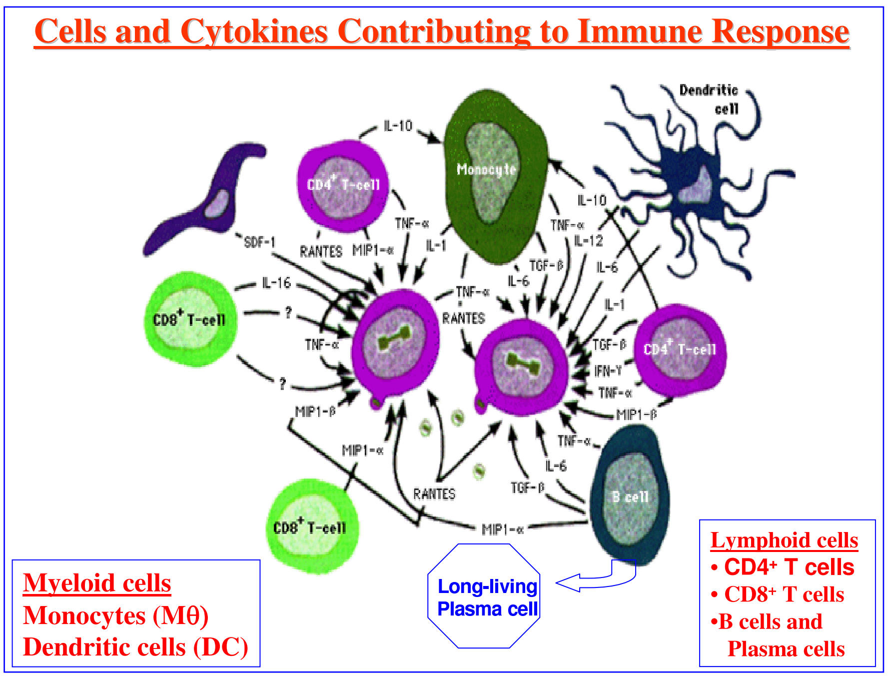


Figure 7.2: The principal leukocytes that drive the adaptive immune response to infection are CD4+ T, CD8+ T, B, and plasma cells of the lymphoid origin as well as monocytes and dendritic cells of the myeloid origin. Other leukocytes (not shown here) that influence the course or magnitude of adaptive immune response are natural killer (NK) and regulatory T cells of lymphoid origin, and polymorphic neutrophils, eosinophils and mast cells of myeloid origin as well as platelets. Cells release chemokines (RANTES, MIP1) and cytokines that are either chemotactic (MCP1-3, IL-8) or inflammatory (IL-2, IFN- , TNF- , IL-1, IL-6, IL-12) or anti-inflammatory (IL-10, TGF-).

pathogen properties necessary to attain a disease-free state during an infectious process have not been understood. These limitations have led to major setbacks in progress towards development of vaccine (for example, against HIV, HCV or malaria) and novel immunotherapeutic and other antimicrobial agents, prevention and treatment of autoimmunity and related disorders, and prevention and management of drug resistance. Mathematical modeling holds great promise in providing new, counterintuitive insights into the dynamical processes intrinsic to mechanism and control of infectious diseases.

Pioneered by Daniel Bernoulli in 1760 [5], mathematical modeling has an illustrious history in predicting and rationalizing the spread or control of infectious diseases in a population. Current literature is rich with epidemiological models, which have greatly added to our understanding of outbreaks, epidemics and pandemics of diverse pathogens. Notably, the principles enunciated by Hamer in 1906 [10] and later extended by Ronald Ross [21] in 1911 and Kermack and McKendrick in 1927 [13], form true foundations of mathematical epidemiology today. In those days, when influenza virus was not yet discovered and the knowledge of immunological protection against an infectious disease was minimal, the limited criteria that differentiated one disease from the other were mostly captured in their values of R0 and estimates of incubation period. These models were limited in their abilities to inform about the disease process. In 1980s Robert May and Roy Anderson consolidated concepts in mathematical epidemiology [1] and provided new insights into spread of HIV infection.

As insights into immune responses to infection and the nature of pathogen diversity grew, some epidemiological models began incorporating specific features of the pathogen and even immune responses to infection in attempts to enhance their predictability. The initial efforts were met with limited success owing to a sparse understanding of immunological mechanisms and complexities of interaction between a pathogen and the immune system. Not until late 1980s that several investigators (most notably Alan Perelson, Robert May, Robert Nowak, Roy Anderson, and Simon Levin) developed simple yet insightful in-host models to describe the dynamical behavior of pathogens and specific immune cells and antibodies, in particular during HIV and influenza infection (see [16, 17, 19, 20] for reviews). These contributions actually laid the foundations of mathematical immunology. In 1995, a counterintuitive finding, realized through use of relatively simple mathematical models, that the replication rate of HIV is great in magnitude and thus, the current antiretroviral therapies would be inadequate in eliminating HIV from infected patients [12, 23], made resounding impacts both clinically and in understanding of the mechanism of HIV pathogenesis. This finding also highlighted the important roles mathematical models may play in uncovering the intricacies of an infectious process and in evaluating the adequacy of therapeutic strategies. This pioneering work triggered a surge of new, more complex immunological models addressing different aspects of HIV infection.

The dawn of the 21st century witnessed remarkable leaps in immunological modeling emanating from collaborative research between Rustom Antia (a mathematical modeler) and Rafi Ahmed (a prominent immunologist) at Emory University in Atlanta that aimed to decipher the mechanisms of development of antiviral CTLs and CTL memory in mouse models of LCMV infection [2, 3, 4, 9]. These models were preceded or followed by several notable contributions in immunology through mathematical modeling ranging from basic T cell biology and diversity of T cell repertoire to control of viral, bacterial or parasitic infection by investigators such as Rob DeBoer at Utrecht University , Denise Kirschner at University of Michigan at Ann Arbor, Dominik Wodarz at University of California at Irvine (see [6, 7, 8, 14, 15, 24, 25, 26] for examples) and others such as those mentioned in the preceding paragraph. These investigators modeled selected immunological processes implicit in control of infectious diseases in vivo. However, several key mechanisms and their fundamental principles that drive these processes and, in turn, impact the infectious process have not been fully understood. In many cases, important new insights can be attained through in-depth explorations with mathematical and computational modeling. Consequently, a comprehensive model that provides a theoretical framework for assessing or monitoring immunological control of an infectious process has not been developed, but crucial for making needed advances.

Open questions in immunology and immunological control of infectious diseases

Several unanswered questions pertaining to the immunological control of infection have hampered the development of a comprehensive immunological model that could monitor dynamic interactions between a pathogen and the immune system during an infection. To appreciate them, a brief account of the immune response to infection may be warranted. The monocytes, dendritic cells, natural killer (NK) cells, and neutrophils are non-specific components of the innate immunity (Fig.2) which, during the course of infection, either facilitate the formation or directly serve as antigen-presenting cells (APC). APCs engulf the pathogen and the dying infected cells through phagocytosis and in response (i) release inflammatory and chemotactic cytokines and chemokines, and (ii) process short antigenic peptides from the pathogen and display them on their surface together with MHC class I or II molecules. While chemotactic cytokines recruit additional leukocytes to the site of infection, the pathogen peptide-bearing APCs trigger onset of cellular immune response that encompasses selection, activation and clonal expansion of pathogen-specific CD4+ T (helper) and CD8+ T (cytotoxic) cells from a diverse T cell repertoire, and their direct and cytokine-mediated effects on mitigating the infectious process. Aided by the helper T cell response, the pathogen and its proteins trigger activation and pathogen-specific antibody secretion from B cells bearing receptors that specifically bind to them. Some of these B cells, in turn, differentiate into antibody-secreting, long-lived plasma cells. The antibody and other soluble factor response are commonly referred to as humoral immune response. The following is a select list of outstanding questions that so far remain unaddressed:

1. The formation and dynamics of diminution of the diversity of T and B cell repertoires as a function of age, vaccination, and acute or chronic infection;
2. The relationship between the rate of pathogen replication and suppression of infection by innate immune response and the dynamics of generation of APCs;
3. Conditions causing pathologic inflammation from chemotaxis and its possible control by immunological memory;
4. The mechanism of control of T cell proliferation (the finite number of cell divisions) during antigen-driven clonal expansion of T cells and factors affecting this process;
5. The mechanism of formation of effector and central memory T cells and relationship between them;
6. The mechanisms of homeostatic proliferation and long-term sustenance of nave and memory T cells, and the waning of T cell memory;
7. The relationship between the incubation period of an infectious disease and the ability of the immune response to suppress primary or secondary infection, to prevent death;
8. The impact of memory CTLs in controlling the spread viral infection in vivo and in the population;
9. The impact of pathogen mutation vis--vis the ability of CTLs to suppress infection;
10. The relationship between immunological control of infection and pathogen evolution including the development of drug resistance; and
11. A model providing theoretical framework for novel vaccine designs.

In view of the aforesaid and in attempts to link immunological characteristics of a disease to its epidemiological models, this workshop was structured under the following interrelated themes:

1. Organization and function of the immune system

   (Insights in immune organization and principles inherent in a typical immune response)
2. Mechanism of a disease process

   (Enhancing understanding of the mechanism of a specific disease or its key principles)
3. Assessment of novel drug or immune therapies

   (Evaluating a therapeutic agent based on its impact on pathogen dynamics in vivo)
4. Pathogen evolution: changing virulence, drug resistance or extinction

   (Elucidating genetic and immunological forces intrinsic to pathogen evolution)

5. Vaccine design and evaluation

   (Utilizing in-host models to aid the design and preclinical evaluation of vaccines)

6. Specific in-host processes affecting the spread or control of disease in population

   (Factoring biological characteristics of a disease that affect its spread or control)

7. Vaccination or antimicrobial strategies for population-wide control of disease

   (Specialized epidemic models to evaluate vaccination and antimicrobial strategies)

8. Novel mathematical techniques and barriers to in host immune modeling

   (Descriptions of new mathematical techniques, parameter estimation strategies, and existing barriers to modeling immune response and pathogen dynamics in vivo)

Recent developments

The workshop consisted of the 8 themes listed above. Each session consisted of a plenary talk, contributed talks and a discussion period. The workshop allowed younger researchers and postdocs to present their results and experienced researchers to present an overview of their work. The presentations were of very high quality and stimulated interesting discussions. Speakers and abstracts of plenary talks are listed below in alphabetical order:

Rob deBoer
Utrecht University
Estimating the killing efficacy of cytotoxic T cells
In order to defend our bodies against viruses, Cytotoxic T Lymphocytes (CTL) continuously migrate through nearly all tissues searching for infected cells. When a CTL finds a suitable target, they form a cytotoxic synapse, and the target cell is killed after some time. It is unclear how many target cells a CTL kills per day, how many CTL are required to clear a viral infection, and what proteins should be targeted for optimal protection. Mathematical modeling has demonstrated that the cytotoxic control of rapidly expanding pathogens requires a large initial population of CTLs, and we have shown that this can explain the failure of current HIV vaccines. We have analyzed experimental data on estimating the killing rate of CTL, and argue that these experiments readily deliver the (fast) death rate of target cells, but that estimating the killing rate per CTL requires more information on the functional form of the killing term. We propose to analyze in vivo movies in which one can observe how cells find each other in vivo, and how long they are attached before the target cell is killed. The efficacy of CTL is not only determined by their numbers and the rate at which they kill. Immune response to particular viral epitopes seem much more protective than those to others. One factor is the time course at which the epitope is expressed, and we will review experimental, bioinformatic and modeling results that in HIV infection targeting early epitopes may be most protective and can explain otherwise surprising observations.

Peter Bretscher
University of Saskatchewan
Macroimmunology and immunological intervention
The activation of CD4 T cells is required to initiate most immune responses, and the cytokine profile of the activated CD4 T cells determines the qualitative nature of the ensuing immune response. Two questions will be addressed: what determines whether antigen activates or inactivates CD4 T cells, and how is their Th1/Th2 phenotype determined? The prevalent views are that pathogen-associated molecular patterns (PAMPs) are required to activate CD4 T cells, and that their nature determines the Th1/Th2 phenotype of the response. Arguments against these views will be presented. A valid understanding of how the Th1/Th2 phenotype of a response is determined must account for why various variables of immunization affect this Th1/Th2 phenotype. There are several quantitative variables of immunization affecting this phenotype. The ability of the ”Threshold Hypothesis”, that attempts to delineate how the Th1/Th2 phenotype of a response is determined, to account for these quantitative variables of immunization will be presented, and contrasted with non-quantitative theories based upon the activity of PAMPs. The quantitative framework developed has been successfully tested experimentally and allows one to understand how broad parameters of the immune response determine its Th1/Th2 phenotype. Detailed information is not necessary to understand how this phenotype is determined and to control its nature. Our ”macro-immunological” framework has allowed us to develop strategies to prevent or cure those infectious diseases where the Th1/Th2 phenotype of an immune response against the pathogen is critical to whether the pathogen is contained or causes chronic/progressive disease. Evidence as to the effectiveness of strategies based upon the ”Threshold Hypothesis” will be presented. The quantitative considerations, underlying the framework developed, should provide a rich context for mathematical modeling to gain greater insight into the underlying processes.

Matt Keeling
University of Warwick
Presented by: **Jane Heffernan** (York University)
Immuno-epidemiolgy: bringing together within-host and between-host dynamics for measles
One of the major challenges in the study of infectious diseases is bringing together immunological models with epidemiological ones. Here we develop and parameterise a within-host model for the dynamics of measles (an acute childhood infection), whose resultant dynamics can be used to drive an epidemiological model. Under the assumption of life-long immunity the population-level dynamics of these two models are identical; however, there is some evidence to suggest that immunity to measles wanes slowly. Such waning immunity can only be mechanistically captured by a within-host model. Prior to vaccination, waning of immunity is epidemiological irrelevant as repeated re-exposure to the virus leads to multiple boosting events. We show however that with high levels of vaccination immunity can wane to such an extent that large-scale epidemics can ensue. We discuss the implications of this observation and the insights that immuno-epidemiology can bring to infection control.

Alan Perelson
Las Alamos National Laboratory
Stochastic models of early HIV-1 infection
Not every encounter with HIV/SIV leads to infection. In the case of needle stick injury about 1 out of every 200 sticks leads to infection. Within discordant couples, one of whom is HIV positive and one whom is not, HIV infection occurs in about 1 out of every 1000 sex acts. Further, when HIV/SIV infection results, there can be a long eclipse phase during which time HIV/SIV remains at undetectable levels. To explain the low frequency of successful infections and the length of the eclipse phase we are developing stochastic models of early infection. I will present some simple models and preliminary analytic and simulation results.

Beni Sahai
Cadham Provincial Laboratory
Immunological control of influenza infection and basis for creation of a universal vaccine
In addition to annual epidemics, influenza A viruses of unpredictable genetic constitution and origin are responsible for major pandemics that have dotted human history with dire outcomes. Infection with a virulent influenza A strain in the absence of any preexisting immunity can be fatal. The immunity needed to prevent death can persist independent of that needed to avoid infection. Learning their separation is crucial for limiting influenza-related deaths, devising vaccination strategies, and pandemic planning. While delineating the key features of immunological control of influenza infection in vivo, this presentation aims to provide new insights into dynamics of spread of infection in the respiratory tract and the importance of preexisting memory CTLs in limiting illness and preventing death. These insights constitute the basis for creation of a universal vaccine capable of eliciting lasting immunity against influenza A viruses irrespective of their inter-strain differences.

Identification of key areas for future work

The workshop consisted of the 8 themes listed above. Each session consisted of a plenary talk, contributed talks and a discussion period. In the discussion period the section chairs briefly summarized the key points to each talk and lead with some questions to open the discussion. Some key areas for future work were identified in both areas of immunology and mathematics.

Areas for future work in immunology

- Clonal expansion of T cells leading to antiviral CTL response and T cell memory
- Perturbations of T cell diversity during chronic infection
- Defining conditions for CTL escape
- Rationalizing appearance of HIV mutants with multiple mutated CTL epitopes
- CTL failure during early HIV infection
- Un-helpful CTL responses during some viral infections (HIV, HBV, HTLV-1)
- Immunological control of influenza infection and basis for a universal vaccine
- Shaping of evolution and dynamics of variant surface antigens of malaria parasite by anti-malaria immune response
- Impact of preexisting memory CTLs on viral dynamics in vivo and the spread of infection in population

- Antiviral suppression of viral dynamics in vivo and the spread of infection in population: HIV and pandemic influenza
- Principles and hurdles to eradication of an infectious disease through vaccination
- Relating in-host and between-host dynamics

Areas for future work in mathematics

- Optimizing control strategies
- Stochastic vs Deterministic Models
- Tradeoffs between complex biological systems and simple models
- Model structure at the in-host and between-host interface

Outcome of the Meeting

The workshop fostered new contacts and collaborations. Participants learned about the immune system and how it is modelled, identified areas for future work and discussed possible avenues to reach these goals. A possible proceedings of the workshop or special issue on the key ares of future work was discussed and future collaborative meetings and workshop were envisioned. Two groups of individuals emphasized the need for future workshops in the areas of mathematical immunology. One group emphasized the need to further understand the dynamics of the immune system at the cellular and mechanistic level. They stated that they had used some free time during the workshop to write a proposal for a future meeting. Another group decided to propose a future workshop on immuno-epidemiology. They stated that comprehensive models will improve the accuracy of epidemiological predictions, they are key to further understanding the evolution of resistance and virulence and had the capacity to determine optimal control strategies including vaccine and drug therapies.

A goal of our workshop was to bring together participants in varying stages of their research careers. A quick survey of the junior participants shows that overall, their workshop experience was very positive and has lead to research projects stemming from the key areas of research discussed above. Some have also already published their studies in internationally recognized journals.

Sessions in mathematical immunology have been associated with every major mathematical biology conference in recent years, sponsored by SMB, ECMTB, CMS, CAIMS, JSMB, SIAM. However, it is rare to find such senior researchers in the area at these conferences together. It is also rare to have focussed discussions identifying areas for future research.

Summer schools and workshops in mathematical epidemiology and mathematical biology have recently started to include courses in modelling immunology. A future goal of our workshop participants is to have a summer school focussing solely on the immune system, its interaction with pathogens and how this can be modelled.

Our workshop has given a conducive environment for discussion and collaboration in this growing field. We thank BIRS for their hospitality and this great opportunity.

List of Participants

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Chapter 8

Probabilistic Models of Cognitive Development (09w5100)

May 24 - May 29, 2009

Organizer(s): Fei Xu (University of British Columbia), Tom Griffiths (University of California Berkeley)

This workshop focused on some of the major issues in the study of cognitive development, especially from the computational modeling point of view. Forty participants from developmental psychology, computational cognitive science, philosophy, and education engaged in five days of talks, poster presentations, and many discussion sessions. It was the first time that these researchers were brought together in a forum, and the workshop was a huge success. Currently the organizers are putting together a special issue of the journal *Cognition*, one of the most prestigious journals in cognitive science, based on contributions from the workshop participants.

Overview of the Field

This workshop aimed to capitalize on a major new direction in research on formal models of human cognition. The question of how people come to know so much about the world on the basis of their limited experience has been at the center of the study of the mind since it was first asked by Plato. This question takes a modern form in the nature-nurture debate, which has guided the study of cognitive development from infants to middle childhood over the last few decades. Nativists, favoring strong innate constraints provided by nature, have emphasized competences found in young infants (e.g., constraints on word learning, early understanding of the physical world) whereas empiricists, who focus on the role of experience and nurture, have emphasized learning mechanisms (e.g., keeping track of frequencies and correlations). However, this debate has been hard to resolve without formal tools for evaluating what might plausibly be learned from experience, and what kind of constraints are necessary to support the inferences that children make.

In recent years, several researchers in the cognitive sciences have argued that the nature-nurture framework may have set up a false dichotomy. A more fruitful and productive research strategy may be to find principled ways of combining prior constraints with statistical information in the input. In particular, a number of researchers have begun to use the principles of Bayesian statistics to establish a formal framework for investigating empirical phenomena in development and building computational models of developmental processes. The technical advances that have been made in the use of probabilistic models over the last twenty years in statistics, computer science, and machine learning have made this research enterprise possible, providing psychologists with a set of mathematical and computational tools that can be used to build explicit models of psychological phenomena. By indicating the conclusions that a rational learner might draw from the data provided by experience, Bayesian models can be used to investigate how nature and nurture contribute to human knowledge.

Recent Developments and Open Problems

The goal of this workshop was to explore a new approach for studying cognitive development: analyzing childrens learning from the perspective of rational statistical inference. Bayesian statistics and probability theory provide the formal tools that allow us to investigate what prior constraints and what input data are needed in order to justify a particular conclusion. Most importantly, this approach indicates how the kind of prior constraints that might be provided by nature should be combined with the data provided by experience when a learner evaluates a set of alternative hypotheses. More formally, imagine that a learner has a set of hypotheses $\mathcal{H} = \{h_1, h_2, \ldots, h_n\}$ about the structure of her environment, and has degrees of belief in those hypotheses that can be expressed through a "prior" probability distribution $p(h)$, where $p(h_i)$ indicates her degree of belief in h_i. She is then provided with some data d, and needs to revise these beliefs in light of evidence. Bayes' rule indicates that the resulting distribution over hypotheses, $p(h|d)$, known as the "posterior" distribution, is given by

$$p(h|d) = \frac{p(d|h)p(h)}{\sum_{h' \in \mathcal{H}} p(d|h')p(h')} \tag{8.1}$$

where $p(d|h)$, the "likelihood," indicates the probability of observing the data d if the hypothesis h were true.

Under Bayes' rule, the posterior probability of a hypothesis is proportional to the product of its prior probability and its likelihood, with the ultimate beliefs of the learner being the result of combining her prior dispositions with the evidence provided by the data. This has direct implications for understanding cognitive development, where innate constraints can be viewed as influencing the prior probability of hypotheses, or even which hypotheses are considered. This approach thus provides a natural compromise between the nativist position, in which strong innate constraints are the key to learning, and the empiricist position, where these constraints are taken to be extremely weak. By exploring the consequences of using different prior distributions, we can determine what kind of constraints are necessary in order to explain the state achieved by adults from the data available to children. This probabilistic approach has already been shown to be productive in studying cognition in human adults, providing accounts of how people make predictions, generalize from examples, form categories, and learn causal relationships [2, 1, 6, 9]. Explaining the inferences that people make requires going beyond the simple formal ideas expressed in the last two paragraphs. Probabilistic models of cognition use cutting-edge tools from statistics and computer science tools that have largely been developed over the last two decades. Understanding human causal learning requires a formal language for representing and reasoning about causal relationships, which is provided by causal graphical models [8, 7]. The properties of with recursive generative systems, such as linguistic syntax, can be captured using probabilistic context-free grammars and other structured statistical models from computational linguistics [4]. Performing probabilistic inference in large, structured models requires using modern Monte Carlo methods, such as Markov chain Monte Carlo [5]. Finally, capturing the flexibility of human mental representations, and the capacity for these representations to increase in complexity when warranted by the data leads to the use of ideas from nonparametric Bayesian statistics, such as the Dirichlet process [3]. Applying these statistical tools in novel contexts can often lead to new insights, and we hope that new formal methods will result from tackling some of the most difficult problems in cognitive development.

Presentation Highlights

At our workshop, a number of formal problems were presented and discussed extensively, with the aim to develop further the mathematical/computational tools for modeling cognitive development.

1. *Iterated learning.* A basic question for the cognitive sciences is how information is changed when it is transmitted from person to person. In real cases of cultural evolution, children are often the agents of transmission, meaning that this question has relevance to understanding how cognitive development links to culture. Recent accounts have emphasized the effects of innate constraints on learning on

cultural transmission, arguing that universals in languages, religious concepts, and social conventions can be explained in terms of the structure of the human mind. However, these claims have not been supported through detailed mathematical analysis.

Griffiths presented a formal account of cultural evolution by iterated learning that makes it clear in what sense the expectations of learners influence the outcome of cultural evolution. Formally, imagine a sequence of learners, each of whom observes data d_i, forms a hypothesis h_i about the process that generated those data, and then generates new data d_{i+1} that is presented to the next learner. This process can be shown to define a Markov chain, and if the hypotheses are selected by sampling from a Bayesian posterior distribution, the stationary distribution of this Markov chain is the prior of the learners. This means that we should expect that the distribution over hypotheses that are selected will converge to the prior distribution as the process of cultural transmission continues.

This mathematical result is interesting in providing a connection between constraints on learning and the outcome of cultural evolution, but also suggests a way that we can explore the question of what constraints guide human learning. By simulating this process of cultural transmission in the laboratory and seeing what hypotheses emerge in the minds of the participants, we can estimate the priors used by human learners. This process is analogous to estimating a complex probability distribution via Markov chain Monte Carlo, a deep connection that opens the possibility of other randomized algorithms being relevant to exploring the subjective probability distributions maintained by people.

2. *Probabilistic models for the diagnosis of learner knowledge.* Michael Lee gave a presentation illustrating how probabilistic models can be used to infer the knowledge that young children have about numbers. As preschool children begin to understand the relationship between the size of a set of objects (e.g. two mice) and its numerical label ("two"), they go through several stages. These stages correspond roughly to understanding the referents of the words "one", "two", and "three", at each point not understanding the meaning of any of the terms for larger sets, and then transitioning to a complete understanding of the mapping between the terms used for numbers and the size of the sets they describe. Lee considered the problem of how a researcher or clinician could solve the problem of diagnosing the knowledge that a given child has from the way they perform on tasks that require producing sets of objects of different sizes in response to verbal requests.

   The basic idea behind the method that was used to solve this problem was to estimate the distribution over responses produced by children identified as being at each of these stages of knowledge. This information could then be used to apply Bayesian inference, providing a probability distribution over the level of knowledge that a learner seems to possess. Formally, each level of knowledge is a hypothesis h about the learner, and the behavior that the learner produces is data d. The probability of each hypothesis based on the data is given by Bayes' rule (Equation 1). The likelihood, $p(d|h)$, encodes the probability of the learners behavior if the hypothesis is true, and is estimated from the responses of children whose level of knowledge has been diagnosed by an expert. The prior, $p(h)$, reflects expectations about the relative probability of those knowledge states, and can be left uniform if the goal is simply to identify the amount of evidence in favor of each knowledge state.

   A similar strategy for assessing the knowledge of learners can be used in other cases where there are common patterns of understanding or errors that need to be diagnosed. If those patterns of understanding can be identified with a probability distribution over responses, Bayesian inference can be used to work backwards from responses to a picture of the knowledge that the learner has. All that is required is specification of a set of hypotheses about the knowledge of the learner, and estimation of the likelihood function $p(d|h)$ that characterizes the behavior associated with each knowledge state.

3. *Enriching our view of learning to include multiple levels of abstraction.* Many of the probabilistic models that were used to explain aspects of cognitive development shared the use of hierarchical Bayesian inference. In its standard form, Bayesian inference represents a way for a learner to optimally update his or her beliefs about a set of hypotheses in light of data. The basic computations are exactly like those described above: probabilistic models of cognition identify the hypotheses that a learner might entertain, and describe the beliefs of the learner in terms of a probability distribution over those hypotheses, with Bayes rule acting as a learning algorithm for updating those distributions. Hierarchical Bayesian

inference assumes a richer representation on the part of the learner, with knowledge at multiple levels of abstraction.

Many learning problems require making inferences not just about the current problem, but about more general principles that organize a domain. For example, many of the hierarchical Bayesian models presented at the workshop assumed that learners considered not just hypotheses that could be used to explain the most recently observed data, but higher-level theories that captured regularities linking the current hypothesis with hypotheses from past learning opportunities. To take some examples from the presentations: a learner could be forming hypotheses about the meaning of particular words based on labels provided by a parent, but simultaneously developing a theory about the kinds of objects that tend to share a label (such as shape being an important cue about whether two objects can be labeled with the same word); alternatively, a learner could be forming hypotheses about the causal relationships that exist in a particular physical system, while simultaneously forming a theory about how causal relationships operate in that system (such as causes deterministically producing their effects, or combining additively).

The hierarchical Bayesian approach provides a richer picture of learning than that assumed in many computational approaches, with the learner considering not just the solution to a particular problem but also forming generalizations about what solutions to these problems look like. In this way, a learner can form "overhypotheses" that guide future inferences. The ability to make inferences at multiple levels of abstraction provides a way to understand how a child can "learn to learn": as the theory of the domain becomes more accurate, it provides information that reduces the amount of data required to evaluate a particular hypothesis. Formally, we assume that we have random variables at three levels – the data d, the hypotheses h, and a higher level "theory" t. Different learning situations will involve different observed data (say d_1 and d_2), and inferring different hypotheses to explain those data (say h_1 and h_2), but the same theory t can apply across those situations. As a consequence, the prior that is used in one situation is informed by the data observed in the other.

4. *Formalizing pedagogical reasoning.* By providing a way to describe learning, Bayesian inference also provides a way to formalize optimal teaching. The presentation by Patrick Shafto focused on a framework for formalizing pedagogical reasoning. This framework can be built up in three steps. First, we consider the problem of a teacher selecting what data d to provide a student in order to best support learning of a hypothesis h. This problem is solved by finding the d that maximizes $p(h|d)$. Next, we assume that the learner knows that he or she is being taught, and takes this into account in assessing which data d the teacher would produce if h were the intended hypothesis. Intuitively, the learner should expect data that are diagnostic of the hypothesis, sharpening the original likelihood function. Finally, we allow the teacher to take into account this modification of the likelihood function by the learner, potentially changing the data provided to better discriminate between hypotheses. This set of assumptions results in the definition of a system of equations that characterize optimal pedagogical reasoning on the part of teachers and learners. Shafto presented results suggesting that adults produce behavior consistent with a solution to this system of equations when learning about simple categories and causal relationships.

The basic idea behind this approach is that the teacher should choose data that are maximally informative about the target hypothesis h that the learner should acquire. This gives

$$p_{\text{teacher}}(d|h) \propto p_{\text{learner}}(h|d)^{\alpha} \tag{8.2}$$

where the exponent reflects the degree of noise in the teacher's production of data. Accordingly, the learner should use this distribution when inferring hypotheses, with

$$p_{\text{learner}}(h|d) \propto p_{\text{teacher}}(d|h)p(h) \tag{8.3}$$

which is just Bayes' rule substituting the teacher's behavior for the likelihood. This defines a system of equations that can be solved by iteration, with the iterative solution predicting a pattern of behavior by both teachers and learners.

In its current form, this research on optimal pedagogical reasoning could potentially guide the development of more efficient automated tutoring systems. However, it seems that this approach has significant potential for further impact on education, provided similar results hold with children and in more complex learning situations. One of the other presentations at the workshop, by Elizabeth Bonawitz, provided preliminary evidence that this pedagogical framework can be used to explain inferences that children make in relatively complex causal reasoning situations.

5. Probabilistic inference on logical reprentations. Goodman and colleagues have recently developed new formal tools for understanding how probabilistic inference can be done on logical representations. Statistical and logical methods have both been influential in cognitive science. The compositionality of logical representations is key for capturing productivity and systematicity of human thought, while probabilistic or statistical inference is crucial for capturing flexible reasoning under uncertainty. Yet compositionality and statistics have rarely been combined in a meaningful way.

   This line of research explore this unification through the probabilistic language of thought hypothesis (PLoT): that mental representations which subserve higher-level cognition are compositional, their meaning is probabilistic, and their function follows from the probabilistic inferences they support. Further, these representations describe generative processes—causal models of the world that may be used to make many different predictions, explanations, and actions.

   To formalize the PLoT hypothesis, Goodman and colleagues have turned to the stochastic λ-calculus. The stochastic λ-calculus is a formal system that extends untyped λ-calculus with stochastic primitive operations and a primitive conditioning operator. The evaluation of an expression in stochastic λ-calculus (which can be understood in terms of a definitional interpreter, or in terms of reduction rules) results in a randomly sampled value. Intuitively this induces a distribution on return values. Indeed, it can be shown that any expression in stochastic λ-calculus that halts almost-always induces a computable distribution on return values, and that any computable distribution can be thus represented. Thus stochastic λ-calculus is universal for representing probability distributions, and for probabilistic reasoning, yet it is a compositional representation language. This compositionality is particularly important when we attempt to understand conceptual development in terms of the PLoT: we view concept learning as induction of expressions in stochastic λ-calculus.

Scientific Progress Made

The primary scientific progress resulted from direct contact between empirically-focused developmental psychologists and researchers pursuing mathematical models of human cognition. This interaction has led to several new collaborations, as well as a greater understanding of these models by the broader developmental psychology community.

Outcome of the Meeting

In addition to less tangible outcomes resulting from the interaction between these researchers, the meeting has led to the production of a special issue of the journal *Cognition* focusing on probabilistic models of cognitive development. *Cognition* is a leading journal in the cognitive sciences, and the ideal venue for this kind of work. The special issue is edited by Dr. Xu and Dr. Griffiths, and will include a tutorial introduction as well as approximately ten short empirical papers presenting these ideas, authored by the attendees of the meeting.

List of Participants

Abelev, Maxim (University of British Columbia)
Aslin, Richard (University of Rochester)
Baldwin, Dare (University of Oregon)
Bonatti, Luca (University of Nantes)
Bonawitz, Liz (Massachusetts Institute of Technology)
Buchsbaum, Daphna (University of California Berkeley)
Chater, Nick (University College London)
Ciancetta, Matthew (California State University Chico)

Colunga, Eliana (University of Colorado)
Danks, David (Carnegie Mellon University)
Denison, Stephanie (University of British Columbia)
Dewar, Kathryn (University of British Columbia)
Feldman, Naomi (Brown University)
Frank, Michael (Massachusetts Institute of Technology)
Gerken, LouAnn (University of Arizona)
Girotto, Vittorio (University Iuav of Venice)
Goldwater, Sharon (University of Edinburgh)
Gomez, Rebecca (University of Arizona)
Goodman, Noah (Massachusetts Institute of Technology)
Gopnik, Alison (University of California Berkeley)
Griffiths, Tom (University of California Berkeley)
Johnson, Scott (University of California Los Angeles)
Johnson, Mark (Brown University)
Kemp, Charles (Carnegie Mellon University)
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Lee, Michael (University of California Irvine)
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Mareschal, Denis (Birkbeck College)
McClelland, Jay (Stanford University)
Newport, Elissa (University of Rochester)
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Regier, Terry (University of Chicago)
Saffran, Jenny (University of Wisconsin - Madison)
Shafto, Patrick (University of Louisville)
Shultz, Thomas (McGill University)
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Tenenbaum, Josh (Massachusetts Institute of Technology)
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Chapter 9

Complex Analysis and Complex Geometry (09w5033)

May 31 - Jun 05, 2009

Organizer(s): Rasul Shafikov (University of Western Ontario), Dan Coman (Syracuse University), Finnur Larusson (University of Adelaide), Stefan Nemirovski (Steklov Institute)

Overview of the Field

Complex analysis and complex geometry can be viewed as two aspects of the same subject. The two are inseparable, as most work in the area involves interplay between analysis and geometry. The fundamental objects of the theory are complex manifolds and, more generally, complex spaces, holomorphic functions on them, and holomorphic maps between them. Holomorphic functions can be defined in three equivalent ways as complex-differentiable functions, as sums of complex power series, and as solutions of the homogeneous Cauchy-Riemann equation. The threefold nature of differentiability over the complex numbers gives complex analysis its distinctive character and is the ultimate reason why it is linked to so many areas of mathematics.

Plurisubharmonic functions are not as well known to nonexperts as holomorphic functions. They were first explicitly defined in the 1940s, but they had already appeared in attempts to geometrically describe domains of holomorphy at the very beginning of several complex variables in the first decade of the 20th century. Since the 1960s, one of their most important roles has been as weights in a priori estimates for solving the Cauchy-Riemann equation. They are intimately related to the complex Monge-Ampère equation, the second partial differential equation of complex analysis. There is also a potential-theoretic aspect to plurisubharmonic functions, which is the subject of pluripotential theory.

In the early decades of the modern era of the subject, from the 1940s into the 1970s, the notion of a complex space took shape and the geometry of analytic varieties and holomorphic maps was developed. Also, three approaches to solving the Cauchy-Riemann equations were discovered and applied. First came a sheaf-theoretic approach in the 1950s, making heavy use of homological algebra. Hilbert space methods appeared in the early 1960s, and integral formulas around 1970 through interaction with partial differential equations and harmonic analysis. The complex Monge-Ampère equation came to the fore in the late 1970s with Yau's solution of the Calabi conjectures and Bedford and Taylor's work on the Dirichlet problem.

Most current work in complex analysis and complex geometry can be seen as being focused on one or both of the two fundamental partial differential equations, Cauchy-Riemann and Monge-Ampère, in the setting of Euclidean space or more general complex manifolds. The past ten years have seen an increasing thrust towards extending both the theory and its applications to singular spaces, to almost complex manifolds, and to infinite-dimensional manifolds.

Today, as before, complex analysis and complex geometry is a highly interdisciplinary field. The foundational work described above has been followed by a broad range of research at the interfaces with a number of other areas, such as algebraic geometry, functional analysis, partial differential equations, and symplectic geometry, to name a few. Complex analysts and complex geometers share a common toolkit, but find inspiration and open problems in many areas of mathematics.

Recent Developments and Open Problems

1. Analytic methods in complex algebraic geometry are based on increasingly sophisticated ways of solving the Cauchy-Riemann equation (often also called the $\bar{\partial}$-equation) with L^2-estimates using plurisubharmonic weights in geometric settings. Yum-Tong Siu has long been a leader in this area. His announcement of an analytic proof of the finite generation of the canonical ring of a smooth complex projective variety of general type [27] came on the heels of an algebraic proof by Birkar, Cascini, Hacon, and McKernan. This is a milestone in algebraic geometry.

Bo Berndtsson and Mihai Paun use analytic methods to obtain a nearly optimal criterion for the pseudo-effectivity of relative canonical bundles and give several applications in algebraic geometry [2]. Shigeharu Takayama uses analytic techniques, including multiplier ideal sheaves, to extend Siu's celebrated result on the invariance of plurigenera from the smooth case to the case of fibres with canonical singularities [28]. So far, there is no known algebraic proof of the full result.

Currents are differential forms with distribution coefficients; closed currents satisfying a certain positivity condition are objects of fundamental importance that generalize both Kähler forms and analytic subvarieties. New work of Tien-Cuong Dinh and Nessim Sibony advances the basic theory of currents (intersections, pullbacks, etc.) and has many potential applications ([12], [13]).

2. Pluripotential theory and the Monge-Ampère equation. Pluripotential theory on compact Kähler manifolds, based on the notion of a quasiplurisubharmonic function, has been developed by Vincent Guedj and Ahmed Zeriahi since 2004 (starting with [18]). Whereas plurisubharmonic functions have a positive Levi form by definition and are constant on compact manifolds, quasiplurisubharmonic functions are allowed to have a negative Levi form down to a fixed lower bound and lead to a fruitful pluripotential theory in a compact setting. Guedj, Zeriahi, and Philippe Eyssidieux posted a major application of this theory [14] (posted in March 2006). They extended the work of Aubin and of Yau on the complex Monge-Ampère equation to certain singular settings and proved that the canonical model of a smooth complex projective variety of general type (proved to exist soon afterwards by Birkar et al. and by Siu—this is equivalent to finite generation of the canonical ring) has a Kähler-Einstein metric of negative Ricci curvature. This is only one example, albeit a very important one, of current work on the complex Monge-Ampère equation.

The highly nonlinear nature of the Monge-Ampère operator presents many challenges. Proper understanding of its maximal domain of definition in the local case of a domain in Euclidean space was obtained only recently in work of Zbigniew Błocki [3]. Surprisingly, very recent work of Guedj, Zeriahi, and Dan Coman has shown the domain of definition to be much larger in the global case of a compact Kähler manifold [8].

Coman and Evgeny Poletsky have derived new Bernstein, Bezout, and Markov inequalities using pluripotential theory and applied them to transcendental number theory [9]. Their results have been generalized by Alexander Brudnyi [7]. In a series of papers, the earliest posted in 2002, Charles Favre and Mattias Jonsson have made a deep study of the singularities of plurisubharmonic functions and multiplier ideals in two dimensions using the novel concept of a tree of valuations [15]. In a new paper [6] with Sebastien Boucksom, they have extended some of their work to higher dimensions. A connection with probability theory appears in recent work of Thomas Bloom and Bernard Shiffman [4] and of Robert Berman [1], who use various techniques of pluripotential theory to study zeros of random polynomials and, more generally, random sections of holomorphic line bundles.

3. The Cauchy-Riemann equation on singular spaces. Existing methods for solving the Cauchy-Riemann equation are largely restricted to smooth spaces. Consequently, the central problem of classical several complex variables, the Levi problem, which asks whether Steinness is a local property and was solved for manifolds decades ago, is still open for singular spaces. Progress in this area has been slow and difficult.

Recently, John Erik Fornæss, Nils Øvrelid, and Sophia Vassiliadou have been able to solve the Cauchy-Riemann equations with L^2-estimates in certain singular settings ([16], [17]).

4. Almost complex geometry. In a seminal paper of 1985, Mikhail Gromov introduced almost complex structures and pseudoholomorphic curves into symplectic topology. Interaction between complex geometry and symplectic geometry began in earnest with the work of Sergey Ivashkovich and Vsevolod Shevchishin in the late 1990s [20]. There is now a growing body of work concerned with extending concepts and results from complex analysis and complex geometry to the almost complex case. Often the non-integrable case requires new methods that shed light on the integrable case. Notable new work includes a paper by Bernard Coupet, Alexander Tumanov, and Alexander Sukhov on proper pseudoholomorphic discs [11], a long paper on fundamentals of local almost complex geometry by Coupet, Sukhov, and Hervé Gaussier [10], a paper by Xianghong Gong and Jean-Pierre Rosay on removable singularities of pseudoholomorphic maps [48], and a paper by Ivashkovich and Shevchishin on almost complex structures that are merely Lipschitz [21].

5. Infinite-dimensional complex geometry. Complex analysis in infinite dimensions languished outside the mainstream until László Lempert commenced a major research program in the mid-1990s. Generalized loop spaces (spaces of smooth maps from a compact smooth manifold into a finite-dimensional complex manifold) are examples of infinite-dimensional complex manifolds; their importance in physics provides strong motivation for Lempert's program. Fundamental notions, including Dolbeault cohomology, coherence of analytic sheaves, and holomorphic approximation, have been brought into an infinite-dimensional setting by Lempert, in part with coauthors Endre Szabó, Ning Zhang, and Imre Patyi ([24], [15], [25]). Imre Patyi has studied the Oka principle for infinite-dimensional complex manifolds [26].

Presentation Highlights

Zbigniew Błocki (Jagiellonian University)
On geodesics in the space of Kähler metrics
Our main result is that geodesics in the space of Kähler metrics (as considered by Mabuchi, Donaldson and Semmes) are (fully) $C^{1,1}$, provided that the bisectional curvature is nonnegative. Existence of such geodesics (without curvature assumption) with bounded mixed complex 2nd derivatives was proved by X. X. Chen. It boils down to solving a homogeneous complex Monge-Ampere equation on a compact Kähler manifold with boundary. We also discuss slightly more general equations of this kind.

Sebastien Boucksom (Institut de mathématique de Jussieu)
Equilibrium measures and equidistribution of Fekete points on complex manifolds
Fekete points are optimal configurations of points in polynomial interpolation. It is a classical result that Fekete points confined within a given compact set of the complex plane equidistribute towards the potential-theoretic equilibrium measure of the compact set. I will present a joint work with Robert Berman where we extend this result to the higher-dimensional case by a variational principle, working in the more geometric setting of sections of a line bundle over a compact complex manifold

Debraj Chakrabarti (Notre Dame University)
CR functions on subanalytic hypersurfaces
We consider the problem of local one-sided holomorphic extension of continuous or smooth CR functions from hypersurfaces with singularities, in particular from the class of subanalytic hypersurfaces, which include the real-analytic ones. We discuss the obstructions to the existence of such extension, which turn out to be different from those in the classical smooth case.

Bruno De Oliveira (University of Miami)
Symmetric differentials, differential operators and the topology of complex surfaces
The space of symmetric differentials of order 1, i.e. holomorphic 1-forms, are intimately connected with the topology of a complex surface. On the other hand, the same does not happen for symmetric differentials of higher order. Examples of this difference are: There are families of algebraic surfaces where $h^0(X_t, S^m\Omega^1_{X_t})$ is not locally constant for $m > 1$, a simply connected surface X can have nontrivial symmetric differentials

of order $m > 1$ (in fact Ω^1_X can be ample). To regain the connection with the topology we need to consider a special class of symmetric differentials, we call these differentials closed, they are locally of the form $df_1...df_m$. Opposite to closed differential forms we will show that there is no collection of differential operators characterizing closed symmetric differentials, but as we will see this can be done if we ask to be closed around a general point. A special case of a topological result to be presented is that if a complex surface X has a nontrivial closed symmetric differential of order 2 then $\pi_1(X) \neq 0$.

Xianghong Gong (University of Wisconsin)
Regularity in the CR embedding problem
We will prove a new regularity on the local embedding of strongly pseudoconvex CR manifolds of dimension at least 7. This is joint work with Sidney Webster.

Vincent Guedj (Université Aix-Marseille)
Variational approach to complex Monge-Ampère equations
I will present a new variational approach to Monge-Ampere equations on compact complex manifolds, which enbles to construct singular solutions to the Dirichlet problem without relying on Yau's fundamental existence result. This is joint work with R. Berman, S. Boucksom and A. Zeriahi.

Gordon Heier (University of California, Riverside)
On complex projective manifolds of negative holomorphic sectional curvature
It is a long-standing open problem to show that a complex projective manifold with a Kähler metric of negative holomorphic sectional curvature has an ample canonical line bundle. In this talk, partial results towards this problem will be presented. This is joint work in progress with Bun Wong.

Alexander Isaev (Australian National University)
Infinite-dimensionality of the automorphism groups of homogeneous Stein manifolds
Let X be a Stein manifold of dimension greater than 1 homogeneous with respect to a holomorphic action of a complex Lie group. We show that the Lie algebra generated by complete holomorphic vector fields on X is infinite-dimensional, i.e. it is impossible to introduce the structure of a Lie transformation group on the group of holomorphic automorphisms of X. The well-known examples of complex linear space and affine quadric fit into this general situation. The work is joint with Alan Huckleberry.

Sergey Ivashkovich (Université de Lille-1)
Vanishing cycles in holomorphic foliations by Riemann surfaces and foliated shells
The purpose of this talk is the study of vanishing cycles of holomorphic foliations by Riemann surfaces on compact complex manifolds. The notion of a *vanishing cycle* was implicitly introduced by S. Novikov in his proof of the existence of compact leaves in smooth foliations by surfaces on the three-dimensional sphere. Later it appeared as an obstruction to the simultaneous uniformizability of the object known as a *skew cylinder*, introduced by Ilyashenko, which is proved to be an extremely useful tool in foliation theory. Our main result consists in showing that a vanishing cycle comes together with a much richer complex geometric object—we call this object a *foliated shell*. A number of related statements will be given and several open questions will be discussed.

Burglind Juhl-Jöricke (IHES)
Envelopes of holomorphy and holomorphic discs
The envelope of holomorphy of an arbitrary domain in a Stein manifold is identified with a connected component of the set of equivalence classes of analytic discs immersed into the Stein manifold with boundary in the domain. This has several corollaries, in particular, in case of dimension two for each of its points the envelope of holomorphy contains an embedded (non-singular) Riemann surface passing through this point with boundary contained in the natural embedding of the original domain into its envelope of holomorphy. The method has applications also for the case of projective manifolds.

Nikolay Kruzhilin (Steklov Mathematical Institute)
Holomorphic maps of Reinhardt domains

Nonbiholomorphic proper maps from bounded Reinhardt domains are considered. The structure of the boundary of the source domain and the boundary behavior of the map are investigated. The role of the target domain is discussed.

Frank Kutzschebauch (Universität Bern)
A solution of Gromov's Vaserstein problem
It is standard material in a linear algebra course that the group $\mathrm{SL}_m(\mathbb{C})$ is generated by elementary matrices $E + \alpha e_{ij}, i \neq j$, i.e., matrices with 1's on the diagonal and all entries outside the diagonal are zero, except one entry. Equivalently every matrix $A \in \mathrm{SL}_m(\mathbb{C})$ can be written as a finite product of upper and lower diagonal unipotent matrices (in interchanging order). The same question for matrices in $\mathrm{SL}_m(R)$ where R is a commutative ring instead of the field $\mathbb{C}$ is much more delicate. For example if R is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a space X the problem amounts to find for a given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ a factorization as a product of upper and lower diagonal unipotent matrices

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

where the G_i are maps $G_i : X \to \mathbb{C}^{m(m-1)/2}$. Since any product of (upper and lower diagonal) unipotent matrices is homotopic to a constant map (multiplying each entry outside the diagonals by $t \in [0,1]$ we get a homotopy to the identity matrix), one has to assume that the given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ is homotopic to a constant map or as we will say null-homotopic. In particular this assumption holds if the space X is contractible. This very general problem has been studied in the case of polynomials of n variables. For $n = 1$, i.e., $f : X \to \mathrm{SL}_m(\mathbb{C})$ a polynomial map (the ring R equals $\mathbb{C}[z]$) it is an easy consequence of the fact that $\mathbb{C}[z]$ is an Euclidean ring that such f factors through a product of upper and lower diagonal unipotent matrices. For $m = n = 2$ the following counterexample was found by COHN: the matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices. For $m \geq 3$ (and any n) it is a deep result of SUSLIN that any matrix in $\mathrm{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices. In the case of continuous complex valued functions on a topological space X the problem was studied and solved by THURSTON and VASERSTEIN. It is natural to consider the problem for rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$. Explicitly this problem was posed by GROMOV in his groundbreaking paper where he extends the classical OKA-GRAUERT theorem from bundles with homogeneous fibers to fibrations with elliptic fibers, e.g., fibrations admitting a dominating spray. In spite of the above mentioned result of VASERSTEIN he calls it the *Vaserstein Problem: Does every holomorphic map $\mathbb{C}^n \to SL_m(\mathbb{C})$ decompose into a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $SL_m(\mathbb{C})$?* In the talk we explain a complete solution to GROMOV'S *Vaserstein Problem.* This is joint work with B. Ivarsson.

László Lempert (Purdue University)
The uniqueness of geometric quantization
This is joint work with Szöke, and in progress. In geometric quantization (as in most other schemes of quantization) one associates with a Riemannian manifold a Hilbert space. The manifold represents the classical configurations of a mechanical system, and the Hilbert space is to represent its quantum states. Often the Hilbert space depends on additional choices, and these choices form a smooth or complex manifold S. The uniqueness problem asks whether there is a natural isomorphism between the Hilbert spaces H_s and H_t corresponding to different choices $s, t \in S$.

In the 1990s Axelrod, Della Pietra, and Witten suggested to view the H_s as fibers of a Hilbert bundle H over S, define a connection on H, and use parallel transport to identify its fibers. In the talk I will briefly explain what is unsatisfactory, from the mathematical point of view, in their work. Then I will discuss the mathematical structures to which their idea leads, and properties of these structures. Finally I will tackle the issue of uniqueness when geometric quantization is based on so called adapted Kähler structures.

Vakhid Masagutov (Purdue University)
Homomorphisms of infinitely generated analytic sheaves

We prove that every homomorphism $\mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^F$, with E and F Banach spaces and $\zeta \in \mathbb{C}^m$, is induced by a $\mathrm{Hom}(E, F)$-valued holomorphic germ, provided that $1 \leq m < \infty$. A similar structure theorem is obtained for the homomorphisms of type $\mathcal{O}_\zeta^E \to \mathcal{S}_\zeta$, where $\mathcal{S}_\zeta$ is a stalk of a coherent sheaf of positive depth. We later extend these results to sheaf homomorphisms, obtaining a condition on coherent sheaves which guarantees the sheaf to be equipped with a unique analytic structure in the sense of Lempert-Patyi.

Laurent Meersseman (PIMS/Université de Bourgogne)
Uniformization of deformation families of compact complex manifolds
Consider the following uniformization problem. Take two holomorphic (parametrized by the unit disk) or differentiable (parametrized by an interval containing 0) deformation families of compact complex manifolds. Assume they are pointwise isomorphic, that is for each point t of the parameter space, the fiber over t of the first family is biholomorphic to the fiber over t of the second family. Then, under which conditions are the two families locally isomorphic at 0?

After recalling some known results (positive and negative) on this problem, I will give a sufficient condition in the case of holomorphic families. I will then show that, surprisingly, this condition is not sufficient in the case of differentiable families. These results rely on a geometric study of the Kuranishi space of a compact complex manifold.

Joël Merker (École Normale Supérieure)
Effective algebraic degeneracy
In 1979, Green and Griffiths conjectured that in every projective algebraic variety X of general type, there exists a certain *proper* subvariety Y with the property that every *nonconstant* entire holomorphic curve $f\colon \mathbb{C} \to X$ landing in X must in fact lie inside Y. For projective hypersurfaces X, Siu showed in 2004 that there is an integer d_n such that every generic hypersurface X in $\mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geqslant d_n$, such an Y exists. The talk, based on Demailly's bundle of invariant jet differentials and on a new construction of explicit slanted vector fields tangent to the space of vertical jets to the universal hypersurface (realizing an idea of Siu), will present a recent complete detailed proof (joint with Diverio and Rousseau) of such a kind of algebraic degeneracy statement, with the *effective* degree bound :

$$d \geqslant n^{(n+1)^{n+5}}.$$

In the early 1980's, Lang conjectured a deep correspondence between degeneracy of entire holomorphic curves and finiteness/non-denseness of rational points on projective algebraic varieties that the whole subject is, unfortunately, still unable to put in concrete form.

Imre Patyi (Georgia State University)
On holomorphic domination
We discuss the question of flexible exhaustions of pseudoconvex open sets in a Banach space by sublevel sets of the norm of holomorphic vector valued functions; this has applications to sheaf and Dolbeault cohomology of complex Banach manifolds. We show that if X is a Banach space with a Schauder basis (e.g., $X = C[0,1]$), $D \subset X$ is pseudoconvex open, $u : D \to (-\infty, \infty)$ is continuous, then there are a Banach space Z and a holomorphic function $h : D \to Z$ such that $u(x) < \|h(x)\|$ for $x \in D$; in this case we say that holomorphic domination is possible in D. On a different note we also show that many complex Banach submanifolds of the Banach space ℓ_1 of summable sequences admit many nowhere critical numerical holomorphic functions.

Evgeny Poletsky (Syracuse University)
Functions holomorphic along holomorphic vector fields
We will discuss the following generalization of Forelli's theorem: Suppose F is a holomorphic vector field with singular point at p, such that F is linearizable at p and the matrix is diagonalizable with eigenvalues whose ratios are positive reals. Then any function that has an asymptotic Taylor expansion at p and is holomorphic along the complex integral curves of F is holomorphic in a neighborhood of p. We also present an example to show that the requirement for ratios of the eigenvalues to be positive reals is necessary.

Jean-Pierre Rosay (University of Wisconsin)
Pluripolar sets in almost complex manifolds

The notion of plurisubharmonicity makes sense for functions defined on almost complex manifolds. Pluripolar sets are sets on which a plurisubharmonic function is $-\infty$. I shall discuss the notion of pluripolarity and the important question of logarithmic singularities versus weaker singularities.

The Chirka function with pole at a point has been efficiently used for localization of the Kobayashi metric (Gaussier-Sukhov and Ivashkovich-Rosay). I shall discuss more recent results on pluripolarity with applications to uniqueness results (Ivashkovich-Rosay).

Alexandre Sukhov (Université de Lille-1)
Constructions of pseudoholomorphic discs
We establish an existence and study the properties of J-complex curves with prescribed boundary conditions in almost complex Stein manifolds.

Sophia Vassiliadou (Georgetown University)
Hartogs extension theorems on complex spaces with singularities
I will discuss some generalizations of the classical Hartogs extension theorem to complex spaces with singularities and present an analytic proof using $\overline{\partial}$-techniques (joint work with Nils Øvrelid).

Jörg Winkelmann (Universität Bayreuth)
On Brody curves
We discuss a number of properties of Brody curves which underline that the class of Brody curves is rather "delicate", for example, the property of a variety of admitting a non-degenerate Brody curve does not behave well in families.

Aaron Zerhusen (Illinois Wesleyan University)
Local solvability of the $\bar{\partial}$-equation in certain Banach spaces.
In a sharp contrast to the situation in finite dimensions, Imre Patyi has shown that the $\bar{\partial}$-equation is not always solvable, even locally, in an infinite dimensional Banach space. On the other hand, László Lempert has shown that in ℓ_1, the Banach space of 1-summable sequences, the $\bar{\partial}$-equation is solvable for (0,1)-forms on pseudoconvex domains. I will discuss how Lempert's result leads to a proof of local solvability of the $\bar{\partial}$-equation in a large class of Banach spaces which includes any L_1 space and the dual space of any L_∞ space.

Scientific Progress Made

Almost all the talks generated many interesting questions from the audience, related to the results presented in the talks. Some of the questions were about new possible directions of research, while some pointed to possible connections of the exposed results to other fields of mathematics.

Aside from such questions, there were quite a few longer discussions between groups of participants regarding not only the topics presented in the lectures but also other important open questions. We note here a few such discussions.

A very recent result of Berman and Boucksom [5] shows that in higher dimensions the arrays of Fekete points in a compact set equidistribute to the equilibrium measure of that compact set. This was known in dimension one (Fekete's Theorem), and had been conjectured to be true in any dimension once the right analogues of equilibrium measures were introduced in pluripotential theory. This result provides important applications of pluripotential theory in approximation theory. Discussions about it went on throughout the duration of the workshop between Boucksom, Levenberg, Bloom, Bos and others.

The new variational method of solving Monge-Ampère type equations on compact Kähler manifolds presented in the talk of V. Guedj also generated much discussion during the workshop. The notion of foliated shells and their applications presented by S. Ivashkovich, and the recent characterization of envelopes of holomorphy of domains in Stein manifolds using analytic discs presented by B. Jöricke [22], were considered very interesting and discussed by some participants.

An interesting open question in complex geometry is to study whether in a complex manifold, analytic discs with boundary have a Stein neighborhood. The existence of such neighborhoods is useful in applications. The question was discussed a number of participants, including Ivashkovich, Poletsky, Rosay, Shcherbina.

One usually cannot expect major theorems to be proved during a five-day workshop. However, the organizers are confident that the ideas generated by the talks presented and by the many discussions that took place during the week will lead to important progress at least in some of the many topics covered by the conference.

Outcome of the Meeting

Although a single workshop cannot do justice to the breadth and depth of contemporary complex analysis and complex geometry, the organizers believe it was beneficial to bring together a group of experts from diverse subfields to discuss recent results and work in progress, and to share ideas on open questions. We chose a coherent collection of interrelated topics for the workshop, representing some of the most vibrant developments in the subject today.

There were 41 participants, ranging from leading experts to graduate students (5 in total) and recent PhDs (another 5). Among the participants were 6 female mathematicians.

The program consisted of 24 talks, each of 45 minutes, with a break of at least 15 minutes in between talks. Five of the talks were by recent PhDs or graduate students. There were three full days, in which the presentations ended by 5:00 pm, while the remaining two days consisted of just a morning session. This allowed ample time for questions and discussions.

There is general agreement among the participants that the workshop was both inspiring and stimulating. All very much appreciated the excellent facilities and the hospitality at BIRS. The beautiful scenery and the unseasonably warm weather during the week helped make the workshop a success.

List of Participants

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Chapter 10

Advances in Stochastic Inequalities and their Applications (09w5004)

Jun 07 - Jun 12, 2009

Organizer(s): David M. Mason (University of Delaware), Luc Devroye (McGill University), Gabor Lugosi (ICREA and Pompeu Fabra University)

Overview of the Field

Stochastic inequalities play a crucial role in a wide variety of areas of mathematical science. Among these areas are learning theory, empirical processes, nonparametric function estimation, combinatorial optimization, high-dimensional geometry, random graphs, and Gaussian processes. Stochastic inequalities include concentration inequalities for functions of independent random variables, deviation inequalities for independent sums and their extension to functions of independent random variables such as U-Statistics, decoupling inequalities, as well as sharp moment inequalities for the norm of independent sums of Banach space valued random variables. Also partial extensions of these inequalities to dependent situations such as martingale and weakly dependent sequences have been accomplished and are important in applications.

Outcome of the Meeting

The objective of the workshop was to bring together a strong group of mathematicians who have made important contributions to stochastic inequalities and their applications. We witnessed a lively interdisciplinary exchange of ideas and methods that will surely lead to further progress in the particular research areas of the participants and will eventually lead to new developments.

The workshop featured 32 talks on a wide range of aspects of stochastic inequalities and their applications. Various speakers with different background presented their work in a way accessible to all participants which was important to help ideas penetrate across different areas and triggered interesting and fruitful discussions. In the next section we describe the some highlights of these talks.

Presentation Highlights

The 32 presentations considered different kinds of stochastic inequalities arising in different fields and various applications were developed. The topics included inequalities for Markov chains, inequalities for empirical processes, concentration inequalities, inequalities for regression, density estimation, and statistical learning theory, multivariate central limit theorems, U-statistics and chaoses, random matrix inequalities, inequalities for dependent random variables, applications for operations research, applications for high-dimensional geometry, and the Gaussian correlation conjecture. The titles and abstracts of the talks are listed below.

Tail inequalities for additive functionals and empirical processes of geometrically ergodic Markov chains

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I will present some Bernstein style tail inequalities for additive functionals and empirical processes of geometrically ergodic Markov chains. The bounds are expressed in terms of the asymptotic variance and and the L_∞ norm of the function defining the functional. The proofs are based on the classical regeneration method and some new inequalities for empirical processes of independent random variables with finite exponential Orlicz norms.

An invariance principle for set-valued M-estimators through the boundary empirical process.

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In current researches [1] with John Einmahl we combine several tools from the empirical process theory – all derived from concentration, symmetrization and moment inequalities – to describe the oscillation behavior of various kinds of set-valued M-estimators C_n in R^d. Among these empirical minimizers are excess mass sets, minimum volume sets, shorth sets – or maximum probability sets – selected in a class $\mathcal{C}_n$ by means of an i.i.d. sample having law P in R^d. When P has a density f these sets estimate a level set C of f. We do not study the volume $\lambda(C_n \Delta C)$ of the symetric difference between an empirical C_n and a target set C as in [8] but the set $C_n \Delta C$ itself, provided C is a convex body. To deal with weak convergence of random sets C_n we investigate a new kind of limit theorems. For this we use the cylinder description of the boundary empirical measure introduced in [6] and [7]. Also, the probability bounds for the strong gaussian approximation from [2,3] and for the stability of empirical minimizers from [4] both play a crucial role. It turns out that we can describe the joint limit law of the above error sets $C_n \Delta C$ in terms of an auxiliary Brownian motion indexed by functions describing the boundary ∂C, drifted by a deterministic process driven by the second order of P around ∂C. Extensions are on the way in clustering or quantization type problems such as the optimal k-balls covering, or in the trimmed k-means problem introduced in [5].

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Self-bounding functions, Talagrand's convex distance inequality and related questions

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Using the (modular) entropy method, Boucheron, Lugosi and Massart (2003), Maurer (2006) obtained transparent proofs of parts of Talagrand's convex distance inequality (1995). The argument relied on a simple observation: the (random) Efron-Stein estimate of the variance of the convex distance is upper-bounded d by 1. This was not enough to handle the fluctuations of the convex distance to very small sets. By relating the squared convex distance to (general) weakly self-bounding functions, it is possible to recover the full power of Talagrand's convex distance inequality using a transparent and modular proof. This amounts to check that the squared convex distance (and many other weakly self-bounding functions) satsifies a Bernstein-like inequality.
Joint work with G. Lugosi and P. Massart.

Spectrum of large random Markov chains

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We consider the spectrum of random Markov chains with very large finite state space. The randomness of these chains, which appears as a random environment, is constructed by putting random weights on the edges of a finite graph. This approach raises stimulating open problems, lying at the interface between random matrix theory and random walks in random environment. Part of this work is in collaboration with Ch. Bordenave (Toulouse, France) and P. Caputo (Rome, Italy).

Exponential inequalities for self-normalized processes with applications

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We prove the following exponential inequality for a pair of random variables (A,B) with $B>0$ satisfying the following *canonical assumption*, $E[\exp(\lambda A - \frac{\lambda^2 B^2}{2})] \leq 1$ for λ in *R*.

$$P\left(\frac{|A|}{\sqrt{\frac{2q-1}{q}(B^2+(E(|A|^p])^{2/p})}} \geq x\right) \leq c_q x^{-\frac{q}{2q-1}} e^{-x^2/2}$$

where $C_q = (\frac{q}{2q-1})^{\frac{q}{2q-1}}$, $x>0$ and $1/p+1/q=1$ for $p \geq 1$. Applying this inequality, we obtain subgaussian bounds for the tail probabilities for self-normalized martingale difference sequences. We propose a method of hypothesis testing for the L^p-norm ($p \geq 1$) of A (in particular, martingales) and some stopping times.

Multivariate Bahadur-Kiefer Representations

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The classical Bahadur-Kiefer representation (Bahadur (1967), Kiefer (1970)) gives a limit law (as $n \to \infty$) for the statistic $\|\alpha_n + \beta_n\|$, where α_n (resp. β_n) denotes a uniform empirical (resp. quantile) process based upon a sample of n uniformly distributed on $(0,1)$ observations. Here, we set $\|f\| := \sup_{0 \le t \le 1} |f(t)|$ for the sup-norm of a bounded function f on $[0,1]$. In this paper, we provide a multivariate extension of this result. We consider $d \ge 1$ pairs of empirical and quantile processes $\{\alpha_{n;j}, \beta_{n;j}\}$, $j = 1, \dots, d$ and focus our interest in the case where these d pairs are mutually independent, corresponding to the situation where each individual pair is generated by an independent sequence of i.i.d. uniform $(0,1)$ random variables. Setting $\mathbf{t} = (t_1, \dots, t_d)$, we establish limit laws (as $n \to \infty$) for statistics of the form $\sup_{\mathbf{t} \in [0,1]^d} \left| \sum_{j=1}^d \Psi_j(\mathbf{t}) \{\alpha_{n;j}(t_j) + \beta_{n,j}(t_j)\} \right|$, where $\Psi_1, \dots, \Psi_d$ are suitable continuous functions on $[0,1]^d$. Besides providing some extensions of the classical Bahadur-Kiefer representation, these results allow us to obtain optimal rates of strong approximation of empirical copula processes by sequences of Gaussian processes.

Markov Chain Coupling for Stochastic Domination of Order Statistics

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For the order statistics $(X(1:n), X(2:n), \cdots, X(n:n))$ of a collection of independent, not necessarily identically distributed random variables and for any $i \in [n]$, the conditional distribution $(X(i+1:n), \cdots, X(n:n) \mid X(i:n) > s)$ is shown to be stochastically increasing in s using a coupling of Markov chains. (Joint work with Olle Häggström.)

Uniform in bandwidth consistency of kernel regression estimators at a fixed point

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We show that the empirical process approach developed by Einmahl and Mason (2000, 2005) for proving uniform consistency results for kernel regression function estimators on compact sets can be adapted so as to also give optimal results for pointwise convergence. Our results are uniform in bandwidth and uniform over certain function classes. As in the previous work, we need good exponential deviation and moment inequalities for general empirical processes. As we are dealing with the pointwise convergence of such estimators it is sufficient to use a Bernstein type exponential inequality in terms of the strong second moments which is due to Yurinskii (1976) rather than the more elaborate inequality of Talagrand (1994) in terms of the weak second moments. The moment inequality we need seems to be new and might be of independent interest. Combining these tools we can obtain results with optimal convergence rates for function classes having an envelope function with finite moment generating function. This is different from the corresponding results on uniform convergence rates over compact sets where the function classes had to be bounded. One might wonder whether one can extend these results to the finite moment generating function case as well. We would be able to answer this question in the affirmative if we had a Bernstein type inequality for unbounded function classes in terms of the weak second moments. (This is joint work with Julia Dony, Free University of Brussels (VUB).)

A limit theorem for the distribution of the absolute deviation of linear wavelet density estimators

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The Smirnov-Bickel-Rosenblatt limit theorem for the sup-norm deviation of a convolution kernel density estimator is extended to some wavelet density estimators. This is joint work with Richard Nickl.

Optimal Rates in the Multivariate Central Limit Theorem for Balls

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We discuss the connections between asymptotic approximations of quadratic forms with generalized χ^2-limits in the multivariate central limit theorem with classical lattice point counting problems of Hardy and Landau. We describe recent optimal approximation bounds, which are valid starting at dimension 5 in the multivariate CLT obtained with A. Zaitsev. This is related to joint work with G. Margulis on the local equidistribution of values of indefinite quadratic forms on lattices.

Oracle Inequalities in Sparse Recovery Problems

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Numerous problems in Statistics and Learning Theory can be reduced to penalized empirical risk minimization over linear spans or convex hulls of large dictionaries of functions. The goal is to recover a sparse approximation of a target function (such as regression functions or Bayes classification rules) based on noisy observations at random locations. Convex complexity penalties are often used in empirical risk minimization to find such a sparse solution and sharp oracle inequalities with error terms that depend on the degree of sparsity of the problem have to be proved. The talk will deal with such inequalities in several problems including ℓ_1-norm penalized empirical risk minimization over linear spans and entropy penalized empirical risk minimization over convex hulls. Talagrand's concentration inequalities and other bounds for empirical and Rademacher processes are among the main tools in these problems.

Limit Theorems for High Dimensional Data

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We establish limit theorems for high dimensional data that is characterized by small sample sizes relative to the dimension of the data. In particular, we provide an infinite-dimensional framework to study statistical models that involve situations in which (i) the number of parameters increase with the sample size (that is allowed to be random) and (ii) there is a possibility of missing data. Under a variety of tail conditions on the components of the data, conditions for the law of large numbers, as well as various results concerning the rate of convergence in these models are obtained. We also present central limit theorems in this setting, some which involve data driven coordinate-wise normalizations.

Estimates of moments and tails for some multidimensional chaoses

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We present two sided estimates on moments and tails of random variables of the form

$$\sum a_{i_1,\ldots,i_d} X_{i_1} \cdots X_{i_d},$$

where X_i are independent symmetric random variables with logarithmically concave tails and $d \leq 3$. For $d > 3$ we are able so far to derive upper bounds only in the special cases (including exponential and Gaussian

random variables). Estimates are exact up to constants depending on d only. As a tool we show a new bound for suprema of certain empirical processes.
The talk is based on joint work with Radoslaw Adamczak.

A Gaussian Inequality for Absolute Value of Products

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We will discuss the inequalities

$$E|X_1X_2\cdots X_n| \leq \sqrt{\text{perm}\Sigma} \leq (EX_1^2X_2^2\cdots X_n^2)^{1/2}$$

for any centered Gaussian random variables $X_1, \cdots, X_n$ with the covariance matrix Σ. The first inequality is due to the speaker and the second inequality is due to Frenkel (2008). Various implications, examples, applications and conjectures will also be presented. This is a joint work with Ang Wei.

On the behavior of random matrices with independent columns

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The talk is based on joint works with R. Adamczak, O. Guédon, A. Pajor, and N. Tomczak-Jaegermann. We discuss behavior of several parameters of a random $n \times N$ matrix A, whose columns are independent random vectors in $\mathbf{R}^n$ satisfying some natural conditions. In particular, we obtain estimates for the spectral norm of A (i.e. the largest singular value of A or, equivalently, the operator norm $\|A : \ell_2^N \to \ell_2^n\|$); the smallest singular value; the norm of A on the set of all m-sparse vectors (i.e. vectors having at most m nonzero coordinates), which is denoted by A_m. Our estimates hold with overwhelming probability, that is, the probability tending to one as the dimension grows to infinity. In particular, we obtain that for isotropic log-concave i.i.d. random vectors X_i's

$$\text{Prob}\left(\exists m \leq N \,:\, A_m \geq C\left(\sqrt{n} + \sqrt{m}\log\frac{2N}{m}\right)\right) \leq \exp\left(-c\sqrt{n}\right),$$

where c and C are absolute positive constants. Note here that $A_N = \|A\|$.

We apply our results to solve several problems. First, we provide asymptotically sharp answer to the question posed by R. Kannan, L. Lovász, M. Simonovits: *Let K be an isotropic convex body in $\mathbf{R}^n$. Given $\varepsilon > 0$, how many independent points X_i uniformly distributed on K are needed for the empirical covariance matrix to approximate the identity up to ε with overwhelming probability?* Namely, we show that it is enough to take $N \approx C(\varepsilon)n$ vectors. Then we turn to applications to compressed sensing and convex geometry. We investigate RIP (Restricted Isometry Property) of random matrices with independent columns and show that the matrix A, considered above, satisfies RIP. Thus, as was shown in works of E. Candes and T. Tao, and D. L. Donoho, such a matrix can be used to solve exact reconstruction process of m-sparse vectors via ℓ_1 minimization as well as to construct neighborly polytopes.

Concentration of measure and mixing for Markov chains.

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We discuss certain Markovian models on graphs with local dynamics. We show that, under suitable conditions, such Markov chains exhibit strong concentration of measure over long time intervals. Further, with additional assumptions, we also have both rapid convergence to equilibrium and strong concentration of measure in the stationary distribution.

A high dimensional Wilks phenomenon

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A theorem by Wilks asserts that in smooth parametric density estimation the difference between the maximum likelihood and the likelihood of the sampling distribution converges toward a chi-square distribution where the number of degrees of freedom coincides with the model dimension. This observation is at the core of some goodness-of-fit testing procedures and of some classical model selection methods. This paper describes a non-asymptotic version of the Wilks phenomenon in bounded contrast optimization procedures. Using concentration inequalities for general functions of independent random variables, it proves that in bounded contrast minimization (as for example in Statistical Learning Theory), the difference between the empirical risk of the minimizer of the true risk in the model and the minimum of the empirical risk (the excess empirical risk) satisfies a Bernstein-like inequality where the variance term reflects the dimension of the model and the scale term reflects the noise conditions. From a mathematical statistics viewpoint, the significance of this result comes from the recent observation that when using model selection via penalization, the excess empirical risk represents a minimum penalty if non-asymptotic guarantees concerning prediction error are to be provided. From the perspective of empirical process theory, this paper describes a concentration inequality for the supremum of a bounded non-centered (actually non-positive) empirical process. Combining the now classical analysis of M-estimation (building on Talagrand's inequality for suprema of empirical processes) and versatile moment inequalities for functions of independent random variables, this paper develops a genuine Bernstein-like inequality that seems beyond the reach of traditional tools.

Inequalities for Self-Bounding Random Variables

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The talk applies inequalities for self-bounding random variables to variance related objects. I give a result on the concentration of the empirical variance of a sample of independent, bounded variables, and show how it can be used to give tight empirical versions of Bernstein's inequality. A related result concerns the eigenvalues of the normalized Gramian generated by n independently drawn datapoints in a high-dimensional ball. Here the estimation error for the k-th largest eigenvalue can be bounded with high probability in terms of the largest eigenvalue and a remainder of order n^{-1}.

Quantitative asymptotics of graphical projection pursuit

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In 1984, Diaconis and Freedman proved a limit result stating roughly that, given a large number n of data points in a high dimension d, most one-dimensional projections of the data would look approximately Gaussian. In this talk, I will present a quantitative version of the theorem, in the form of a concentration inequality for the bounded-Lipschitz distance between the empirical distribution of a random projection of the data and a suitably scaled Gaussian distribution. I will also present a multivariate version, considering projections of the data onto subspaces of dimension k. In particular, I will discuss the issue of how k may grow with n and d for this normal-projections phenomenon to persist. The method of proof is by a combination of Stein's method, the concentration of measure phenomenon, and Dudley's entropy bound, and is likely to have many other applications.

Concentration of polynomial functions of random matrices

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In the spirit of results of Guionnet and Zeitouni and of free probability theory, we prove concentration inequalities for noncommutative polynomials of large independent random matrices. This is joint work with S. Szarek.

A Bernstein type inequality and moderate deviations for weakly dependent sequences

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In this talk I shall present a joint work with M. Peligrad and E. Rio, concerning a tail inequality for the maximum of partial sums of a weakly dependent sequence of random variables that is not necessarily bounded. The class considered includes geometrically and subgeometrically strongly mixing sequences. The result is then used to derive asymptotic moderate deviation results. Applications include classes of Markov chains, functions of linear processes with absolutely regular innovations and ARCH models.

Adaptive Confidence Bands in Density Estimation

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Given a sample from some unknown continuous density $f :\rightarrow$, we construct fully adaptive estimators for f and prove an exact Smirnov-Bickel-Rosenblatt type limit theorem for it. This allows to obtain adaptive confidence bands which are honest for all densities in a 'generic' subset of the union of t-Hölder balls, $0 < t \leq r$, where r is a fixed but arbitrary integer. The proofs are based on a precise analysis of the stochastic behaviour of certain linear wavelet or kernel density estimators, in particular exponential inequalities and extremal type limit theorems.

Weak vs. strong parameters for vector-valued Rademacher sums

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Weak and strong parameters (moments and tails) of vector-valued Rademacher sums are related by a deviation inequality. Estimates obtained imply asymptotic equality of the optimal constants in the Khinchine and Khinchine-Kahane inequalities. Aslo, they form a counterpart to the classical results of Talagrand about concentration on the discrete cube - the new bounds being of interest when weak parameter is rather large with respect to the strong one. The work is unpublished yet but not new and some parts of it were presented already back in 2005 and 2006; however, the presentation was restricted to the convex geometry circles and I think that both the results and their quite elementary proofs may be of some interest also for people working on stochastic inequalities.

Functional central limit theorem via martingale approximation

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Martingale approximation as a tool to obtain asymptotic results goes back to Gordin and the theory was developed by many mathematicians including Philipp, Kipnis, Varadhan, Hall, Heyde, Maxwell, Woodroofe, Zhao, Wu, Volny, Dedecker, Merlevède, Rio among others. We shall stress the characterization of stochastic processes that can be approximated by martingales for deriving the conditional functional central limit theorem. The results are easily applicable to a variety of examples, leading to a better understanding of the structure of several classes of stochastic processes and their asymptotic behavior. The approximation brings together many disparate examples in probability theory. It is valid for classes of variables defined by familiar projection conditions, various classes of mixing processes including the large class of strong mixing processes and to classes of reversible and normal Markov operators. The main tool in analyzing all these examples are maximal inequalities. Joint work with Mikhail Gordin.

On the Bennett-Hoeffding inequality

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The well-known Bennett-Hoeffding bound for sums of independent random variables is refined, by taking into account truncated third moments, and at that significantly improved by using, instead of the class of all increasing exponential functions, the much larger class of all generalized moment functions f such that f and f'' are increasing and convex. It is shown that the resulting bounds have certain optimality properties. Comparisons with related known bounds are given. The results can be extended in a standard manner to (the maximal functions of) (super)martingales. The proof of the main result is much more difficult than those of the previous results; it uses an apparently new method that may be referred to as infinitesimal spin-off.

Estimation of convex-transformed densities

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A convex-transformed density is a quasi-concave (or a quasi-convex) density which is a composition of monotone and convex functions. We consider nonparametric estimation in a scale of such families of densities on R^d indexed by a real parameter s. The value $s = 0$ corresponds to log-concave densities, while values of $s \neq 0$ correspond to heavier tailed densities or densities concentrated on particular subsets of R^d according as $s \leq 0$ or $s > 0$. Many parametric and non-parametric families of densities can be included in a suitable family of convex-transformed densities: normal, gamma, beta, Gumbel and other log-concave densities, multivariate Pareto, Burr, Student t, Snedecor etc. We study the properties of nonparametric estimation in these classes of convex-transformed densities, including existence and consistency of the maximum likelihood estimator, and asymptotic minimax lower bounds for estimation.

Stochastic Limit Theorems with Deterministic Analogs, Stationary Analogs, or No Analogs

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If you choose n points at random in the unit square, the classic theorem of Beardwood, Halton, and Hammersley tells you that the length of the shortest tour through these points is asymptotic to a constant times the square root of n. Now, consider the purely deterministic case where for each n we look at the worst case point configuration. Again, we get a sequence of lengths that are asymptotic to the square root of n. We consider several examples of such analog pairs, and also some cases where the analogy fails. In particular, we consider some instructive instances of failure where independent samples are replaced by sequences from a stationary ergodic process. Naturally, there are links with large deviation inequalities, Orlicz norm bounds, discrepancy theory, and empirical processes.

Vector-valued tangent sequences and decoupling

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The class of UMD spaces was extensively studied by Burkholder in the eighties. It is the right setting for vector-valued harmonic analysis and stochastic integration theory as developed in that same period. In more recent years progress on research in PDEs and harmonic analysis as well as progress in stochastic integration and SPDEs incited new interest in UMD spaces (see Kunstmann and Weis (2004) and references therein, and Neerven, Veraar and Weis (2007-now)). More precisely, decoupling inequalities related to UMD spaces proved to be useful. This talk focusses on such inequalities.
Hitczenko (1989) and McConnell (1989) proved decoupling inequalities for tangent martingale differences with values in a UMD space. Cox and Veraar (2007) considered a one sided version of the decoupling inequalities and showed that it also holds for L^1-spaces (which are not UMD). This inequality can be interpreted as a probabilistic Banach space property, which we refer to as *the decoupling property*. In the talk we present some recent results such as p-independence of the property and constants, and we give examples of other spaces with the decoupling property.
We dicuss several open problems, explaining their importance to harmonic and stochastic analysis.

A Comparison of Three Methods for Bounding Moments of Sums

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Moment inequalities for sums of independent random vectors are important tools for statistical research. Nemirovski and coworkers (1983, 2000) and Pinelis (1994) derived one particular type of such inequalities: For certain Banach spaces $(\mathbb{B}, \|\cdot\|)$ there exists a constant $K = K(\mathbb{B}, \|\cdot\|)$ such that for arbitrary independent and centered random vectors $X_1, X_2, ..., X_n \in \mathbb{B}$, their sum S_n satisfies the inequality $E\|S_n\|^2 \le K \sum_{i=1}^n E\|X_i\|^2$. We present and compare three different approaches to obtain such inequalities: The results of Nemirovski and Pinelis are based on deterministic inequalities for norms. Another possible vehicle are type and cotype inequalities, a tool from probability theory on Banach spaces. Finally, we use a truncation argument plus Bernstein's inequality to obtain another version of the moment inequality above. Interestingly, all three approaches have their own merits. (Talk based on joint work with Lutz Dümbgen, Sara van der Geer, and Mark Veraar.)

An Approach to the Gaussian Correlation Conjecture

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From the original question by Dunnett and Sobel (1955) to the present formulation given by Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel (1970), many specific cases of the Gaussian Correlation Inequality have been proved. The methods of proof are quite varied. For example, Zbyněk ˇSidák (1967) used mainly algebraic and calculus methods to obtain the case when one of the sets is a symmetric strip. Loren Pitt (1977) used a geometric approach together with of a special case of what is often called the Gradient Conjecture to prove the conjecture in dimension 2. Gilles Hargé (1999) uses a semigroup and Dario Cordero-Erausquin (2002) uses a Mass Transport approach via a Theorem of Caffarelli (2000). Our approach is inductive.

Brunn-Minkowski type inequalities for Gaussian Measure.

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In this talk we will present a joint work with Richard Gardner. We will discuss the Brunn-Minkowski type inequalities for Gaussian Measure in R^n. The best-known of these are EhrhardÕs inequality, and the weaker logarithmic concavity inequality. We obtain some results concerning other inequalities of this type, as well as a best-possible dual Gaussian Brunn-Minkowski inequality (where the Minkowski sum is replaced by radial sum).

List of Participants

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Chapter 11

Multimedia, Mathematics & Machine Learning II (09w5056)

Jul 05 - Jul 10, 2009

Organizer(s): Rabab Ward (University of British Columbia), Li Deng (Microsoft Research), Jeffrey Bloom (Dialogic Research Inc.)

Overview of the Workshop and the Field

Following the success of the previous Workshop of Multimedia and mathematics during July 23-28, 2005, the current workshop continues the intensive study of the field and brings together the earlier participants plus additional new prominent researchers, with the expanded theme including machine learning. The expanded theme is to push the state of the art in multimedia processing techniques and multimedia technologies by exploring modern mathematical, pattern recognition, and machine learning methods that have cross-media generality. In particular, we bring prominent researchers as well as tutorial lecturers who have rich working/research experiences in one or more media types and who share the experiences on commonality and differences in the mathematical and machine learning techniques for processing different types of media contents.

Multimedia technologies represent rich applications and interactions among a variety of information sources including audio/music, speech, image/graphics/animation, video, and text/documents/language. They also span over wide ranging information processing tasks including coding/compression, analysis, communication/networking/security, synthesis, user interface, perception/recognition/understanding, and retrieval/mining. Future multimedia technology development will require an increasing level of intelligence, for which mathematical representation, modeling, and learning will play an increasingly important role. This is one of the reasons that we include machine learning, providing a rich set of practical algorithms derived from rigorous mathematical analysis. This forms one principal element in the workshops theme.

The main component of the workshop is a series of presentations and ensuing discussions. Due to the multidisciplinary nature of the subject, we invited two tutorial speakers who are experienced in research with multiple media contents. The remaining presentations are focused on state of the art research in a wide range of subjects on multimedia with related machine learning techniques.

Presentation Highlights

Tutorials

The first tutorial was given by Prof. Bernd Girod of Stanford University, entitled “Mobile Image Matching Recognition Meets Compression.” This is a prime example of integrating recognition and coding tasks that are both common in multimedia research. Specifically, the application is on image retrieval with handheld mobile devices, such as camera phones or PDAs, which are expected to become ubiquitous platforms for visual search and mobile augmented reality applications. For mobile image matching, a visual data base is typically stored at a server in the network. Hence, for a visual comparison, information must be either uploaded from the mobile to the server, or downloaded from the server to the mobile. With relatively slow wireless links, the response time of the system critically depends on how much information must be transferred in both directions. The tutorial reviews recent advances in mobile matching, using a “bag-of-visual-words” approach with robust feature descriptors. The results demonsrate that dramatic speed-ups are possible by considering recognition and compression jointly. Real-time implementations for different example applications are described, such as recognition of landmarks or CD cover, to show the benefit from image processing on the phone, the server, and/or both.

The second tutorial was given by Prof. Hermann Ney of RWTH Aachen University in Germany, entitled “Statistical Methods for image, speech, and language processing: Achievements and open problems.” This tutorial gives an overview of the statistical methods underlying the dramatic progress in statistical methods for recognizing image and speech signals and for translating spoken and written languageover the last two decades. In particular, it focuses on the remarkable fact that, for all three tasks, the statistical approach makes use of the same four principles: 1) Bayes decision rule for minimum error rate; 2) probabilistic alignment models, e.g. Hidden Markov models, for handling strings of observations (like acoustic vectors for speech recognition and written words for language translation); 3) training criteria and algorithms for estimating the free model parameters from large amounts of data, and 4) the generation or search process that generates the recognition or translation result. The author points out that most of these methods had originally been designed for speech recognition. However, it has turned out that, with suitable modifications, the same concepts carry over to both language translation and image recognition, which in both cases results in systems with state-of-the-art performance. This tutorial elegantly summarizes the achievements and the open problems in this extremely fertile area of statistical modelling.

Research-Oriented Presentations

In addition to the two tutorials, there are numerous high-quality presentations at the workshop that are focused on research ideas and applications in various areas of multimedia including image/video, audio/speech, and multimedia security. We now give a summary of these presentations.

Prof. Lina Karam of Arizona State University gave a talk on “Adaptive Rate-Distortion Based Wyner-Ziv Video Coding.” Her talk starts with a brief introduction to the area of Distributed Video Coding (DVC), which is also known as Wyner-Ziv Video Coding. Two novel adaptive DVC systems are then presented: a pixel-domain DVC system with a rate-distortion based BitpLAne SelecTive decoding (BLAST-DVC), and a transform-domain DVC system with a rate-distortion based Adaptive QuanTization (AQT-DVC). Coding results and comparisons with existing DVC schemes and with H.264 interframe and intraframe coding are presented to illustrate the performance of the proposed systems.

In the presentation entitled “Rectification-based View Interpolation and Extrapolation for Multiview Video Coding: R-D Analysis and Applications,” Prof. Jie Liang of Simon Fraser University applis view interpolation in an emerging area of multiview view coding (MVC). Existing schemes assume all cameras are aligned. These methods do not perform well when neighboring cameras point to different directions. In this talk, the author first derives the theoretical R-D performance of the rectification-based view interpolation. He then applies it to H.264 MVC. To further improve the coding efficiency, he develops a rectification-based view extrapolation for MVC. Finally, he investigates the application of the view interpolation in multiple-description coding of multiview images.

In the presentation entitled “High dimensional consensus,” Pro.f Jos M. F. Moura of Carnegie Mellon University considers distributed algorithms that can arise when a large number of agents cooperate to reach a common decision or in sensor networks where a large number of sensors cooperate to process large amounts of collected data. In the last few years there has been intensive research in distributed algorithms for such problems. The presenter describes a general class of distributed algorithms, high dimensional consensus (HDC). He shows how a number of problems of interest including distributed inference (like detection, estimation, or classification,) distributed localization, or several types of consensus algorithms can be cast in the framework of HDC. He discusses the convergence of HDC under a broad set of conditions: deterministic as well as random, as when there is noise in the intersensor communications or links among sensors fail at random times. Finally, he address tradeoffs among network and application parameters and their impact on resource allocation, convergence rate, and topology design.

In the presentation entitled “Rotation-invariant wavelet-based matching of local features, with enhanced tolerance to shifts in location and scale,” Prof. Nick Kingsbury of University of Cambridge describesa technique for using dual-tree complex wavelets to obtain rich feature descriptors of keypoints in images. The main aim has been to develop a method for retaining the full phase and amplitude information from the complex wavelet coefficients at each scale, while presenting the feature descriptors in a Fourier-domain form that allows for efficient correlation at arbitrary rotations between the candidate and reference image patches. The feature descriptors are known as Polar-Matching matrices. Recently, he modified the previously proposed approach so that it can be more resilient to errors in keypoint location and scale. These multi-scale feature descriptors are potentially useful for object detection, recognition, classification and tracking in images and video

Prof. Tsuhan Chen of Cornell University presented “A Graphical-Model Framework for Using Context to Understand Images of People.” The motivation of this research is that when we see other humans, we can quickly make judgements regarding many aspects, including their demographic description and identity if they are familiar to us. We can answer questions related to the activities of, emotional states of, and relationships between people in an image. We draw conclusions based not just on what we see, but also from a lifetime of experience of living and interacting with other people. The presenter proposes contextual features and graphical models for understanding images of people with the objective of providing computers with access to the same contextual information that humans use.

In the presentation of “Genomic/Proteomic Signal Processing for Cancer Classification and Prediction,” Prof. Ray Liu of University of Maryland, College Park discussed an important topic of cancer classification and prediction. DNA microarray and proteomic mass spectrum technologies make it possible to simultaneously monitor thousands of genes/protein expression levels and distribution. A topic of great interest is to study the different expression profiles from cancer patients and normal subjects, by classifying them at gene/protein expression levels. Currently, various clustering methods have been proposed in the literature to classify cancer and normal samples based on microarray data, and they are dominantly data-driven approaches. In this talk, an alternative model-driven approach, named ensemble dependence model, is presented aiming at exploring the group dependence relationship of gene clusters. Because of the limited size of current data, it is not feasible to examine the regulation relationship between all genes. Also, both the microarray gene expression and mass spectrum data are noisy. However, if they are clustered in a right way, the noise level in the resulting cluster expression will be reduced, thus the ensemble dependence dynamics of gene clusters will be revealed. Under the framework of hypothesis-testing, genes dependence relationship as a feature to model is employed to classify cancer and normal samples. The classification scheme is then applied to several real cancer data sets. It is noted that the method yields very promising performance.

A group of presentations at the workshop are focused on speech and audio processing and on the general relationship across various media area, following Prof. Ney’s tutorial.

In the presenation of “From Recognition to Understanding — Expanding the Traditional Scope of Signal Processing,” Dr. Li Deng of Microsoft Research at Redmond first observes that the traditional scope of signal processing as defined in the SPS constitution includes the ”signal” classes of audio, video, speech, image, communication, musical, and ”others”, and includes the ”processing” classes of filtering, coding, transmitting, estimating, detecting, analyzing, recognizing, synthesizing, recording, and reproducing. He argues that in our modern information society, we immerse ourselves with the signal processing techniques and applications that go far beyond the above scope. In the presentation, he constructs a ”matrix” which succinctly represents the traditionally defined scope of signal processing and uses this matrix representation to argue for natural expansion of the signal processing scope. In particular, he advocates the extension of the ”signal” coverage from typical numerical type to symbolic type such as text and documents, and for extension of the ”processing” class from recognition to understanding. He then present a case study which demonstrates principled ways in which the commonly used speech recognition techniques are naturally extended to handle the more challenging problem of speech understanding (with new, problem-specific processing steps added in an integrative manner).

The presentation of “Speech Recognition and Machine Translation: A Comparative Overview” given by Dr. Xiaodong He of Microsoft Research, Redmond summarizes substantial progress made over the last decade in both research and real-world applications of speech recognition and machine translation. Despite conspicuous differences, many problems in speech recognition and in machine translation share a wide range of similarities, and it is of great interests to see techniques in these two fields can be successfully cross-fertilized. In this talk, he discusses the similarities and difference between speech recognition and machine translation. As case studies, three specific technologies that have been successfully applied to both fields are discussed in details: hidden Markov model, template based modeling and decoding, and system combination. Through these examples, he compares the properties of speech and language, and shows how generic sequential pattern recognition technologies could be extended and applied to address the particular needs of speech recognition and machine translation.

In the audio processing area, Prof. George Tzanetakis of University of Victoria gave an entertaining talk on “Computational Ethnomusicology - Expanding the reach of Music Information Retrieval to the musics of the world.” Music Information Retrieval (MIR) is a relatively new research area in multimedia. MIR deals with the analysis and retrieval of music in digital form. It reflects the tremendous recent growth of music-related data digitally available and the consequent need to search within it to retrieve music and musical information efficiently and effectively. Most of existing work in MIR has focused on western classical and popular music as these types of music have the largest commercial interest. In this talk Dr. Tzanetakis describes two case studies in Computational Ethnomusicology that explores how MIR techniques can be applied to the study of non-Western music for which there is no standardized written reference (which is a large percentage of the music of world if not of album sales). The first case study is an automatic analysis of micro-timing in complex Afro-Cuban percussion music using rotation-aware dynamic programming. The second case study is a content and context aware web visualization interface for the study of religious chant. In addition to the technical challenges these projects presented he also discusses the social challenges of Interdisciplinary collaborations between engineering and humanities.

The second interesting talk in the area of audio processing is titled “Machine Hearing - A Research Agenda and an Approach,” by Richard F. Lyon of Google Inc. at Mountain View. He points out that the field of machine hearing is still in its infancy, in comparison with the thriving field of machine vision. This unfortunate situation, combined with the availability of good front-end auditory models, provides us the opportunity to make quick progress by leveraging techniques from machine vision to help make progress in research and applications in machine hearing. His project at Google aims to help machine hearing become a first-class academic and commercial field. His group developed applications that will do something useful with all that uninterpretable audio media out there, such as sound tracks of amateur movies. There are three main tactics that help us: (1) Leveraging techniques already developed in the machine-vision and machine-learning fields; (2) Productive interaction with the wider field of hearing research, to keep models honest and motivate better experiments; (3) Focus on applications for which the challenge has to do with what things sound like, as opposed to specialized domain knowledge (”non-speech non-music audio”). He presented some results showing how very-high-dimensionality feature spaces can effectively connect auditory representations to simple but powerful machine learning techniques for a range of applications such as sound ranking from text queries.

The third audio-processing talk, titled “Acoustic Scenes, Complex Modulations, and a New Form of Filtering,” is by Prof. Les Atlas of University of Washington. Be it in a restaurant or other reverberant and noisy environment, normal hearing listeners segregate multiple sources, usually strongly overlapping in frequency, well beyond capabilities expected by current computational approaches. What is it that we can learn from this common observation? As is now commonly accepted, the differing dynamical modulation patterns of the sources are key to these powers of separation. But until recently, the theoretical underpinnings for the notion of dynamical modulation patterns have been lacking. He has taken a decades-old and loosely defined concept, called ”modulation frequency analysis,” and developed a theory which allows for distortion-free separation (filtering) of multiple sound sources with differing dynamics. A key result is that previous assumptions of non-negative and real modulation are not sufficient and, instead, coherent and sparse separation approaches are needed to separate different modulation patterns. These results may have an impact in separation and representation of multiple simultaneous sound streams for speech, audio, hearing loss treatment, and underwater acoustic applications. This research also suggests exciting new and potentially important open theoretical questions for general signal representations, extending beyond acoustic applications and potentially impacting other areas of engineering and physics. The final talk in the area of audio/speech processing is titled "Deep-structured learning in speech processing" by Dr. Dong Yu of Microsoft Research, Redmond. In this talk he reports recent investigations on ways to learn complexy sequential decision boundaries in speech processing by composing multiple-layers of simple learners. He shows that this hierarchical structure allows one to learn and use long-range dependencies hidden in the signals and use features that cannot be easily incorporated in the hidden Markov model.

There are a large number of image or video processing related presentations. The first one is "Material classification using visible and near-infrared images," by Prof. Sabine Ssstrunk of EPFL. Recently, the presenter's research group have shown the advantages of simultaneously capturing visible and near infrared (NIR) radiation in digital photography applications, such as white balancing, shadow detection, dehazing, and face rendering. In this talk, she presented the on-going research using NIR images in conjunction with visible images for material classification. As many colorants are transparent to NIR, it is possible to reproduce the intrinsic lightness and texture characteristics of a material. The researchers are thus currently analyzing visible and NIR images according to their lightness and texture. The results are the input of a classifier in the form of feature vectors, and the probability of that data belong to a material category is then calculated. They achieve good classification results on a limited set of material classes.

The next presentation is "Visualizing and Understanding Challenges in the Design of Light Field Displays," by Dr. Amir Said of Hewlett Packard. While attempts to recreate three-dimensional views are nearly as old as photography, no solution has been able to generate consistent interest and wide acceptance of their quality. New analysis techniques are presented that can easily and more naturally show why the problem is not impossible, but can be indeed very challenging. Sequences of display simulations are shown, which can provide a much more intuitive appreciation of the difficulties, and facilitate understanding how design limitations impact visual quality.

Dr. Jeffrey Bloom of Dialogic Research Inc. presented the next talk of "Understudied Constraints Imposed by Watermarking Applications." As video watermarking becomes more mature and more widely known and accepted, a number of security and non-security applications are emerging. When watermarked content is traveling through a network or series of networks, there is a need to embed and/or detect watermarks at various points in the distribution chain. Traditional watermarking research concentrated only on the input and output of the network. This leaves a number of scenarios understudied. We will discuss two such scenarios: embedding in an entropy-encoded bitstream and detection in a compressed domain that differs from the embedding domain due to transcoding.

The final series of presentations are on multimedia security, starting with Prof. Edward J. Delp (Purdue University) talk on "Multimedia Security: A Viewpoint from a Walking Wounded." This talk describes current research issues in multimedia security involving data hiding, device forensics, biometrics, DRM, and authentication. He ”predicts” the future as to where this is all going. This talk also presents a brief overview of the research done in the Video and Image Processing Laboratory at Purdue University. Projects described include video compression, media indexing, multimedia security, language translation, mobile applications, and medical imaging.

The next talk in this series is "Information Management and Security in Media-Sharing Social Networks" by Professors Mehrdad Fatourechi and Jan Wang of University of British Colombia and Prof. Hong Zhao of University of Alberta. Digital media has profoundly changed our daily life during the last decade. For example, the wide adoption of broadband residential access and recent advances in video compression technologies has fueled increasing popularity in delivery of TV services via Internet. We have also witnessed the emergence of large-scale multimedia social network communities such as Facebook and YouTube. This proliferation of digital multimedia data creates a technological revolution to the entertainment and media industries and introduces the new concept of web-based social networking communities. However, the massive production and use of digital media also pose new challenges to the scalable and reliable sharing of multimedia over large and heterogeneous networks, demand effective management of enormous amount of unstructured media objects that users create, share, distribute, link and reuse, and raise critical issues of protecting intellectual property of digital media data. The proper management and protection of digital multimedia at such an unprecedented scale are beyond the capability of current technologies and demand new solutions.

This collaborative research effort between University of British Columbia and University of Alberta tackles the emergent technical challenges (e.g. information management and content protection) in large-scale media social networks. The aim is to establish a multimedia management and security framework to provide effective management, secure and reliable sharing of digital media in large-scale social networks. In particular, this talk addresses a summary of our recent research efforts in the following areas: content-based fingerprinting for media indexing and content recognition; understanding and analyzing the impact of human factors on multimedia systems; and building an automated network-service monitoring paradigm.

Dr. Darko Kirovski of Microsoft Research, Redmond gave the next talk of "Realizing the Uniqueness of Optical Media." When a DVD is stamped it is physically unique. He proposes a scheme to detect and deploy this uniqueness for the benefit of Digital Rights Management.

The next talk of "Connectivity and Security in Directional Multimedia Sensor Networks" by Prof. Deepa Kundur of Texas A&M University discussed the recent increased interest in the development of untethered sensor nodes that communicate directionally via directional radio frequency (RF) or free space optical (FSO) communications. Directional wireless sensor networks, such as the original Smart Dust proposal that employs broad-beamed FSO communications have the potential to provide gigabits per second speeds for relatively low power consumption suitable for multimedia sensing systems. Two significant challenges shared by the class of directional networks are connectivity and routing security, especially for random deployments. In this talk, two issues are addressed: 1) the feasibility of employing directional communications paradigms in large-scale security-aware broadband randomly and rapidly deployed static multimedia sensor networks; 2) the implications of link directionality to network connectivity and secure ad hoc multihop routing and highlight approaches in network design to mitigate compromising between the two.

The talk of "Bounds on Biometric Security" by Dr. Ton Kalker of Hewlett Parkard gives an overview of some recent work on trade-offs between the capacity and security of biometric systems. He shows that it is possible to formulate bounds for a number of cases, and that some of the classical schemes (for example fuzzy commitment) are sub-optimal. He also sketches many open questions. A highly entertaining presentation, entitled "Cognitive Sensors Networks: The New Frontier for DSP," was given by Prof. Magdy Bayoumi of University of Louisiana at Lafayette. Computers, communication, and sensing technologies are converging to change the way we live, interact, and conduct business. Wireless sensor networks reflect such convergence. These networks are based on collaborative efforts of a large number of sensor nodes. They should be low-cost, low-power, and multifunction. These nodes have the capabilities of sensing, data processing, and communicating. Sensor networks have a wide range of applications, from monitoring industrial facilities to control and management of energy applications, to military and security fields. Because of the special features of these networks, new network technologies are needed for cost effective,low power, and reliable communication. These network protocols and architectures should take into consideration the special features of sensor networks such as: the large number of nodes, their failure rate, limited power, high density, etc. Moreover, applications and impact of Sensors Networks are going to a higher and wider levels through the development of cognitive capabilities of these networks. Cognitive Sensors Networks, CSN, represent a transformational impact on technologies, applications, and expectations. In this talk the impact of wireless sensor networks will be addressed, several of the design and communication issues will be discussed, and a case study of a current project of using such networks in drilling and management off-shore oil and natural gas in the gulf region are given. The main criteria, expectations, and objectives of CSN are also highlighted

One special presentation was made on the topic of "The H- INDEX," by Prof. Yucel Altunbasak of Georgia Tech, with open disucssions. A while ago, with the help of some Ph.D. students, the presenter compiled H-indices for image processing researchers. He received several feedback about publishing this list. The most common feedback were: 1) How do you define an image processor? 2) How do you resolve the different H-index characteristics of image processing, computer Vision, and computer-science oriented people? 3) ISI-based and Google-based H-index numbers are vastly different. Which one do you use?, and most important of all, 4) what is the use of publishing such a list? Would it benefit the society? Further feedback from the society is still needed regarding 1) if publishing such a list (online) would benefit the society or be harmful because of possible misinterpretation (esp. by students), and 2) how best to do this project.

Outcome of the Meeting

The workshop II follows the workshop I over 4 years ago, and provides a timely and updated cross-disciplinary bridge among the relatively new area of multimedia, the well-established discipline of mathematics, and the emerging area of machine learning that heavily depends on mathematics. For many researchers in a specific area of multimedia, the workshop provided an excellent opportunity to broaden their perspective, exceeding the level achieved 4 years ago. A series of the workshop's high-quality presentations made clear the surprisingly similar mathematical and machine-learning approaches applied to speech, audio, image, and video-processing research. The presentations and intensive discussions enabled participants to examine the variety of approaches in different media areas, an invaluable opportunity made possible by the mixed forma and linformal styles of the workshop.

We would like to have BIRS to continue sponsoring cross-disciplinary workshops such as the series of two that we organized. Cross-disciplinary research sharing similar mathematical approaches, which now expand significantly to machine learning approachesm, stands to benefit the most from such workshops. The different branches of media processing research make it impossible to gain expertise in every sub-area, and our BIRS workshops helped immeasurably to foster an awareness of new trends in the various sub-disciplines. This is particularly important to some industrial researchers whose work has a relatively short-term scope. This was the case 4 years ago, and not is still the case although to somewhat less extent.

Most researchers in multimedia cannot afford the time-consuming process of mastering the subtleties of all the multimedia processing techniques. Our BIRS workshop provided an ideal opportunity to make close connections among them and to deepen our understanding of problem areas. The workshop succeeded in its aim to bring mathematicians, machine learning researchers. engineers, and scientists to interact and get exposed to each others ideas and advances in these disciplines. As different multimedia technologies have evolved and continue to evolve at a very rapid rate, the exact definition of multimedia remains illusive, even though multimedia technologies are now being widely deployed in industries in a multitude of applications.

The cross-fertilization among the different disciplines, academics and practitioners, engineers and mathematicians encouraged by the workshop was very useful in exposing the different communities to a new range of challenging and timely technical advances, the underlying mathematical problems and applications, and implementation challenges. This workshop II advances what was achieved by Workshop I by expanding the mathematical approaches to include the most relevant machine learning aspects. This expansion is particularly beneficial for multimedia research because of its cross-disciplinary nature and because of the intricacy on many of its sub-areas.

List of Participants

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Chapter 12

Multiscale Analysis of Self-Organization in Biology (09w5070)

Jul 12 - Jul 17, 2009

Organizer(s): Benoit Perthame (Laboratoire J. L. Lions, Université Pierre et Marie Curie)
Thomas Hillen (University of Alberta)

We have planed our workshop with the main goal to favorize discussions and collaborative researches. In practice we have left much free time available and avoided too long session swith many talks. This has led to the following particularities in the workshop organization

- Two short courses have been asked to A. Friedman and M. Ward (the video of this course is available on-line on the BIRS website)
- Poster sessions have been organized so that everybody can present his research work (and forteen posters were presented)
- Only six talks per day have been programmed.

Overview of the Field

The mathematical modelling of biological systems has rapidly grown over the past decades. Positions in mathematical biology are announced in many Universities and relevant contributions are reported in the highest international journals. Most of the research is done on a model-computation-result and prediction basis. There are, however, very interesting mathematical problems related to these biological models. In this workshop we want to focus on the mathematical and analytical side of modelling, where we particularly focus on the use of integro-differential equations and partial differential equations for multi scale analysis of self organization in Biology. Here questions on finite-time blow-up, global existence, pattern formation, regularity and homogenization play an important role. Some of the models discussed here are brand new and their mathematical properties are basically unknown (for example integro differential equations).

1) Integro differential equations are used to model development and evolution. There has not been much analysis of integro-differential equations. Numerically, these models show interesting pattern formations, such as emerging pattern and pulse splitting. Such numerical observations motivate questions about the underlying instabilities, the bifurcational structure and the relevance of the distribution kernel of the integral operator. Many results on travelling waves are available.

2) Our second topic relates to multiscale analysis and homogenization. This is particularly relevant if microscopic details of individual cells, such a cell motors, are used to design models for cell and population movement. In poarticular the problem of cell locomation based on the underlying biochemical networks is wide open. Also homogenization techniques are widely used in the domain to infer multiagent behaviors.

3) models for cross diffusion show a very rich menu of spatial pattern formation. These range from finite-time blow-up, over Turing-like patterns to merging and emerging patterns. While Turing patterns are well described, the other phenomena are not quite so well understood. New mathematics is needed to follow a solution after blow-up, and new methods are needed to properly understand merging and emerging dynamics.

Recent Developments and Open Problems

The field can be organized in overlaping themes that all contribute to give a global understanding of model behavior and thus to assert the reliability of the description of the underlying biological processes: Modeling, Analysis of PDEs, Asymptotic methods, Numerics.

Modeling

The workshop has asserted the numerous interactions between mathematicians and biologist in the area of the workshop. To quote only a few, let us mention
• Tumor growth and tissue mechanics. This topics was at the heart of A. Friedman's course on free boundary problems in mathematical biology and a mechanical view, based on a mixture theory approach has been presented in the talk by A. Tosin, see also [25].
• Evolution biology (for instance microscale movement and the evolution of dispersal)
• Biofilms and spatially structured microbial depositions on surfaces (this is a wide subject with several important applications and several possible mathematical approaches). See [12]
• Motor proteins, Kinesin-Microtubule Interactions, oriented transport along filaments, cytoskeleton dynamics (this is a very important biophysical subject where modeling has been progressing very strongly in the last years and which is now ready for new and challenging mathematical analysis)
• Auto-organization of cell communities (Group Dynamics in Phototaxis see [15], From individual to collective behaviour of cells and animals see [5])
• Biological invasions as described in the talk of Mark Lewis
• Role of noise.

Analysis of PDEs

This was a central subject in the workshop. Only to quote a few examples we can mention
• The analysis of the Keller-Segel system and several extensions proposed recently (see [26, 18]). This nonlinear Fokker-Planck system poses difficult mathematical questions because finite blow-up can happen and thus critical spaces and critical nonlinearities occur. These are extremely challenging questons in parabolic PDEs on which an important international community is working.
• Aspects of bistability have been widely disccused. How do that arise from elementary Fisher equations was described in the talk of R. Cantrell; it turns out that boundary conditions can explain it.
• Traveling waves are a central subject (ecological invasion for instance).
• Regularity: global existence vs singularity appearance. This arises for parabolic equations as the afore mentioned Keller-Segel system or the biofilm models [12], but also in kinetic equations [2], in free boundary problems [13, 14].
• Pattern formation and bifurcation analysis

Asymptotic methods

The biological modeling often leads to introduce small parameters in the models. These can come from time scales (for instance jump time in bacterial movement vs organization time of the colony) or space scales (leading often to small diffusion coefficients).

The workshop has shown an important interest in the use of methods from homogenization and in new questions. We can give two examples. Models for multicellular organisms are derived by homogenizing the equations for the exchanges for a single cell in the talk by A. Marciniak (see [21]). It is also a way to explain the oriented motion of molecular motors (talk of P.E. Souganidis and [24]).

Numerics

Numerics is a fundamental tool to communicate the outcome of a mathematical model and this was clear in most of the talks with applications. It is also a way to vizualize and better understand the meaning of mathematical results (as those numerical solutions presented by M. Ward). Finally, the computation of these singular phenomena arising in PDEs we have encountered are challenging in terms of mathematical algorithms (for instance optimal transportation can be used for computing the Keller-Segel system up to its blow-up as in [8]).

Presentation Highlights

Most remarkable are the two courses of three lectures given during this workshop by A. Friedman and M. Ward. We present them here together with the abstracts of the talks.

Course by A. Friedman

The three lectures of A. Friedmann have treated of 'Free boundary problems in mathematical biology'. He gave a brief overview of some general free boundary problems, such as variational inequalities and Hele-Shaw problems. Then he focussed on mathematical models of tumor growth, wound healing, cartilage growth, etc. For some of the models he could describe the shape of the free boundary, bifurcation of the free boundary, and asymptotic stability results. Open problems were also described. See [13, 14].

Course by M. Ward

The three videos of M. Ward's courses on the topics 'Traps, Patches, Spots, and Stripes: An Asymptotic Analysis of Localized Solutions to Some Diffusive and Reaction-Diffusion'. These are based on his recent works (see for instance [7, 19, 20] and the references therein). The content was 'A survey of the development and application of singular perturbation methods to treat a variety of both linear and nonlinear PDE models of diffusion and reaction-diffusion type with localized solutions is presented. Many of the problems considered have certain key common elements, notably related to the Neumann Green's function and the reduced-wave Green's function, and their regular parts. We highlight some of these key elements, and suggest some open problems and possible further directions.

In the first lecture we focus on three different linear diffusive problems; the narrow escape problem for diffusion from within a sphere to small traps on its boundary, the analysis of free diffusion on the boundary of a sphere with small traps, and the determination of the persistence threshold for the diffusive logistic model in a highly patchy spatial environment. For the first two problems we derive two discrete variational problems, related to classical Fekete points, that are central to determining the mean first passage time.

In the second lecture, we study localized spot-type solutions to certain non-variational reaction-diffusion systems, notably the Gray-Scott and Schnakenburg models, in a two-dimensional spatial domain. The dynamics of spots will be determined, and three ifferent (but generic) types of instabilities for spot-patterms will be discussed and analyzed: spot self-replication, spot-annihilation, and spot oscillations. Phase diagrams indicating parameter regimes where these instabilities occur will be constructed. The asymptotic analysis to construct quasi-equilibrium spot patterns is shown to be rather similar to that used to treat the linear diffusive problems in the first lecture.

In the third lecture, we highlight some results for the analysis of localized stripe solutions to some reaction-diffusion systems in planar domains. In many instances a stripe or ring pattern is unstable to a breakup instability, which leads to the disintegration of the stripe or ring into a sequence of spots. In other cases, a stripe is de-stabilized by a transverse or zigzag instability, leading to a wriggled stripe. In certain cases, this wriggled stripe is the precursor to a complicated space-filling labyritnhian pattern. Our analysis of stripe stability involves a combination of singular perturbation theory, the spectral theory of nonlocal eigenvalue problems, and numerical eigenvalue computations.'

Talk by P. Bates

Kinesin-Microtubule Interactions: Transport and Spindle Formation

This talk consists of two parts: Pattern formation in families of microtubules under the action of kinesin and the detailed motion of kinesin along a microtubule.

Microtubules are long cylindrical structures (lengths being tens of microns and diameter approximately 25 nm) comprised of tubulin dimers, which self-assemble, 13 protofilaments being required side-to-side to form the circular cross section. In the first set of results, microtubules are represented as stiff, polar rods which are subject to diffusion in position and orientation and also subject to pair-wise interaction, mediated by kinesin molecular motors. The concentration of kinesin is represented by a parameter that feeds into the probability of an interaction occurring when two microtubules collide. The probability of an interaction also depends on the location of the collision point along the lengths of the microtubules, because kinesin accumulates at the positive end of each microtubule. With collision rules in place, Monte-Carlo simulations for large numbers of freely moving microtubules are performed, adjusting parameters for concentration of kinesin and polarity of the microtubules. From these studies, a phase diagram is produced, indicating thresholds for phase change to occur. Simulation results are compared to those from in vitro experiments.

The second part of the talk involves modeling the fine scale dynamics of a kinesin motor as it walks along a microtubule. The two heads of the kinesin molecule alternately bind and unbind to the microtubule with certain mechanisms providing a directional bias to the Brownian motion expected. One bias is the shape of the head and the shape of the binding site, along with the companion electrostatic charges. The second bias is that, utilizing ATP capture and transferal of phosphors for energy, part of the polymeric leg (neck-linker) of the bound head becomes attached towards the front of that head (the ”zipped” state). The trailing head detaches from the microtubule. It then becomes subject to the biased entropic force due to the zipped state of the leading head and also preferentially (because of shape orientation) attaches in front of the currently attached head at which time ADP is released and a conformational change occurs, strengthening the binding. This motion is modeled using stochastic a differential equation. Simulations are performed with different lengths of neck-linkers and the mean speeds of progression obtained. These are compared with experimental results. (with Zhiyuan Jia)

Talk by N. Bournaveas

Kinetic models of chemotaxis

Chemotaxis is the directed motion of cells towards higher concentrations of chemoattractants. At the microscopic level it is modeled by a nonlinear kinetic transport equation with a quadratic nonlinearity. We’ll discuss global existence results obtained using dispersion and Strichartz estimates, as well as some blow up results. (joint work with Vincent Calvez, Susana Gutierrez and Benoit Perthame).

Talk by S. Cantrell

How biased density dependent movement of a species at the boundary of a habitat patch may mediate its within-patch dynamics.

In this talk we will discuss some reaction-diffusion models for the propagation of a species density in a bounded habitat. The particular models we will consider are of diffusive logistic type in the interior of the patch, subject to a nonlinear condition on the boundary of the patch of the form

$$a(u) * grad(u).n + (1a(u)) * u = 0.$$

Here a(u) is a non-decreasing nonnegative function of the species density that takes values between 0 and 1 when u is between 0 and the local carrying capacity of the species under the logistic growth law, which is presumed to be constant on the patch. When a(u) is identically constant, the prediction of the model is that all nonnegative nontrivial initial species density profiles evolve to 0 in the case of extinction or to a unique positive equilibrium profile in the case of survival. By way of contrast, in the case when a(u) is non-constant, the dynamics at the scale of the patch may be more complicated. In particular, such a(u) may mediate Allee effects at the scale of the patch, consistent with empirical results for the Glanville fritillary butterfly. The models demonstrate how meso-scale effects locally at the boundary of a habitat patch may mediate macro-scale effects on the patch as a whole.

This work is joint with Chris Cosner and Salome Martinez.

Talk by J.A. Carrillo

Some kinetic models in swarming

I will present a kinetic theory for swarming systems of interacting, self-propelled discrete particles. Starting from the particle model [11], one can construct solutions to a kinetic equation for the single particle probability distribution function using distances between measures [10].

Moreover, I will introduce related macroscopic hydrodynamic equations. General solutions include flocks of constant density and fixed velocity and other non-trivial morphologies such as compactly supported rotating mills. The kinetic theory approach leads us to the identification of macroscopic structures otherwise not recognized as solutions of the hydrodynamic equations, such as double mills of two superimposed flows.

I will also present and analyse the asymptotic behavior of solutions of the continuous kinetic version of flocking by Cucker and Smale [9], which describes the collective behavior of an ensemble of organisms, animals or devices. This kinetic version introduced in [16] is obtained from a particle model. The large-time behavior of the distribution in phase space is subsequently studied by means of particle approximations and a stability property in distances between measures. A continuous analogue of the theorems of [9] will be shown to hold for the solutions on the kinetic model. More precisely, the solutions concentrate exponentially fast their velocity to their mean while in space they will converge towards a translational flocking solution.

The presentation is based in works in collaboration [4, 5, 6].

Talk by C. Cosner

Microscale movement and the evolution of dispersal

The dispersal of organisms is clearly a significant aspect of many ecological processes, but the evolution of dispersal is still not well understood. In the setting of reaction-advection-diffusion models and their discrete analogues there is evidence that in spatially variable but temporally constant environments the dispersal strategies that are evolutionarily stable are those that allow populations to distribute themselves to match the distribution of their resources. Such strategies produce population distributions where fitness is zero everywhere (since all resources are used) and there is no net movement at equilibrium. Those features characterize populations that are distributed according to the ideal free distribution, where each individual locates itself to maximize its fitness. Whether or not a diffusion process derived from a simple random walk can support such an ideal free dispersal strategy depends on microscale assumptions about local movement probabilities. Classical physical diffusion as described by Ficks law cannot support such strategies without additional advection terms, and in fact if dispersal strategies are restricted to classical diffusion there is selection for lower diffusion rates. However, changing the assumptions about microscale movement can lead to diffusion models that can support some type of ideal free dispersal. At the mesoscale, adding advection to classical diffusion can achieve similar results. This talk will describe these ideas and some of their implications.

Talk by H.J. Eberl

van Leeuwenhoek's and Hilbert's Microscopes: A spatially structured model of biofouling

When studying microbial population and resource dynamics, mathematical biologists and experimental microbiologists have traditionally focused on suspended bacterial populations. In fact, the well-developed theory of the chemostat can be considered one of the biggest success stories in Mathematical Biology. However, it is becoming more and more accepted now that most bacterial populations live in fact as spatially structured microbial depositions on surfaces, usually in aqueous environments. These biofilms play beneficial roles in environmental processes (pollution degradation), and detrimental roles in industrial (biofouling, biocorrosion) and medical contexts (bacterial infections, health risks). In the past decade a variety of mathematical models of these biofilms have been proposed, focusing on different aspects and time-scales of biofilm processes and utilizing a variety of mathematical model concepts (ranging from individual based models to cellular automata and models of continuum mechanics). We discuss in some detail a density-dependent diffusion-reaction model for population and resource dynamics in a single-species/single-substrate biofilm and show in some examples how this modeling concept can be applied to more involved biofilm processes. This model is a meso-scopic model of spatial organisation; we will also comment on but not present solutions to multi-scale challenges in biofilm modeling.

Talk by R. Erban

From individual to collective behaviour of cells and animals

In this talk, we focus on two model systems: flagellated bacteria and locust nymphs. In both cases, the individual behaviour can be described as a biased random walk, although the nature of the bias and the corresponding mathematical models differ. We present methods for inferring collective properties from the individual-based models.

Flagellated bacteria are modelled as the velocity jump process with internal dynamics. We show that this framework can be used for relating the coefficients of macroscopic partial differential equations (which describe the evolution of the density of cells) to parameters of the intracellular signal transduction mechanism. Moreover, we also show that the velocity jump process with (metabolic) internal variables can be used to study travelling waves in the density of cells.

Locusts are modelled using a modified self-propelled particle model. Systematic analysis of the experimental data reveals that individual locusts appear to increase the randomness of their movements in response to a loss of alignment by the group. We show how properties of individual animal behaviour can be implemented in the self-propelled particle model to replicate the group-level dynamics seen in the experimental data.

Talk by B. Kawohl

Convex sets of constant width, or why geometry can be of vital importance.

When does a steel pipe have an exactly circular cross section? When it features constant exterior width from each angle? That could easily by verified with a big caliper or slide gauge, and this what used to happen in the process of assembling booster rockets for the space shuttle. The authors of the corresponding manuals had overlooked that there are geometric shapes, so-called sets of constant width, that are not circles. This was a contributing factor to the Challenger disaster in 1986. In my talk I will point out that these odd sets show up in our daily life, and that there are interesting mathematical questions connected with them. The talk is directed at a general audience.

Talk by D. Kinderlehrer

Aspects of modeling transport in small systems with a look at motor proteins

Motion in small live systems has many challenges. Prominent environmental conditions are high viscosity and warmth. It is difficult to move and maintaining a course is compromised by immersion in a highly fluctuating bath. We discuss some possibilities for motor proteins, which transduce chemical energy into directed mechanical energy. Such nanoscale motors, like conventional kinesin, have a role in intracellular transport, separating the mitotic spindle, and many other cellular functions. Our approach is to formulate a dissipation principle connected to the Monge-Kantorovich mass transfer problem. We show how this leads to a system of evolution equations. We then discuss how various elements of the system must be related in order that transport actually occur. Finally, what opportunities do these ideas offer? We examine some 'hybrid variational problems' and discuss unresolved issues.

Talk by P. Laurençot

Global existence and blowup for the parabolic-elliptic Keller-Segel system with nonlinear diffusion

Whether solutions to the parabolic-elliptic Keller-Segel system with nonlinear diffusion are global or blow up in finite time is investigated in one space dimension and in several space dimensions for radially symmetric initial data. The study mainly relies on an alternative formulation of the problem and virial identities.

Talk by D. Levy

Group Dynamics in Phototaxis

Microbes live in environments that are often limiting for growth. They have evolved sophisticated mechanisms to sense changes in environmental parameters such as light and nutrients, after which they swim or crawl into optimal conditions. This phenomenon is known as "chemotaxis" or "phototaxis." Using time-lapse video microscopy we have monitored the movement of phototactic bacteria, i.e., bacteria that move towards light. These movies suggest that single cells are able to move directionally but at the same time, the group dynamics is equally important. Following these observations, in this talk we will present a hierarchy of mathematical models for phototaxis: a stochastic model, an interacting particle system, and a system of PDEs. We will discuss the models, their simulations, and our theorems that show how the system of PDEs can be considered as the limit dynamics of the particle system. Time-permitting, we will overview our recent results on particle, kinetic, and fluid models for phototaxis.

This is a joint work with Devaki Bhaya (Department of Plant Biology, Carnegie Institute), Tiago Requeijo (Math, Stanford), and Seung-Yeal Ha (Seoul, Korea).

Talk by M. Lewis

Mathematical challenges in the modelling of biological invasions

Biological invaders are introduced locally, and then spread spatially into new environments, often impacting ecosystems. Models for invasions track the front of an expanding wave of population density. The underlying equations are often systems of parabolic partial differential equations and related integral formulations.

I will structure this talk around three challenges in the analysis of biological invasions where mathematical theory has provided new insight:

(i) Reid's paradox of rapid plant migration. How were trees were able to migrate very quickly behind retreating ice sheets after the last ice age?

(ii) Multispecies competition paradox. Why do classical mathematical methods, based on linearization, fail to predict the rate of competitive spread of one species into another?

(iii) Reid's paradox in multispecies communities. Pollen data indicates that secondary species can spread very quickly into regions already occupied by a close competitor. How can this spread occur so quickly?

Each of these challenges will be addressed using mathematical analysis to provide insight regarding the behaviour of the biological models. I will finish by suggesting some new mathematical challenges where biological invasion theory and mathematical models meet.

Talk by A. Marciniak-Czochra

Hysteresis-driven pattern formation in a developmental system

It is becoming increasingly clear that multistability plays an important role in cell signalling. Coupled with the diffusion process, it may give rise to spatial patterns in chemical and biological systems, such as

Liesegang rings formed by precipitating colloids and bacterial growth patterns. Processes containing switching between different pathways or states lead to new types of mathematical models, which consist of nonlinear partial differential equations of diffusion, transport and reactions, coupled with dynamical systems controlling the transitions. Diffusion tries to average different states and is the cause of spatio-temporal patterns. Based on these concepts we propose a model for pattern formation in a fresh-water polyp, hydra, a simple organism, which can be treated as a prototype for axis formation in higher organisms. The proposed model shows how the hysteresis in intracellular signalling may result in spatial patterning. In particular, it demonstrates that bistability in the dynamics of the growth factor controlling cell differentiation explains the experimental observations on the multiple head formation in hydra, which is not possible to describe using classical Turing-type models. Depending on the type of nonlinearity stationary and oscillatory patterns are found. The model is discussed in the context of recent experimental findings of the Wnt and Dkk overexpression during regeneration.

Talk by C. Schmeiser

Analysis and qualitative properties of a two-dimensional continuum model for cytoskeleton dynamics in the lamellipodium.

A recently developed continuum model for the dynamics of the actin cytoskeleton in lamellipodia will be presented. It is derived from a microscopic description of the bending, polymerization and depolymerization of individual cross-linked actin filaments taking into account substrate adhesion and mechanical effects of the leading edge. The model can be seen as a generalized gradient flow, however, equipped with a number of peculiarities like nonconvexity of the energy functional and of the manifold of admissible states, as well as energy gain and loss through (de)polymerization and the building and breaking of cross-links and adhesions. Aspects of the existence and numerical analysis and of qualitative properties of simplified model problems will be presented (joint work with D. Oelz and N. Sfakianakis).

Talk by A. Tosin

Tumor growth by a mixture theory approach: modeling and analytical issues

Resorting to the theory of deformable porous media, we address tumors as a mixture of abnormal and healthy cells within a porous extracellular matrix (ECM), which is wet by a physiological extracellular fluid. In the talk, we will focus mainly on the modeling of the mechanical interactions between a growing tumor and the host tissue, their influence on tumor growth, and the attachment/detachment mechanisms between cells and ECM. Then, by weakening the role of the extracellular matrix, we will derive a system of PDEs describing the evolution of the cell density coupled to the dynamics of some nutrient, e.g., oxygen, whose higher and lower concentration levels determine proliferation or death of cells, respectively, and we will briefly discuss some related analytical issues.

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Talk by D. Wrzosek

Chemotaxis models with volume filling effect and singular diffusion.

A quasilinear parabolic system of Keller-Segel type in which it is assumed that 1) there is a critical threshold value the density of cells cannot exceed and 2) the diffusion of cells becomes singular when the density approaches the threshold. The structure of the model includes recent models by Wang and Hillen (2007) with fast diffusion and that of Lushnikov (2008) with superdiffusion. It is proved that for some range of parameters describing the relation between the diffusive and the chemotactic part of a cell flux there are global-in-time classical solutions which in some cases are separated from the threshold uniformly in time. For the case of fast diffusion existence and uniqueness of global weak solutions and stationary solutions are studied. Applications of general results to particular models are shown.

Scientific Progress Made

Outcome of the Meeting

The vitality of the field of PDEs applied to biology and medicine has been demonstrated. Not only many interactions are beeing developed presently, but a wide variety of PDEs; of course parabolic equations are often central in the domain but other types of PDEs are used
• Kinetic equations as in the various microscopic descriptions of cell movement which are available nowadays (talk of N. Bournaveas and [2], talk of J. Carrillo and [5], work of T. Hillen and [17, 26])
• Geometrical motions and free boundary problems
• Fluid equations for tissue growth, hyperbolic equations for cell motion
• Gradient flows(talk of Ch. Schmeiser) and optimal transportation (talk of D. Kinderlehrer)

Also many mathematical challenges underly the full mathematical structure of many models
• From stochastic particle dynamics to macroscopic nonlinear equations (talks of P. Bates, D. Levy see also [15], R. Erban and [27])
• Blow-up, regularity in nonlinear parabolic equations (talks of P. Laurençot, D. Wrzosek and reference [18]) and regularity vs unstability of free boundary problems (course of A. Friedman).
• Asymptotic methods: spike dynamics and homogenization

Mark Lewis has pointed an important aspect of mathematical biology. The impact of mathemtical results can be much higher if formulated in such a way biologists can understand and use them. He gave striking examples in this direction.

List of Participants

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Carrillo, José Antonio (ICREA)
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Cuadrado, Silvia (Universitat Autonoma de Barcelona)
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Chapter 13

Permutation Groups (09w5030)

Jul 19 - Jul 24, 2009

Organizer(s): Robert Guralnick (Univeristy of Southern California) Katrin Tent (Universität Münster) Cheryl Praeger (University of Western Australia) Jan Saxl (University of Cambridge)

Overview of the Field

The theory of permutation groups is essentially the theory of symmetry for mathematical and physical systems. It therefore has major impact in diverse areas of mathematics. Twentieth-century permutation group theory focused on the theory of finite primitive permutation groups, and this theory continues to become deeper and more powerful as applications of the finite simple group classification, and group representation theory, lead to astonishingly complete classifications and asymptotic results.

Recent Developments and Open Problems

The theory of permutation groups is a classical area of algebra. It originates in the middle of the nineteenth century, with very considerable contributions by most of the major figures in algebra over the last two centuries, including Galois, Mathieu, Jordan, Frobenius, Burnside, Schur and Wielandt. In the last twenty years, the direction of the subject has changed substantially. The classification of finite simple groups has had many applications, many of these through thorough investigation of relevant permutation actions. This in turn led to invigoration of the subject of permutation groups, with interesting new questions arising and techniques developed for tackling them. Interestingly, some topics arose in more than one context, forming new connections. The concept of exceptionality was first suggested by work on covers of curves; it then appeared independently in homogeneous factorizations of graphs, and more recently it has found applications in investigations of line-transitive linear spaces. The concept of derangements in groups (that is, fixed-point-free permutations) and their proportions is classical; it has applications to images of rational points for maps between curves over finite fields, in probabilistic group theory and in investigating convergence rates of random walks on groups. Recently a conjecture of Boston and Shalev on the proportion of derangements in simple group actions has been settled; interestingly, this conjecture fails in the slightly more general case of almost simple groups, through examples of exceptional actions mentioned above. This area continues to be very lively. The topic of fixed point ratios and minimal degrees of elements in permutation groups is classical, going back over 100 years, but there has been significant progress in the last fifteen years both for finite and algebraic groups. It has had applications in arithmetic algebraic geometry, besides leading to significant insights in group theory – a striking example is the solution of Wielandt's conjecture on the characterization of subnormal subgroups. The question of base size of permutation actions is of importance in computational group theory as well as in the study of the graph isomorphism problem. Recent research has thrown much light on the base sizes of actions of almost simple groups in particular. The concept of quasiprimitive permutation groups is also classical, but there has been revived interest in the subject through investigations of groups of graphs and designs. Algebraic graph theory has developed greatly over the last ten to twenty years; there are interesting connections to association schemes and representation theory. Some of these in turn found an application in the study of derangements mentioned above, as well in the study of random walks on groups. Expander graphs is a fairly quickly developing field and there are connections to permutation groups here as well.

We now discuss some of these and related areas and recent developments in more detail.

Permutation groups of small genus.

This area has been motivated by the well-known conjecture of Guralnick and Thompson concerning composition factors of monodromy groups of covers of curves of small genus to the Riemann sphere. The proof of this conjecture has been recently completed, through the work of a number of authors. Work on obtaining explicit lists of composition factors in the very small genus cases continues. These methods and their refinements have led to a complete classification of the monodromy groups of maps from the generic curve of genus $g > 2$, greatly extending classical work of Zariski. There are related conjectures of Guralnick concerning the corresponding situation in positive characteristic, which will require considerable extension of existing methods.

Exceptionality of permutation groups.

The concept of exceptional permutation groups arose in the context of investigations of exceptional polynomials, which arose originally in the work of Dickson, Schur, Davenport, Fried and others. These are polynomials over finite fields which induce permutations on infinitely many finite extension fields. This can be translated into a question relating orbitals of a permutation group and its automorphism group. An answer led to major progress in our knowledge of exceptional polynomials in the work of Fried, Guralnick and Saxl. As a direct consequence, new families of exceptional polynomials were discovered by Lenstra, Zieve and others. Another application appeared in the recent memoir of Guralnick, Müller and Saxl on the rational function analogue of a question of Schur concerning polynomials with integer coefficients which induce permutations on residue fields for infinitely many primes. The solution involved a substantial amount of permutation group theory and algebraic geometry. At about the same time, exceptional permutation groups arose also in the work of Praeger and others on homogeneous factorizations of complete graphs. There is a further recent application in the study of line-transitive linear spaces.

Derangements.

According to a conjecture attributed to Boston and Shalev, there is an absolute lower bound for the proportion of derangements in any action of any simple permutation group. This has been proved recently in an impressive series of papers (and preprints) by Fulman and Guralnick. An important extension, still being investigated, is distribution of derangements in the cosets of the permutation group in its automorphism group. This is connected to the exceptionality condition above. One wants to classify primitive actions in whichmost elements in a coset are not derangements. This would yield information about rational maps and maps between curves over finite fields that are close to being bijective over for arbitrarily large finite fields.

Finite and infinite geometries.

A classification of finite projective planes with automorphism groups primitive on points was obtained by Kantor in the 80's, as a consequence of a classification of primitive permutation groups of odd degree. This was extended to a classification of flag transitive finite linear spaces through the work by Buekenhout and others. Much work has been done recently on line transitive finite linear spaces - there are some interesting problems concerning imprimitivity. Another extension currently under investigation concerns flag transitive finite polygons. One should note that the monumental recent work of Tits and Weiss on Moufang polygons, while concerned with the general problem, does not seem to simplify much in the finite case. However, recent work of Tent on BN-pairs and weak Moufang conditions generalizes many of the classification results to the infinite situation. At the same time, model theory provides techniques to construct (counter-) examples showing that certain results will not generalize to the infinite case.

In this context, also (infinite) split doubly transitive groups should be mentioned. Recent results by Segev, de Medts, Tent, and Weiss seem to make a classification feasible at least in the case of special Moufang sets. This work will also be relevant in the classification of simple groups of finite Morley rank (which again connects the topic to model theory).

Algebraic graph theory.

Successful modern applications of permutation groups in algebraic graph theory date from the late 1980's with proof of the Sims' Conjecture, breakthroughs in the classification of finite distance transitive graphs beginning with the reduction theorem of Praeger, Saxl and Yokoyama, and Weiss's non-existence proof of finite 8-arc transitive graphs of valency greater than 2. These involved use of the simple group classification and built on the theory of finite primitive permutation groups. More recent applications required Praeger's development of the theory of finite quasiprimitive permutation groups. This theory also relies heavily on the finite simple group classification, and has been used successfully to analyse even intransitive finite combinatorial structures such as the Giudici–Li–Praeger theory of locally s-arc transitive graphs. In addition the theory of amalgams and their universal completions forms an important link between infinite graphs and their automorphism groups on the one hand, and classification of finite graphs by their local properties. Much of the geometry associated with the finite simple groups has been elucidated from the study of group amalgams, noting in particular Ivanov's geometric characterisation of the Monster, part of the Ivanov–Spectorov classification of P-geometries and T-geometries. Combining the amalgam approach and the quasiprimitive graph approach is just beginning to pay significant dividends in our understanding of important classes of graphs and group actions. Returning to the finite distance transitive graphs,a related, a slightly more general problem concerns multiplicity free permutation actions. There has been substantial progress towards classification of these actions. Deep character theoretical information on some of these actions has been obtained by Lusztig, Henderson and others. The character tables of the corresponding association schemes have been obtained by Bannai and his coworkers. Some of these actions have been used by Diaconis and others to investigate random walks on groups.

Subgroup structure of finite simple groups.

Theory of primitive permutation groups is closely related to the subgroup structure of finite simple groups and their automorphism groups. There has been impressive progress in this area. For sporadic groups, the answer is almost complete. For alternating groups, the question of maximality was settled in the late eighties through the work of Liebeck, Praeger and Saxl on maximal factorizations of almost simple groups. This reduces the question to classification of maximal subgroups of smaller almost simple groups. For classical groups, Aschbacher's theorem focuses attention on modular representations of almost simple groups; there remain also some questions of non-maximality, currently under investigation. There is some beautiful recent work of Kleshchev and Tiep using very deep ideas in modular representation theory, Hecke algebras and quantum groups. There has been impressive progress in our understanding of subgroup structure of exceptional groups, through the work of Liebeck and Seitz. This is closely linked to subgroup structure of algebraic groups over algebraically closed fields in positive characteristic.

Understanding the maximal subgroups is just the first (but very important) part of understanding the lattice of subgroups. One basic question is a conjecture of Quillen on the contractibility of the subgroup lattice of p-subgroups (the conjecture is that this is the case if and only if there is a nontrivial normal p-subgroup). Another is the question of whether every finite lattice can be embedded in a subgroup lattice of a finite group. This has come up in logic and Banach space theory. There has been considerable progress through the work of Aschbacher, Shareshian, and others.

Infinite permutation groups and model theory

Over the last years there have also been major developments in infinite permutation group theory. One aspect here is the interaction between permutation group theory, combinatorics, model theory, and descriptive set theory, typically in the investigation of first order relational structures with rich automorphism groups. The connections between these fields are seen most clearly for permutation groups on countably infinite sets which are closed (in the topology of pointwise convergence) and oligomorphic (that is, have finitely many orbits on k-tuples for all k); these are exactly the automorphism groups of ω-categorical structures, that is, first order structures determined up to isomorphism (among countable structures) by their first order theory. Here it seems that many new classes of simple groups might be constructed as automorphism groups of such structures. This is particularly interesting when the structures resemble classical objects like projective planes over fields or the like.

Themes of current activity here include the following.

(a) The use of group theoretic means (O'Nan-Scott, Aschbacher's description of maximal subgroups of classical groups, representation theory) to obtain structural results for model-theoretically important classes (totally categorical structures, or much more generally, smoothly approximable structures, finite covers of well-understood structures).

(b) Enumeration and growth rates questions for certain integer sequences associated with oligomorphic groups (e.g. counting the number of orbits on ordered or unordered k-sets – combinatorially well-known sequences frequently arise).

(c) Reconstruction of a first order structure (up to isomorphism, up to having the same orbits on finite sequences, up to 'bi-interpretability') from its automorphism group (typically, presented as an abstract group. Partially successful techniques here include the description of subgroup of the automorphism group of countable index (the 'small index property'), and first order interpretation of the structure in its automorphism group.

(d) Properties which the full symmetric group S on a countable set shares with various other closed oligomorphic groups. We have in mind such properties as: complete description of the normal subgroup structure; uncountable cofinality (that is, the group is not the union of a countable chain of proper subgroups); existence of a conjugacy class which is dense in the automorphism group, or, better, comeagre (or better still, the condition of 'ample homogeneous generic automorphisms'); the Bergman property for a group (a recently investigated property of certain groups G, which state that if G is generated by a subset S, then there is a natural number n such that any element of G is expressible as a word of length at most n in $S \cup S^{-1}$); the small index property. Several of these themes have been linked in recent work of Kechris and Rosendahl motivated partly by descriptive set theory. A closely related issue here is the 'extension property' for a class C of finite relational structures, which stated that if $U \in C$ then U embeds in some $V \in C$ such that every partial isomorphism between substructuresof U extends to an automorphism of V; this condition, proved for graphs by Hrushovski, has connections to the topology on a free group, and to automata theory and issues on the theoretical computer science/finite model theory border.

Presentation Highlights

The quality of the participants at the conference and their presentations was quite remarkable. Speakers included Michael Aschbacher, Persi Diaconis and Alex Lubotzky. A number of participants reported that this was one of the best (in fact, the best) conferences they had ever attended.

Aschbacher and Shareshian reported on their joint work about sublattices of the subgroup lattice of a finite group. The basic (still open) question is whether every finite lattice can be embedded as a sublattice of the subgroup lattice of some finite group. This originally came up in the theory of Banach algebras. A natural problem in that context translates precisely to this problem. It is also relevant to problems in logic. It seems pretty clear that this is not the case (and indeed most finite lattices should not be embeddable in a subgroup lattice of a finite group), but even for fairly simple lattices, this cannot be shown yet. Aschbacher has developed a theory which reduces the problem to various questions about almost simple groups (not just about the sublattices of the simple groups but other properties). This is an amazing insight. Using the classification of finite simple groups, Aschbacher and Shareshian hope to settle the problem. A partial result now exists for alternating groups.

Lucchini gave a related talk about simplicial complex associated to the coset poset of classical groups.

Lubotzky reported on recent joint work with Guralnick, Kantor and Kassabov about presentations of finite simple groups. Given the nature of the conference, he focused on the case of alternating and symmetric groups. The main result is that if G is a simple Chevalley group of rank r over a field of size q, then G has a presentation with an absolutely bounded number of generators and relations with the length of the presentation $O(\log n + \log q)$ which is essentially best possible. There is one possible family of exceptions – there are no know bounded presentations of the groups ${}^2G_2(3^{2k+1}$ (it is not expected to be a counterexample). This result applies to alternating and symmetric groups by viewing them as Chevalley groups over the field of size 1 (as suggested by Tits). For symmetric and alternating groups, another result is that they have presentations with 3 generators and 7 relations (or 2 generators and 8 relations). It had not been widely believed that simple groups had bounded presentations. Another related result is getting bounds on second cohomology groups. Here the result is: Let G be a finite group and V an irreducible faithful G-module. Then

$$\dim H^2(G, V) \leq (18.5) \dim V.$$

It is likely the 18.5 can be further reduced to $1/2$. This answers a conjecture of Holt from about 15 years ago.

Liebeck talked about another classical problem. Which triangle groups surject onto groups of Lie type? Recall a triangle group of type (r, s, t) is the group generated by 3 elements x, y, z such that $xyz = 1$ and $x^r = y^s = z^t = 1$. These come up in geometry in many ways. The special case of $(r, s, t) = (2, 3, 7)$ are known as Hurwitz groups and come as automorphism groups of genus g Riemann surfaces with automorphism groups of maximal cardinality – $84(g - 1)$ with $g > 1$ (this cannot occur when $g = 2$, the smallest case is $g = 3$ – a surface discovered by Klein).

Diaconis gave a beautiful lecture about Gelfand pairs (these are subgroups of H of G such that the permuation module $\mathbb{C}_H^G$ is multiplicity free (equivalently the endomorphism ring of that module is commutative). There are analogs of these problems for Lie groups as well. This is a classic topic studied by quite a number of people. Recently, there has been considerable interest in another aspect of this – absolutely multiplicity free groups (i.e., every irreducible representation of G restricted to H is multiplicity free or dually every irreducible representation of H induced to G is multiplicity free).

Seitz talked about his recent work with Liebeck on the classification of unipotent conjugacy classes in simple algebraic groups and various applications. This is a topic of central impiortance, which has been widely studied over forty years. While there is extensive literature on the subject, the Liebeck-Seitz works really cleans up the subject. This work has applications to permutation groups – especially involving problems on fixed point ratios. He also gave some nice consequences – for example, showing that every unipotent class is rational in a semisimple algebraic group.

de Medts talked about a generalization of Tits method for proving certain groups are simple. He considered automorphism groups of locally finite trees.

Guiudici and Seress gave talks about classical problems in permutation group theory ($3/2$-transitive groups and orbit equivalent permutation groups).

Van Bon reported on his work with Stroth on locally finite, locally s-arc transitive graphs, and in particular their bound $s \leq 9$ for those graphs with all vertices of valency at least 3. It turns out happily, that an exceptional amalgam their work uncovered had been already addressed by construction of an infinite family of examples in the Giudici–Li–Praeger's global approach, further evidence of the confluence of these too approaches.

Peter Cameron, Peter Neumann and Ben Steinberg gave a trilogy of lectures about various aspects of the exciting new area of synchronising groups and their applications to the theory of finite automata, and to combinatorics.

Reichstein gave a talk about essential dimension and group theory. Essential dimension is a classical subject which has had a resurgence in the past decade (primarily through work of Reichstein).

Segev talked about the abelian root groups conjecture for special Moufang sets, another important area where huge amount of progress has been made recently. Tiep talked about recent work on the irreducibility of representations on subgroups. This is a very important part of the maximal subgroup problem. It involves deep representation as well as recent work on Hecke algebras.

Rob Wilson gave a talk on his new way of describing various of the twisted exceptional groups.

Scientific Progress and Outcome of the Meeting

The aim of this workshop was to bring together leading researchers in these related areas as well as those whose research centers on permutation groups. A very successful workshop at Oberwolfach was recently organized on this topic (August 2007) by the same organizers. The range of talks and the results discussed at both conferences were very impressive. The diversity of interests of the participants was rather remarkable. What was even more impressive was the range of discussions and collaborations started and continued during the conferences. It made clear the importance of having such meetings on a regular basis. There were also a fair number of postdocs, women and graduate students.

The Ree groups of type G_2 are the one open case in the theorem about bounded generation for the non-abelian finite simple groups (by Guralnick, Lubotzky, et al) that Alex Lubotzky spoke about in his lecture. Tom De Medts and Richard Weiss, had been working on this case for several months. They gained important new insights from Akos Seress (who has been working on the problem for many years). Even though the problem is still unsolved, it seems no longer out of reach, and the three are now collaborating on it.

De Medts began work with Richard Weiss on Moufang quadrangles of type E_8 mentioned during Weiss's talk. Even though de Medts was already well aware of the important question of trying to find an "exceptional" invariant algebraic structure in the spirit of the classical pseudo-quadratic spaces that arise from Moufang quadrangles of type E_6 and E_7, it was Weiss's lecture that inspired this collaboration. They believe they have indeed discovered a new invariant 32-dimensional algebraic structure, and are working hard to pin it down. This would be a significant step toward solving the larger problem.

Donna Testerman spent a lot of time on a 'book in preparation' with Gunter Malle. She also laid foundations for a joint project with Tim Burness on irreducible embeddings of algebraic groups, to be pursued during a visit by Burness in September.

Several questions that Reichstein asked about representations of p-groups were answered by Rob Wilson and Chris Parker.

Peter Neumann suggested an approach to the solution of a problem on torsion-free uniquely divisible groups raised by Yoav Segev; he was able to answer a question about the number of non-synchronizing degrees of a primitive permutation groups raised by Peter Cameron, and was stimulated by that question to formulate a very strong conjecture about synchronizing semigroups. Csaba Schneider provided solutions to two of questions of Peter Neumann about primitive groups of affine type as synchronizing groups.

We are pleased to report numerous comments made by participants that this was the best workshop they had ever attended. Quoting a participant, 'lively, constructive and informative public discussion [followed] nearly every lecture, and discussion continued long after the formal sessions were over'. This was partly due to the wonderful BIRS facilities. We thank the BIRS for the opportunity to hold such a wonderful meeting at their stunning research facility. Many participants also commented on the BIRS staff and how helpful they were. The organizers in particular are very grateful for all the help received. In particular, the administrator Brenda Williams is singled out for her help both before and during the meeting.

List of Participants

Aschbacher, Michael (California Institute of Technology)
Baginski, Paul (University of California, Berkeley)
Bamberg, John (Ghent University)
Blackburn, Simon (Royal Holloway, University of London)
Burness, Tim (University of Southampton)
Cameron, Peter (Queen Mary University of London)
Damian, Erika (University of East Anglia (UK))
De Medts, Tom (Ghent University)
Diaconis, Persi (Stanford University)

Giudici, Michael (University of Western Australia)
Guest, Simon (Baylor)
Guralnick, Robert (Univeristy of Southern California)
Kantor, William (University of Oregon)
Korchagina Capdeboscq, Inna (Warwick University)
Liebeck, Martin (Imperial College)
Lubotzky, Alex (Hebrew University of Jerusalem)
Lucchini, Andrea (University of Padova)
Lyons, Richard (Rutgers University)
Magaard, Kay (Birmingham)
Malle, Gunter (Universitaet Kaiserslautern)
Muehlherr, Bernhard (Université Libre de Bruxelles (ULB))
Neumann, Peter M (The Queen's College)
Parker, Christopher (University of Birmingham)
Praeger, Cheryl (University of Western Australia)
Pyber, Laci (Renyi Institute of Mathematics Budapest)
Reichstein, Zinovy (University of British Columbia)
Saxl, Jan (University of Cambridge)
Schneider, Csaba (University of Lisbon)
Segev, Yoav (Ben Gurion University)
Seitz, Gary (University of Oregon)
Seress, Akos (The Ohio State University)
Shareshian, John (Washington University)
Steinberg, Benjamin (Carleton University)
Tent, Katrin (Universität Münster)
Testerman, Donna (Ecole Polytechnique Federale de Lausanne)
Tiep, Pham Huu (University of Arizona)
Van Bon, John (Universita della Calabria)
Weiss, Richard (Tufts University)
Wilson, Rob (Queen Mary London)
Ziegler, Martin (Mathematisches Institut Freiburg)

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Chapter 14

Applications of Matroid Theory and Combinatorial Optimization to Information and Coding Theory (09w5103)

Aug 2 - Aug 7, 2009

Organizer(s): Navin Kashyap (Queens University), Pascal Vontobel (Hewlett-Packard Laboratories), Emina Soljanin (Alcatel-Lucent Bell-Labs)

The aim of this workshop was to bring together experts and students from pure and applied mathematics, computer science, and engineering, who are working on related problems in the areas of matroid theory, combinatorial optimization, coding theory, secret sharing, network coding, and information inequalities. The goal was to foster exchange of mathematical ideas and tools that can help tackle some of the open problems of central importance in coding theory, secret sharing, and network coding, and at the same time, to get pure mathematicians and computer scientists to be interested in the kind of problems that arise in these applied fields.

Introduction

Matroids are structures that abstract certain fundamental properties of dependence common to graphs and vector spaces. The theory of matroids has its origins in graph theory and linear algebra, and its most successful applications in the past have been in the areas of combinatorial optimization and network theory. Recently, however, there has been a flurry of new applications of this theory in the fields of information and coding theory.

It is only natural to expect matroid theory to have an influence on the theory of error-correcting codes, as matrices over finite fields are objects of fundamental importance in both these areas of mathematics. Indeed, as far back as 1976, Greene [7] (re-)derived the MacWilliams identities — which relate the Hamming weight enumerators of a linear code and its dual — as special cases of an identity for the Tutte polynomial of a matroid. However, aside from such use of tools from matroid theory to re-derive results in coding theory that had already been proved by other means, each field has had surprisingly little impact on the other, until very recently.

Matroid-theoretic methods are now starting to play an important role in the understanding of decoding algorithms for error-correcting codes. In a parallel and largely unrelated development, ideas from matroid theory are also finding other novel applications within the broader realm of information theory. Specifically, they are being applied to explore the fundamental limits of secret sharing schemes and network coding, and also to gain an understanding of information inequalities. We outline some of these recent developments next.

Background and Recent Developments

Our workshop covered four major areas within the realm of information theory — coding theory, secret sharing, network coding, and information inequalities — which have seen a recent influx of ideas from matroid theory and combinatorial optimization. We briefly discuss the applications of such ideas in each of these areas in turn.

Coding Theory

The serious study of (channel) coding theory started with Shannon's monumental 1948 paper [9]. Shannon stated the result that reliable communication is possible at rates up to channel capacity, meaning that for any desired symbol or block error probability there exists a channel code and a decoding algorithm that can achieve this symbol or block error probability as long as the rate of the channel code is smaller than the channel capacity. On the other hand, Shannon showed that if the rate is larger than capacity, the symbol and the block error probability must be bounded away from zero.

Unfortunately, the proof of the above achievability result is nonconstructive, meaning that it shows *only the existence* of such channel codes and decoding algorithms. Therefore, since the appearance of Shannon's theorem, the quest has been on to find codes with practical encoding and decoding algorithms that fulfill Shannon's promise.

The codes and decoding schemes that people have come up with can broadly be classified into two classes: "traditional schemes" and "modern schemes." In "traditional schemes," codes were proposed that have some desirable properties like large minimum Hamming distance (a typical example of such codes being the Reed-Solomon codes). However, given a code, it was usually unclear how to decode it efficiently. Often it took quite some time until such a decoding algorithm was found (e.g., the Berlekamp-Massey decoding algorithm for Reed-Solomon codes), if at all. In "modern schemes," the situation is reversed: given an iterative decoding algorithm like the sum-product algorithm, the question is what codes work well together with such an iterative decoding algorithm.

"Modern schemes" took off with the seminal paper by Berrou, Glavieu, and Thitimajshima in 1993 on turbo coding schemes [2]. Actually, codes and decoding algorithm in the spirit of "modern schemes" were already described in the early 1960s by Gallager in his Ph.D. thesis [5]. However, these schemes were, besides the work by Zyablov, Pinkser, and Tanner in the 1970s and 1980s, largely forgotten until the mid-1990s. Only then people started to appreciate Gallager's revolutionary approach to coding theory.

Gallager proposed to define codes in terms of graphs. Such graphs are now known as Tanner graphs: they are bipartite graphs where one class of vertices corresponds to codeword symbols and where the other class of vertices corresponds to parity-checks that are imposed on the adjacent codeword symbols. Decoding is then based on repeatedly sending messages with estimates about the value of the codeword symbols along edges, and to locally process these messages at vertices in order to produce new messages that are again sent along the edges. Especially for sparse Tanner graphs the resulting decoding algorithms have very low implementation complexity.

In the last fifteen years, Tanner graphs and iterative decoding algorithms have been generalized to factor graphs and algorithms operating on them, and many connections to techniques in statistical mechanics, graphical models, artificial intelligence, and combinatorial optimization were uncovered. The workshop talk by Kashyap (see Section 14) surveyed the connection between complexity measures for graphical models for a code and the treewidth (and other width parameters) of the associated matroid. On the other hand, the workshop talks by Wainwright and Vontobel (see Section 14) emphasized the connections between message-passing iterative decoding of codes and certain techniques from combinatorial optimization. In particular, they discussed the linear programming decoder by Feldman, Wainwright, and Karger [4], which is a low-complexity relaxation of an integer linear programming formulation of the maximum likelihood decoder. This linear programming decoder (and its variations) has paved the way for the use of tools from combinatorial optimization and matroids in the design and analysis of decoding algorithms.

Secret Sharing

The second major application of matroid-theoretic ideas that we mention here is with respect to secret-sharing schemes. A secret-sharing scheme is a method to distribute shares of a secret value among a certain number of participants such that *qualified* subsets of participants (e.g., subsets of a certain size) can recover the secret from their joint shares, but *unqualified* subsets of participants can obtain no information whatsoever about the secret by pooling together their shares. Secret-sharing schemes were originally motivated by the problem of secure storage of cryptographic keys, but have since found numerous other applications in cryptography and distributed computing.

It is not difficult to show that in a secret-sharing scheme, the size of each of the shares cannot be smaller than the size (information content) of the secret value. An *ideal* secret-sharing scheme is one in which all shares have the same size as the secret value. More generally, the *information rate* of a secret-sharing scheme is the ratio of the size of the secret to the maximum share size.

In a secret-sharing scheme, the collection of qualified subsets of participants is called the *access structure* of the scheme. It is known that for any monotone increasing collection, Γ, of subsets of a finite set, one can define a secret-sharing scheme with access structure Γ. The *information rate* $\rho(\Gamma)$ is defined to be the supremum of information rates among all secret-sharing schemes having access structure Γ. Γ is said to be an *ideal access structure* if it admits an ideal secret-sharing scheme.

Brickell and Davenport [3] began a line of work relating ideal secret-sharing schemes to matroids. They showed that any ideal access structure is induced by a matroid in a very specific sense. However, it is also known that not every matroid gives rise to an ideal access structure; for example, the access structures induced by the Vámos matroid are not ideal. Characterizing the matroids that give rise to ideal access structures has remained an open problem.

There has been some very recent work on computing the information rates of non-ideal access structures using polymatroid techniques, linear programming, and non-Shannon information inequalities. For example, it has been shown that for any access structure Γ induced by the Vámos matroid, $\rho(\Gamma) \leq 19/21$, which shows that such access structures are far from being ideal. This, and other related results, were surveyed in the workshop talks by Padró and Beimel (see Section 14).

Secret-sharing schemes have also been received some recent attention in the quantum domain, a topic covered in the workshop talk by Sarvepalli (see Section 14).

Network Coding

Another novel application of matroid theory and combinatorial optimization within the realm of information theory is in the area of network coding [10]. Network coding is an elegant technique introduced at the turn of the millennium to improve network throughput and performance. Since then, it has attracted significant interest from diverse scientific communities of engineers, computer scientists, and mathematicians in both academia and industry. This workshop explored connections between network coding and combinatorial optimization, matroids, and non-Shannon inequalities.

The area started when the simple but far reaching observation was made that in communication networks, (unlike in their transportation or fluid counterparts), data streams that are separately produced and consumed do not necessarily need to be kept disjoint while they are transported throughout the network [1]. (At the network layer, for example, nodes can perform binary addition of independent bit-streams.) Schemes that employ processing at network nodes of incoming independent data (as opposed to only forwarding) are referred to as network coding. Naturally, the throughput achievable by network coding is in general higher than what can be achieved by schemes that allow only forwarding. Certain standard problems in combinatorial optimization have been crucial in understanding the potential benefits of network coding. Charikar and Agraval as well as Chekuri, Fragouli, and Soljanin characterized the benefits for certain traffic scenarios and throughput measures, as discussed by Chekuri in his workshop talk (see Section 14).

Mathematically, data streams carried by network edges are treated as sequences of symbols which are elements over some finite field. Network nodes map the incoming multiple data streams into a single stream in a possibly different way for each of its outgoing edges. The goal is to choose these maps in way that will allow the intended receivers to recover the original information. In the simplest case of network multi-cast (one in which the source aims at communicating the same information to a set of receivers), it is sufficient that the nodes forward linear combinations of the incoming symbols. The coefficients in these linear combinations can even be chosen uniformly at random from a sufficiently large finite field. In more complex traffic scenarios, such linear network coding is not sufficient, and matroids have been instrumental in demonstrating this fact. In a series of recent papers, Dougherty, Freiling, and Zeger carried out an exploration of the fundamental limits of network coding. They used matroids to systematically construct various networks that demonstrated, for example, the insufficiency of linear network coding and the inachievability of network coding capacity. A survey of these results was given by Sprintson in his workshop talk (see Section 14).

Finally, network coding problems give certain operational meaning to non-Shannon information inequalities. Raymond Yeung, one of the inventors/pioneers in both areas believes that implications of non-Shannon-type inequalities in information theory will be finally understood in the context of network coding. He declared in his talk that "Every constraint on the entropy function is useful in some multi-source network coding problems!" These and other applications of non-Shannon information inequalities, as well as the fundamentals, were addressed in a separate session of the workshop.

Information inequalities

As mentioned above, non-Shannon information inequalities play a key role in computing the information rates of non-ideal secret-sharing access structures. Furthermore, the results of Dougherty *et al.* in the context of network coding also make heavy use of these inequalities. We briefly give some background on information inequalities here. The workshop talks of Yeung, Matúš, and Dougherty (see Section 14) contain a more comprehensive survey of this topic.

Constraints on the entropy function are sometimes referred to as the laws of information theory. It has been known for a long time that the entropy function must satisfy the polymatroid inequalities (non-negativity, monotonicity, and submodularity), and indeed, that these are equivalent to the non-negativity of the Shannon information measures. Inequalities that are implied by the polymatroid inequalities are referred to as *Shannon-type inequalities*. Until recently, Shannon-type inequalities were the only known linear constraints on the entropy function.

A *non-Shannon-type inequality* is a constraint on the entropy function which is not implied by the polymatroid inequalities. In the late 1990s, the discovery of a few such inequalities, starting with the Zhang-Yeung inequality [11], revealed that Shannon-type inequalities alone do not constitute a complete set of constraints on the entropy function.

Linear information inequalities correspond to the supporting hyperplanes of the closed convex cone $\overline{\Gamma}^*_N$ obtained by taking the closure of the set of entropy vectors defined by N random variables. By virtue of the fact that entropy vectors satisfy the polymatroid inequalities, the cone $\overline{\Gamma}^*_N$ is a subset of the closed convex cone Γ_N defined by the polymatroid inequalities. It is a fact that $\overline{\Gamma}^*_N = \Gamma_N$ for $N \leq 3$, but $\overline{\Gamma}^*_N \subsetneq \Gamma_N$ for $N \geq 4$. In fact, Matúš has shown that for $N \geq 4$, $\overline{\Gamma}^*_N$ is not even polyhedral, *i.e.*, it cannot be characterized by finitely many linear inequalities. This means that there are infinitely many distinct non-Shannon inequalities satisfied by entropy vectors defined by $N \geq 4$ random variables. Matúš's study of the cones $\overline{\Gamma}^*_N$ also involves the use of matroid methods in a non-trivial way.

Presentation Highlights

We provide here brief descriptions of the talks presented at the workshop. Slides from most of the talks are available online at `http://robson.birs.ca/∼09w5103/`.

James Oxley kicked off the workshop with a tutorial on matroid theory. His talk introduced the most common ways to define matroids and then presented some fundamental examples, some basic constructions, and some of the main theorems of the subject. A more thorough introduction to matroids is contained in the survey paper "What is a matroid?" available at `http://www.math.lsu.edu/∼oxley/survey4.pdf`.

Coding theory

Oxley's tutorial was followed by a day-long session, consisting of four talks, focusing on the use of matroid theory and combinatorial optimization in coding theory.

Navin Kashyap gave an overview of the applications of matroid methods to the study of graphical models for codes, and to the analysis of decoding methods such as the sum-product algorithm and linear-programming decoding. Among the topics covered were the use of code/matroid decomposition techniques, and various "width" parameters (treewidth, branchwidth) associated with graphs and matroids, in the analysis of graphical models and decoding algorithms for linear codes.

Martin Wainwright's talk began with an overview of the various hierarchies of linear programming (LP) relaxations, as well as related conic programming relaxations (e.g., SOCP and SDP), that can be applied to a given integer program. It then went to cover their on-going applications in coding theory and other areas of applied mathematics, and the connection between such hierarchies and the hypergraph defined by the underlying integer program. He also described some links between these relaxations, and various types of "message-passing" algorithms that are widely used in communication theory as well as many other domains (e.g., statistical physics, computer vision, machine learning, computational biology).

Pascal Vontobel focused on pseudo-codewords, i.e., certain non-zero fractional vectors that play an important role in the performance characterization of iterative message-passing decoders as well as linear programming decoding. This is in contrast to classical coding theory where decoding algorithms are mostly characterized by non-zero codewords. The talk gave an overview of results about pseudo-codewords and their influence on message-passing iterative decoding and linear programming decoding. The topics that were covered included: pseudo-codewords for cycle codes and their relationship to the graph zeta function; pseudo-codewords for finite-geometry-based codes; pseudo-codewords obtained by canonical completion, and how they upper bound the performance of linear programming decoding; the influence of redundant rows in the parity-check matrix on the set of pseudo-codewords; the relationship of pseudo-codewords to other concepts like stopping sets, near-codewords, trapping sets, and absorbing sets.

The final talk in the coding theory session was given by **Thomas Britz**, who presented a brief overview on what is presently known about the support and weight connections between coding and matroid theory, and gave applications of these connections to coding and graph theory. The newest results included an interesting variation of the Tutte polynomial as well as an interesting but ever-evolving dual identity.

The Matroid Minors Project

The morning of the second day of the workshop was devoted to the Matroid Minors Project of Jim Geelen, Bert Gerards, and Geoff Whittle. This project aims to extend the results and techniques of the Graph Minors Project of Robertson and Seymour (see *e.g.* [8]) to matrices and matroids. One of the main goals of this theory is to describe precisely the structure of minor-closed classes of matroids representable over finite fields. This requires a peculiar synthesis of graphs, topology, connectivity, and algebra. In addition to proving several long-standing conjectures in the area, the structure theory is expected to help find efficient algorithms for a general class of problems on matrices and graphs [6].

Bert Gerards presented an overview of the structure theorem (whose proof has just recently been completed by Geelen, Gerards and Whittle) for minor-closed classes of binary matroids. This theorem is a major milestone of the Matroid Minors Project. One important implication of this theorem is that every minor-closed class of binary matroids is characterized by a finite set of excluded minors.

Jim Geelen followed Gerards' talk by surveying some of the applications of the binary matroids structure theorem. It follows from the theorem that there exists an $O(n^7)$ algorithm for testing an n-element binary matroid for the presence of a fixed minor. An application pertinent to coding theory is the interesting result that proper minor-closed families of binary linear codes cannot be asymptotically good. Geelen further presented some open problems concerning minor-closed classes of binary matroids.

Secret sharing

The theme for the afternoon session on the second day was secret-sharing schemes. In a secret-sharing scheme, a secret value is distributed into shares among a set of participants in such a way that the qualified subsets of participants can recover the secret value, while the non-qualified ones do not obtain any information about it. In this situation, the size of every share is at least the size of the secret. If all shares have the same size as the secret, which is the best possible situation, the scheme is said to be ideal. Only a few access structures admit an ideal secret sharing scheme. In general, one is interested in finding schemes with optimal share length for every given access structure. This is a difficult problem that has attracted the attention of many researchers.

Carles Padró discussed several methods to find upper and lower bounds on the share length. He presented the most important results and techniques that have been obtained about this open problem from combinatorics, specially from the use of matroids and polymatroids. He also discussed some combinatorial techniques to construct efficient linear secret sharing schemes.

Amos Beimel, in a talk based on joint works with Noam Livne, Carles Padró, and Ilan Orlov, presented the use of non-Shannon information inequalities for proving lower bounds on the size of shares in secret-sharing schemes. He described two results:

1. A proof, using non-Shannon information inequalities, of lower bounds on the size of the shares in every secret-sharing scheme realizing an access structure induced by the Vámos matroid. This is the first result showing the existence of an access structure induced by a matroid which is not nearly ideal.
2. A proof of the fact that all the information inequalities known to date cannot yield a lower bound of $\Omega(n)$ on the share size.

Pradeep Kiran Sarvepalli talked about the applications of matroids quantum secret sharing, which deals with the problem of distribution of a quantum state among n players so that only authorized players can reconstruct the secret. He presented the first steps toward a matroidal characterization of quantum secret-sharing schemes. This characterization allows one to construct efficient schemes from self-dual matroids that are coordinatizable over a finite field. In the process, he also provided a connection between a class of quantum stabilizer codes and secret-sharing schemes.

Sarvepalli also briefly surveyed the use of matroids in quantum computation and quantum cryptography. He reviewed a recent work by Shepherd and Bremner which claims that even restricted models of quantum computation, such as those consisting of abelian gates, give rise to probability distributions that cannot be sampled efficiently by a classical computer. He sketched their arguments that use the theory of binary matroids to substantiate their claim.

Sarvepalli also considered an open problem related to the classification of a class of quantum states called the stabilizer states. A restricted version is to classify the equivalence classes of a subclass of stabilizer states (namely, the CSS states) under the action of the local unitary group and a subgroup of the local unitary group, called the local Clifford group. Specifically, the problem is to find necessary and sufficient conditions for when a CSS stabilizer state has distinct equivalence classes. Sarvepalli showed that CSS stabilizer states whose equivalence classes are distinct must arise from binary matroids which are neither graphic nor cographic. In doing so, he arrived at a class of minor-closed matroids whose excluded minors have not yet been characterized.

Network Coding

Network coding was the theme for the third day of the workshop, when a tutorial and two survey talks were given, followed by a presentation of an open problem. Network coding was also discussed on the two following days in connection with non-Shannon inequalities, some recent results in wireless networks, and general hardness to find a network coding scheme that achieves, or approximately achieves, capacity.

Emina Soljanin gave a tutorial talk on coding for network multicast (simultaneously transmitting the same information to multiple receivers in the network). She explained sufficient and necessary conditions that the network has to satisfy to be able to support the multicast at a certain rate. For the case of unicast (when only one receiver at the time uses the network), such conditions have been known for the past fifty years, and, clearly, we must require that they hold for each receiver participating in the multicast. The fascinating fact that the main network coding theorem brings is that the conditions necessary and sufficient for unicast at a certain rate to each receiver are also necessary and sufficient for multicast at the same rate, provided the intermediate network nodes are allowed to combine and process different information streams.

Chandra Chekuri surveyed results that seek to understand the potential benefit that network coding offers over more traditional and simpler transmission schemes such as store and forward routing. This was examined by asking the following question: what is the maximum ratio (over all networks) between the rate achievable via network coding and via routing? He restricted his attention to the wireline setting. This question has been answered to a large extent in the multicast setting in both undirected and directed graphs.

In the multiple unicast setting, the benefit is known to be very large in some directed graph instances while the case of undirected graphs is wide open. Combinatorial optimization plays an important role in understanding this question. Steiner-tree packings and integrality gaps of linear programming relaxations for Steiner trees are the key tools in the multicast setting. Multicommodity flow-cut gaps play a role in the multiple unicast setting.

In his talk, **Alex Sprintson** gave an extensive survey of connections between matroid theory and network coding. He presented two ways of constructing new classes of coding networks from matroids. These constructions are instrumental for establishing several important properties of coding networks, such as insufficiency of scalar and vector linear network coding and inachievability of network coding capacity. He also explained the recently introduced problem of index coding, and pointed out its role as an intermediate step from a given matroid to a network whose dependency relations satisfy the given matroidal constraints. He presented recent results in this research area and outlined directions for future work.

The final talk of the session was given by **Randall Dougherty**, who outlined an approach that, if two proof-holes in it can be filled or worked around, will yield a proof that the solvability problem for network coding is undecidable. The idea was to try to represent groups satisfying or not satisfying identities as networks, in order to reduce Rhodes' problem on finite groups to the network coding solvability problem.

Information inequalities

The penultimate day of the workshop was the last "themed" day, the theme this time being information inequalities. Information inequalities are inequalities that must be satisfied by entropies of random variables.

Raymond Yeung's tutorial talk gave the necessary background on information inequalities. It is well-known that the entropy function must satisfy the polymatroidal axioms. All information inequalities implied by the polymatroidal axioms are called Shannon-type inequalities. In 1998, Zhang and Yeung discovered a non-Shannon-type inequality, an information inequality that is independent of the polymatroidal axioms. Since then, many more such inequalities have been found, and connections between the entropy function and a number of fields in information science, mathematics, and physics have been established. Yeung gave several examples of such connections to the fields of probability theory, network coding, combinatorics, group theory, Kolmogorov complexity, matrix theory, and quantum information theory.

Frantisek Matúš considered the problem of characterizing the closed cone, $\overline{\Gamma}^*_N$, formed by taking the closure of the set of entropic points on N random variables. He showed that this cone is not polyhedral, meaning that it cannot be characterized by finitely many linear inequalities, if and only if $N \geq 4$. He also discussed the problem of determining which matroids lie within $\overline{\Gamma}^*_N$, and mentioned that it remains an open problem to identify the excluded minors for this class of "almost entropic" matroids.

The third talk of this session was given by **Andreas Winter**, and was mainly based on a joint paper with N. Linden on quantum (van Neumann) entropy inequalities. Pippenger has initiated the generalization of the programme to find all the "laws of information theory" to quantum entropy. The standard quantum information inequalities derive from strong subadditivity (SSA), which corresponds to the third polymatroidal axiom. SSA of the von Neumann entropy, proved in 1973 by Lieb and Ruskai (who was present at the workshop), is a cornerstone of quantum information theory. All other known inequalities for entropies of quantum systems may be derived from it. In his talk, Winter proved a new inequality for the von Neumann entropy which is independent of strong subadditivity: it is an inequality which is true for any four party quantum state, provided that it satisfies three linear relations (constraints) on the entropies of certain reduced states. He also discussed the possibility of finding an unconstrained inequality (work with N. Linden and B. Ibinson).

Randall Dougherty gave his second talk of the workshop in this session, this talk being on non-Shannon-type information inequalities and linear rank inequalities. He first gave an alternate proof of Zhang and Yeung's non-Shannon-type inequality in four random variables. Zhang and Yeung's original proof used the technique of adding two auxiliary variables with special properties and then applying Shannon-type inequalities to the enlarged list. Dougherty presented a derivation of this inequality by adding just one auxiliary variable. He then used the same basic technique of adding auxiliary variables to give many other non-Shannon inequalities in four variables (which, surprisingly, are all of the same general form). He also derived rules for generating new non-Shannon inequalities from old ones, which can be applied iteratively to generate infinite families of inequalities such as the one used by Matúš to show that the cone $\overline{\Gamma}^*_4$ is not polyhedral.

Dougherty further showed how a variant of this approach (using a different sort of auxiliary variable) allowed one to derive inequalities which always hold for ranks of linear subspaces, but need not hold for entropies of random variables. It is known that the Ingleton inequality and the Shannon inequalities give a complete list of the rank inequalities for four variables (subspaces). Dougherty derived a list of 24 additional inequalities in five variables which, together with the Shannon inequalities and instances of the Ingleton inequality, are complete for rank inequalities on five subspaces. He also gave general many-variable families of rank inequalities.

Short talk sessions

The remaining sessions of the workshop consisted of short talks on several different topics related to the overall theme of the workshop.

Alex Grant presented his work with Terence Chan on quasi-uniform codes and their applications. Quasi-uniform random variables have probability distributions that are uniform over their support. They are of fundamental interest because a linear information inequality is valid if and only if it is satisfied by all quasi-uniform random variables. In his talk, Grant investigated properties of codes induced by quasi-uniform random variables. He proved that quasi-uniform codes (which include linear codes as a special case) are distance-invariant and that Greene's Theorem holds in the setting of quasi-uniform codes. He also showed that almost affine codes are a special case of quasi-uniform codes in the sense that quasi-uniform codes are induced by entropic polymatroids while almost affine codes are induced by entropic matroids. Applications of quasi-uniform codes in error correction and secret sharing were also given.

Serap Savari presented a combinatorial study of linear deterministic relay networks. This network model has gained popularity in the last few years as a means of studying the flow of information over wireless communication networks. This model considers layered directed graphs, and a node in the graph receives a linear transformation of the signals transmitted to it by neighbouring nodes. There is recent work extending the celebrated max-flow/min-cut theorem of Ford and Fulkerson to this model. This result was first established by a randomized transmission scheme over large blocks of transmitted signals. In joint work with S. Tabatabaei-Yazdi, Savari demonstrated the same result with a simple, deterministic, polynomial-time algorithm which takes as input a single transmitted signal instead of a long block of signals. Their capacity-achieving transmission scheme requires the extension of a one-dimensional Rado-Hall transversal theorem on the independent subsets of columns of a column-partitioned matrix into a two-dimensional variation for block matrices. The rank function arising from the study of cuts in their model has an important difference from the rank functions considered in the literature on matroids in that it is submodular but not monotone.

Eimear Byrne presented upper bounds for a particular model of error-correcting codes for coherent network coding. Versions of the Singleton, sphere-packing, and Gilbert-Varshamov bounds for this model were previously given by Yang and Yeung. In her talk, Byrne showed how to extend the classical Plotkin and Elias bounds for the same model.

The final session of the workshop began with a talk by **Dillon Mayhew** on the excluded minors for real-representable matroids. Rota conjectured that if F is a finite field, then there is only a finite number of minor-minimal matroids that are not F-representable. Such matroids are called excluded minors for F-representability. Rota's conjecture contrasts with the long-established fact that there are infinitely many excluded minors for representability over the real numbers. Geelen (2008) conjectured a much stronger fact: if M is any real-representable matroid, then there is an excluded minor, N, for real-representability, such that N contains M as a minor. Mayhew presented a proof of Geelen's conjecture (joint work with Mike Newman and Geoff Whittle).

Michael Langberg discussed the algorithmic complexity of network coding, focusing on how ”hard” it is to find a network coding scheme that achieves, or approximately achieves, capacity. He gave proofs of the fact that deciding whether or not a given instance of a network coding problem (acyclic network plus communication requirements) has scalar linear capacity of 1 is NP-complete. He further showed that it is “hard” (in the sense of being reducible to an open problem in graph colouring) to find a scalar linear code that enables communication with any constant factor of capacity. The same hardness result extends to the problem of finding a vector linear code of a fixed dimension.

Olgica Milenkovic gave a talk which approached the problem of compressive sensing via matroid theory. Compressive sensing is a new sampling technique for sparse signals that has the potential to significantly reduce the complexity of many data acquisition techniques. Most compressive sensing reconstruction techniques are still prohibitively time-consuming, narrowing the scope of practical applications of this method. Milenkovic, in joint work with Wei Dai and Vin Pham Hoa, proposed a new method for compressive sensing signal reconstruction of logarithmic complexity that combines iterative decoding methods with greedy subspace pursuit algorithms. The performance of the method depends on certain characteristics of support weight enumerators of the codes used for constructing the sensing matrix, which can be described via matroid theory.

The final talk of the workshop was given by **Alexander Barg** on the subject of linear codes in the ordered Hamming space. As is well known, the weight distribution of MDS codes in the Hamming metric can be recovered easily from the rank function of a uniform matroid. No such association has been established for the ordered Hamming space (the Niederreiter-Rosenbloom-Tsfasman space), although the weight distribution of MDS codes is also easily found. The question becomes more challenging when one considers codes with distance even one less than the MDS distance. Barg presented his work with Punarbasu Purkayastha which computes such weight distributions for an arbitrary poset metric and characterizes distributions of points in the unit cube that arise from near-MDS codes in the ordered metric.

Outcome of the Meeting

The workshop achieved its stated goal of encouraging interactions between researchers from several different disciplines, for whom there is currently no other forum (conference or workshop) that could serve as a natural meeting point. As a result, the workshop was extremely well received by all the participants, making it an unqualified success. Here we list some of the feedback that we received from the participants.

Thanks for organizing a beautiful workshop. I enjoyed my time during the workshop days no less than a fantastic weekend of hikes. *— Alexander Barg (University of Maryland, College Park)*

Once again, many thanks for the invitation to Banff – it was a very enlightening workshop.
— Eimear Byrne (University College Dublin)

It was very nice to be in Banff, thank you once again for the invitation.
— František Matúš (Institute of Information Theory and Automation Prague)

Thanks for your role in the workshop, it was very educational.
— Dillon Mayhew (Victoria University of Wellington)

Thanks a lot for the invitation to Banff – was a great workshop! Just continue organizing more of these.
— Olgica Milenkovic (University of Illinois, Urbana-Champaign)

Thanks again for the great workshop! *— Michael Langberg (The Open University of Israel)*

Thanks very much for putting together such an interesting workshop. I have enjoyed it very much indeed and am very glad I was invited to speak. *— James Oxley (Louisiana State University)*

Thanks for organizing an interesting and stimulating workshop. I also want to thank you for the opportunity to present at the workshop. I personally benefited a lot from the workshop especially in the sense of gaining a big picture of the associations between various fields. I was glad to have had some useful discussions with some of the workshop participants. *— Pradeep Kiran Sarvepalli (University of British Columbia)*

Thanks again for all of your work in organizing the workshop. *— Serap Savari (Texas A&M University)*

Thank you so much for letting me participate in this workshop. I learned a lot and enjoyed it very much.
— Beth Ruskai (Tufts University)

Best workshop that I've attended for quite a while. *— Alex Vardy (University of California, San Diego)*

Thanks again for organizing the workshop. *— Martin Wainwright (University of California, Berkeley)*

It was indeed a very nice meeting, I learnt a lot of new maths, and enjoyed myself very much!
— Andreas Winter (University of Bristol)

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Rai, Brijesh Kumar (IIT-Bombay)
Ruskai, Mary Beth (Tufts University)
Sarvepalli, Pradeep (University of British Columbia)
Savari, Serap (Texas A&M University)
Smarandache, Roxana (San Diego State University)
Soljanin, Emina (Alcatel-Lucent Bell-Labs)
Sprintson, Alex (Texas A&M University)
Usefi, Hamid (UBC)
Vardy, Alexander (University of California San Diego)
Vontobel, Pascal (Hewlett-Packard Laboratories)
Wainwright, Martin (UC Berkeley)
Whittle, Geoff (Victoria University of Wellington)
Winter, Andreas (University of Bristol)
Yeung, Raymond (The Chinese University of Hong Kong)

Chapter 15

Analysis of nonlinear wave equations and applications in engineering (09w5121)

Aug 9 - Aug 14, 2009

Organizer(s): Vadim Zharnitsky (University of Illinois in Urbana-Champaign), James Colliander (University of Toronto), Michael Weinstein (Columbia University)

Overview of the Field

Nonlinear dispersive wave equations arise naturally in scientific and engineering fields such as fluid dynamics, electromagnetic theory, quantum mechanics, optical communication, nonlinear optics etc. Many important questions (both in theory and applications) are related to the interaction of two effects: energy spreading (dispersion, diffraction) and energy concentrating (nonlinear self-trapping, defect modes,. . .) mechanisms. For example, in Korteweg-deVries equation (KdV)

$$u_t = uu_x + u_{xxx},$$

which describes propagation of long waves in shallow water, the term uu_x steepens the wave and causes it to break, while the term u_{xxx} tends to broaden the wave and smoothes the wave profile. For the so-called soliton solution, which is a localized wave that propagates without distortion, these two effects balance each other. There are several other fundamental equations (nonlinear Schrödinger / Gross-Pitaevskii (NLS / GP), nonlinear Klein-Gordon equations (NLKG)) and their modifications which naturally arise in applications and in which similar balance of nonlinearity and dispersion gives rise to coherent structures (solitary waves, vortices, very long-lived *meta-stable* states). The following are central issues in the field:

- Well-posedness of nonlinear dispersive equations.
- Stability of coherent structures such as solitary waves.
- Interaction among coherent structures.

In the past 2-3 decades, new techniques based on harmonic analysis, variational methods, and dynamical systems have advanced our knowledge in all three directions. However, despite relations between the above questions, there has not been sufficiently strong interaction between them. One of the goals of this workshop was to bring together researchers who would benefit from sharing the ideas. One way in which we achieved such interactions was through two tutorials, which bridged scientific communities:

(I) *Derivations of the nonlinear Schrödinger - Gross Pitaevskii (NLS / GP) equations from the quantum $N-$ body problem (speaker: Benjamin Schlein):* This bridged the mathematical physics community, versed in the quantum N- body problem with the nonlinear PDE and nonlinear waves community, more familiar with NLS / GP as a mathematical description of optical and hydrodynamic phenomena.

(II) *Pore Formation in Polymer Electrolytes (speaker: Keith Promislow):* This provided an introduction to an important class of multi-scale problems of huge interest and receiving broad attention, due to applications to energy problems.

Recent Developments and Open Problems

Nonlinear Optics and Stability Problems

In the last decade the study of variable coefficient generalizations of the basic equations became an active area. This was motivated by advances in fabrication technology, enabling the design of precision *photonic structures*, with applications ranging from optical transmission media, optical storage, pulse compression, or more generally, "light processing", to quantum information science.

For example, in long distance optical fiber communication systems, *dispersion management*, a prescribed variation of the dispersion properties of the optical fiber transmission media, gives rise to a class of NLS equations with time-dependent (periodic, random) coefficients. Also, propagation of light of sufficient intensity in nonlinear and spatially inhomogeneous media as well as the evolution of macroscopic quantum systems (Bose-Einstein condensates) give rise to PDEs of NLS/GP type with spatially dependent variations (*e.g.* compactly supported or periodic) in the linear or nonlinear potentials. Variations in these potentials can be engineered to influence the dynamics of coherent structures.

Stability properties of coherent structures and variable coefficient extensions of standard PDE theory have been under separate study for quite some time. Engineering advances, such as dispersion management in optical fibers and photonic microstructures, motivate studying the interplay between variable coefficients and stability properties of coherent structures.

A mathematical theory of the stability of soliton-like objects in spatially varying media contributes towards a control theory of these objects required for ”light processing”. Currently, there is a need for links between pure mathematicians advancing the rigorous understanding of basic model equations and engineers and applied mathematicians developing new models and applications. One of the goals of this workshop was to encourage this interaction.

Many-body quantum mechanics

In 2001, the Nobel prize was given for the first experimental evidence of existence of Bose-Einstein condensate (BEC). This discovery generated considerable activity in the study of the evolution of BEC. Mathematical physicists derived rigorously macroscopic evolution equations for interacting many body systems. For example, the nonlinear Schrödinger equation and nonlinear Hartree-Fock equations have been rigorously derived as mean field limits of interacting Bose gases.

Fundamental PDE problems

Wave maps and Schrödinger maps are fundamental objects of study in modern PDE analysis. These are natural generalizations of the classical wave equations and Schrödinger equations to non-Euclidean spaces. Wave maps also arise as approximations of Einstein's equations of general relativity.

Dispersive estimates have been an active area of research for a few decades. More recently the emphasis is on Strichartz estimates on non-Euclidean manifolds and with additional terms, e.g. magnetic or potential field.

Presentation Highlights

Application in optics, stability problems

Gadi Fibich described construction of singular solutions of nonlinear evolution equations that become singular on a sphere. The asymptotic profile and blowup rate of these solutions are the same as those of solutions of the corresponding one-dimensional equation that become singular at a point. These results were obtained for the nonlinear Schrödinger equation, the biharmonic nonlinear Schrödinger equation, the nonlinear heat equation, and the nonlinear biharmonic heat equation.

Jared Bronski considered periodic solutions to equations of Korteweg-de Vries type. The stability of periodic wave nonlinear wave-trains is a fundamental problem, whose analytical theory is far less developed that of the solitary wave stability, due to significant mathematical challenges and new phenomena.

Bronski demonstrated a proof of an index theorem giving an exact count of the number of unstable eigenvalues of the linearized operator in terms of the number of zeros of the derivative of the traveling wave profile together with geometric information about a certain map between the constants of integration of the ordinary differential equation and the conserved quantities of the partial differential equation. This index can be regarded as a generalization of both the Sturm oscillation theorem and the classical Lyapunov stability theory for solitary wave solutions for equations of Korteweg-de Vries type (Benjamin, Bona-Souganidis-Strauss, Weinstein, . . .)

In the case of a polynomial nonlinearity this index, together with a related one introduced earlier by Bronski and Johnson, could be expressed in terms of derivatives of hyperelliptic integrals on a finite genus Riemann surface. Since these hyperelliptic integrals satisfy a Picard-Fuchs relation these derivatives can be expressed in terms of the integrals themselves, leading to a closed-form expression in terms of a finite number of moments of the solution.

Boaz Ilan described band-edge solitons of Nonlinear Schrödinger equations with periodic potentials (joint work with M.I. Weinstein). Nonlinear Schrödinger (NLS) / Gross-Pitaveskii equations with periodic potentials admit positive bound states (solitons). For focusing nonlinearities these solitons bifurcate from the zero state with frequencies (propagation constant) lying in the semi-infinite spectral gap and near the spectral band edge. A multiple scale expansion leads to a constant coefficient homogenized / effective medium NLS equation that depends on the band-edge Bloch wave through an effective-mass tensor and nonlinear coupling constant. The multiple scales argument is made rigorous via a Lyapunov Schmidt reduction to Bloch-modes sufficiently near the spectral band edge. To leading order the soliton is constructed from the Bloch wave that is slowly modulated by a ground state of the homogenized equation. In the L^2-critical case, for any non-constant periodic potential the power (L^2 norm) of the soliton is *strictly lower* than the power of the Townes mode, which has the critical power for collapse. The implications to collapse dynamics and self-focusing instability were elucidated using computations of bound states and direct computations of critical NLS equations in 1D and 2D.

Milena Stanislavova presented conditional stability theorems for Klein-Gordon type equations. She considered positive, radial and exponentially decaying steady state solutions of the Klein-Gordon equation with various power nonlinearities. The main result was a precise construction of infinite-dimensional invariant manifolds in the vicinity of these solutions. The precise center-stable manifold theorem for the Klein-Gordon equation includes the co-dimension of the manifold, a formula for the asymptotic phase and the decay rates for even perturbations.

Yoshio Tsutsumi gave a presentation on stability of cavity soliton for the Lugiato-Lefever equation with additive noise. He considered the stability of stationary solution for the Lugiato-Lefever (LL) equation with periodic boundary condition under perturbation of additive noise. The LL- equation is a nonlinear

Schrödinger equation with damping and spatially homogeneous forcing terms, which describes a physical model of a unidirectional ring or Fabry-Perot cavity with plane mirrors containing a Kerr medium driven by a coherent plane-wave field. The stationary solution of (LL) is called a ”Cavity Sliton”. Tsutusmi showed the stability of certain stationary solutions under the perturbation of additive noise from a viewpoint of the Freidlin-Wentzell type large deviation principle.

Roy Goodman described bifurcations of *nonlinear defect modes*. The nonlinear coupled mode equations describe the evolution of light in Bragg grating optical fibers. Defects (localized potentials) can be added to the fiber in order to trap light at a specialized location as a nonlinear defect mode. In numerical simulations these defect modes are seen to lose (linear) stability through several types of bifurcations. Inverse scattering is used to design defects in which the bifurcations can be easily observed and studied via the derivation of finite-dimensional reduced equations. Goodman gave conditions for the existence of Hamiltonian pitchfork and Hamiltonian Hopf bifurcations.

Björn Sandstede gave a presentation on pointwise estimates and nonlinear stability of waves. Over the past decade, pointwise Green's function estimates have proved very useful in establishing nonlinear stability of viscous shock profiles under the assumption of spectral stability. He reported here on recent work with Beck and Zumbrun on extending this approach to the nonlinear stability of time-periodic viscous shocks. Key to the derivation of the required *pointwise* bounds in the time-periodic setting are meromorphic extensions of exponential dichotomies of appropriate time-periodic eigenvalue problems. He also showed how spectral stability of weakly time-periodic shocks can be established near Hopf bifurcation using a spatial-dynamics approach. The motivation for this work comes from sources in reaction-diffusion systems. He also outlined the challenges and hopes for nonlinear stability proofs in this context.

Justin Holmer considered dynamics of KdV solitons in the presence of a slowly varying potential. He studied the dynamics of solitons as solutions to the perturbed KdV (pKdV) equation $\partial_t u = -\partial_x(\partial_x^2 u + 3u^2 - bu)$, where $b(x,t) = b_0(hx, ht)$, $h \ll 1$ is a slowly varying potential. This result refined earlier work of Dejak-Sigal and an estimate on the trajectory of the soliton parameters of scale and position was obtained. In addition to the Lyapunov analysis commonly applied to these problems, a local virial estimate due to Martel-Merle was used. The proof did not rely on the inverse scattering machinery and could be expected to carry through for the L^2 subcritical gKdV-p equation, $1 < p < 5$. The case of $p = 3$, the modified Korteweg-de Vries (mKdV) equation, is structurally simpler and more precise results can be obtained by the method of Holmer-Zworski. This was joint work with Galina Perelman.

Eduard Kirr considered asymptotic stability of nonlinear bound states and resonances for nonlinear Schrödinger equations with *subcritical* nonlinearities. What makes the extension to the subcritical case possible is his recent method for obtaining dispersive estimates for perturbations of linear Schrödinger operators with time-dependent and spatially localized coefficients. The method currently works in dimensions two and higher. Kirr discussed obstacles in extending his method to one space dimension. Also some applications to nonlinear equations, in particular to asymptotic stability and radiative damping of ground states in NLS were presented.

Gideon Simpson presented a poster on numerical simulations of the energy-supercritical NLS equation. These computations were motivated by recent works of Kenig-Merle and Kilip-Visan who considered some energy supercritical wave equations and proved that if the solution is *a priori* bounded in the critical Sobolev space (i.e. the space whose homogeneous norm is invariant under the scaling leaving the equation invariant), then it exists for all time and scatters. Simpson numerically investigated the boundedness of the H^2-critical Sobolev norm for solutions of the NLS equation in dimension five with quintic nonlinearity. It was found that for a class of initial conditions, this norm remains bounded, the solution exists for long time, and it scatters (disperses to zero).

Young-Ran Lee presented a proof of exponential decay of dispersion managed solitons with vanishing average dispersion. It was shown that any L^2 solution of the Gabitov-Turitsyn equation describing dispersion managed solitons with zero average dispersion decays exponentially in space and frequency domains. This confirmed in the affirmative Lushnikov's conjecture of exponential decay of dispersion managed solitons. This work was done jointly with M. Burak Erdogan and Dirk Hundertmark.

Many-body quantum mechanics

Benjamin Schlein gave a two-hour tutorial on derivation of equations of nonlinear Schrödinger / Gross-Pitaevskii type as the mean field limit of N-body quantum problems, as $N \to \infty$. In particular, he showed that the nonlinear Hartree equation can be used to describe the macroscopic properties of the evolution of a many body system in the so called mean field limit. He also explained how Gross-Pitaevskii equation, a cubic nonlinear Schrödinger equation, can be used to describe the dynamics of Bose-Einstein condensates.

Mathieu Lewin considered variational models for infinite quantum systems with an example of the crystal with defects. Describing quantum particles in a quantum medium often leads to strongly indefinite (sometimes unbounded from below) theories, for which it is usually quite hard to establish the existence and the stability of bound states. Two well-known, important examples are relativistic electrons described by the Dirac operator and electrons close to a defect in a quantum crystal. Lewin presented a new method for constructing and studying a variational model for such systems. He concentrated on the Hartree model for the crystal with a defect.

The main idea is to describe at the same time the electrons bound by the defect and the (nonlinear) behavior of the infinite crystal. This leads to a (rather peculiar) bounded-below nonlinear functional whose variable is however an operator of infinite-rank.

Lewin introduced the appropriate functional analytic setting, stated the existence of global-in-time solutions to the associated time-dependent Schrödinger equation, and discussed the existence, properties and the stability of bound states.

Walid K. Abou Salem presented microscopic derivation of the magnetic Hartree equation. He discussed the rigorous derivation of the time-dependent Hartree equation in the presence of magnetic potentials. He also remarked on how to extend the analysis to the Gross-Pitaevskii equation.

Natasa Pavlovic discussed the quintic NLS as the mean field limit of a Boson gas with three-body interactions. She described the dynamics of a boson gas with three-body interactions in dimensions d=1,2. She and her collaborator, Thomas Chen, prove that in the limit as the particle number N tends to infinity, the BBGKY hierarchy of k-particle marginals converges to a limiting Gross-Pitaevskii (GP) hierarchy for which they proved existence and uniqueness of solutions. For factorized initial data, the solutions of the GP hierarchy are shown to be determined by solutions of a quintic nonlinear Schrödinger equation. This was joint work with Thomas Chen.

Thomas Chen discussed some recent developments on the well-posedness of the Cauchy problem for focusing and defocusing GP hierarchies. He surveyed some recent results, all from joint works with Natasa Pavlovic, related to the Cauchy problem for the Gross-Pitaevskii (GP) hierarchy. First, he addressed the local well-posedness theory, in various dimensions, for the cubic and quintic case. He then introduced new conserved energy functionals which were used in the following contexts: (1) In a joint work with N. Tzirakis, to prove, on the L2 critical and supercritical level, that solutions of focusing GP hierarchies with a negative average energy per particle blow up in finite time. (2) To prove the global well-posedness of the Cauchy problem for energy subcritical, defocusing GP hierarchies, based on the conservation of higher order energy functionals. (3) To prove global well-posedness of focusing and defocusing GP hierarchies on the L2 subcritical level, based on a generalization of the Gagliardo-Nirenberg inequalities which they establish for density matrices.

Manoussos Grillakis spoke on precise $N-$ dependent error bounds, satisfied by the NLS / GP approximation to mean-field scaled quantum N body problem, for large N.

Fundamental problems in PDE analysis

Wilhelm Schlag gave a presentation on inverse square potentials and applications. He discussed some recent work on dispersive estimates on a curved background. These problems arise in geometry and physics, and are reduced for fixed angular momentum to a one-dimensional problem with an inverse square potential. For the Schwarzschild case, one obtains local pointwise decay rates which increase with the angular momentum. This was joint work with R. Donninger, A. Soffer, and W. Staubach.

Daniel Tataru presented his recent result on large data wave maps. He proved for large data wave maps from $\mathbb{R}^{2+1}$ into a compact Riemmanian manifold, the following dichotomy: either a solution is global and dispersive, or a soliton like concentration must occur. This was joint work with Jacob Sterbenz.

Energy conversion

Keith Promislow gave tutorial on Pore Formation in Polymer Electrolytes. The efficient conversion of energy from chemical and photonic forms to useful electric voltage requires the development of nanostructured materials with interpercolating structure. In practical applications this is achieved by functionalizing polymers, attaching acid groups to short side chains which extend from long, hydrophobic polymer backbones. When placed in solvent, these functionalized polymers form nanoscale solvent-filled pores lined with the tips of the acid groups, and ideal environment for the selective conduction of properly charged ions.

He presented a family of models, which we call functionalized Lagrangians, which mimic the energy landscape of the functionalized polymer/solvent mixtures. The functionalized energies are higher order, and strongly nonlinear, but with special structure which renders them amenable to analysis. He outlined the properties of the functionalized Lagrangians, and the multi-stage structure of the associated gradient flows.

Geometric PDEs

Stephen Gustafson gave a talk on Schrödinger and Landau-Lifshitz maps of low degree. The Schrödinger (and Landau-Lifshitz) map equations are a basic model in ferromagnetism, and a natural geometric (hence nonlinear) version of the Schrödinger (and Schrödinger-heat) equation. While there has been recent progress on the question of singularity formation for the wave and heat analogues (wave map and harmonic map heat-flow), the Schrödinger case seems more elusive. He presented results on global regularity and long-time dynamics for equivariant maps with near-minimal energy. He emphasized lower degree (2 and 3) maps, for which the analysis is trickier, and the dynamics more complex, phenomena related to slower spatial decay of certain eigenfunctions. This was joint work with K. Nakanishi, and T.-P. Tsai.

Scientific Progress Made

Tataru, harmonic maps

Pavlovic, Chen, Walid, Kunze, mean-field

Lee, exponential tails decay in nonlocal equations

Outcome of the Meeting

Interaction of the two groups...

List of Participants

Abou Salem, Walid (University of Toronto)
Anapolitanos, Ioannis (University of Toronto)
Anton, Ramona (Université Pierre et Marie Curie (Paris 6))
Bronski, Jared (University of Illinois Urbana-Champaign)
Camassa, Roberto (University of North Carolina)
Chen, Thomas (University of Texas at Austin)
Colliander, James (University of Toronto)
Dodson, Benjamin (UC Riverside)
Fibich, Gadi (Tel Aviv University)
Goodman, Roy (New Jersey Institute of Technology)
Grillakis, Manoussos (University of Maryland)
Gustafson, Stephen (University of British Columbia)
Holmer, Justin (Brown University)
Hundertmark, Dirk (University of Illinois)
Ilan, Boaz (University of California, Merced)
Kath, William L. (Northwestern University)
Kirr, Eduard (University of Illinois Urbana-Champaign)
Kunze, Markus (Universitaet Duisburg-Essen)
Lee, Young-Ran (Sogang University)
Lewin, Mathieu (CNRS / University of Cergy-Pontoise)
Liu, Baoping (University of California, Berkeley)
Marzuola, Jeremy (Columbia University and University of Bonn)
Pavlovic, Natasa (University of Texas at Austin)
Pelinovsky, Dmitry (McMaster University)
Promislow, Keith (Michigan State University)
Sandstede, Bjorn (Brown University)
Schlag, Wilhelm (University of Chicago)
Schlein, Benjamin (University of Cambridge)
Shlizerman, Eli (Weizmann Institute)
Simpson, Gideon (University of Toronto)
Stanislavova, Milena (University of Kansas)
Tataru, Daniel (University of California, Berkeley)
Tsutsumi, Yoshio (Kyoto University)
Weinstein, Michael (Columbia University)
Zharnitsky, Vadim (University of Illinois in Urbana-Champaign)

Chapter 16

Emerging issues in the analysis of longitudinal data (09w5081)

Aug16 - Aug 21, 2009

Organizer(s): Lang Wu (University of British Columbia), Liqun Wang (University of Manitoba), Charmaine Dean (Simon Fraser University), Xihong Lin (Harvard University), John Neuhaus (University of California, San Francisco), Grace Yi (University of Waterloo)

Overview of the Field

Longitudinal data arise frequently in practise, either in observational studies or in experimental studies. In a longitudinal study, individuals in the study are followed over a period of time and, for each individual, data are collected at multiple time points. That is, the defining feature of a longitudinal study is that multiple or repeated measurements of the same variables are made for each individual in the study over a period of time. A key characteristic of longitudinal data is that observations within the same individual or cluster may be correlated, which motivates most of the statistical methods for the analysis of longitudinal data. Although there have been extensive methodological developments for the analysis of longitudinal data, there are still many emerging issues arising in practice which motivates further research in this area. In particular, missing data, dropouts, and measurement errors are very common in longitudinal studies, and many of these issues need to be addressed simultaneously in order to draw reliable conclusions from the data. Moreover, longitudinal trajectories of observed data are often very complex. Parametric statistical models may not be flexible enough to capture the main features of the longitudinal profiles, so semiparametric or nonparametric statistical models are particularly attractive. Therefore, statistical analyses of complex longitudinal data can be very challenging, and much research remains to be done.

Specifically, the following problems are common in longitudinal studies: (i) longitudinal data may either be continuous or categorical or a mixture of both; (ii) longitudinal data trajectories may be highly complicated, and there may be large variations between individuals; (iii) there are often missing data or dropouts; (iv) some variables may be measured with errors; (v) longitudinal data may be associated with time-to-event data, and joint modelling may be necessary; and (vi) in some studies the number of variables may be large while the sample sizes may be small. In longitudinal data analysis, new statistical methods are required to address one or more of the above problems since standard methods are not directly applicable. For example, missing data or dropouts are almost inevitable in many longitudinal studies, and ignoring missing data or measurement errors or the use of naive methods may lead to severely biased or misleading results (Little and Rubin, 2002; Carroll et al. 2006).

There has been extensive research for the analysis of longitudinal data in the last few decades. Diggle et al. (2002) and Fitzmaurice et al. (2008) provided a comprehensive overview of various models and methods for the analysis of longitudinal data, among others. Commonly used models for longitudinal data include:

- *mixed effects models*: in these models random effects are introduced to incorporate the between-individual variation and the within-individual correlation in longitudinal data;
- *generalized estimating equations (GEE) models*: in these models the mean structure and the correlation structure are modelled separately without distributional assumptions for the data;
- *transitional models*, in these models the within-individual correlation is modelled via Markov structures.
- *nonparametric models* and *semiparametric models*: in these models the mean structures are modelled semiparametrically or nonparametrically or the distributional assumptions are assumed to be nonparametric, so these models are more flexible than parametric longitudinal models.
- *Bayesian models*: prior information or information from similar studies are incorporated for Bayesian inference, and the advance of Markov chain Monte Carlo (MCMC) methods has led to rapid developments of Bayesian methods.

Each of these modelling approaches offers its own advantages and disadvantages. For example, mixed effects models allow for individual-specific or subject-specific inference but require distributional assumptions, and GEE models are robust to distributional assumptions but may be less efficient. Moreover, transitional models may be particularly attractive for discrete data, Bayesian models borrow information from previous or similar studies, and nonparametric or semiparametric models are appealing for complex longitudinal data.

There is also an extensive literature on missing data and measurement errors. For missing data problems, Little and Rubin (2002) and Molenberghs and Kenward (2007) provided an overview of general models and methods. It is known that naive methods such as the complete-case method and the last-value-carried-forward method for missing data problems often lead to biased or misleading results. Formal methods for missing data include

- multiple imputation methods,
- likelihood inference using EM algorithms,
- single imputation methods with variance adjustments,
- weighted GEE methods,
- Bayesian methods.

These formal missing data methods can incorporate missing data mechanisms and provide valid statistical inference. For measurement error problems, Carroll et al. (2006) provided a overview of commonly used models and methods. It is known that naive methods which ignore measurement errors may lead to biased results and appropriate methods must be used for reliable inference. To formally address measurement errors, two general approaches are often considered:

- functional modelling approach, where no distributional assumptions are made for the true but unobserved covariates. For this approach, commonly used methods include regression calibration and simulation extrapolation (SIMEX).
- structural modelling approach, where models or distributions are typically assumed for the true but unobserved covariates. For this approach, commonly used methods include likelihood methods and Bayesian methods.

These formal methods may correct measurement errors and produce reliable statistical inference. For complex longitudinal data with missing values or measurement errors, further research is needed to extend the general ideas and methods.

In summary, analysis of longitudinal data has received much attention, especially in recent years. Although there have been extensive developments of statistical models and methods for the analysis of longitudinal data, there are many complex issues and problems to be addressed or solved, due to the complexities of longitudinal data in practice. Highly complicated longitudinal data arising in practice are challenging for statisticians, but they also provide great opportunities for research and advance of this important subject.

Recent Developments and Open Problems

There has been extensive research in statistical methods for longitudinal data or clustered data. Recent developments are reviewed in Carroll, Ruppert, Stefanski, and Crainiceanu (2006), Fitzmaurice, Davidian, Molenberghs, and Verbeke (2008), McCulloch, Searle, and Neuhaus (2008), Molenberghs and Kenward (2007), and Wu (2009), among others. It is difficult to give a complete list of recent developments due to the massive literature. In the following, we discuss some of the recent developments and some open problems.

In the analysis of longitudinal data, three types of models are commonly used: mixed effects models, GEE models, and transitional models. We first discuss some recent developments for mixed effects models. Mixed effects models introduce random effects in classical regression models for cross-sectional data to account for within-individual correlation and between-individual variation in longitudinal data. Distributional assumptions are often made for the within-individual random errors and for the random effects in the mixed effects models. In practice, the distributional assumptions for random effects may be difficult to check since the random effects are unobservable. Professors Charles McCulloch and John Neuhaus at the University of California at San Francisco are currently studying how mis-specifications of the random effects distributions may affect estimation and inference for generalized mixed effects models. Lai and Shih (2003) considered nonparametric distributions for the random effects, in which the distributions of the random effects in nonlinear mixed effects models are completely unspecified. Lai, Shih, and Wong (2006) proposed a hybrid estimation method for mixed effects models. However, nonparametric methods often require rich within-individual data. Professor Xihong Lin at Harvard University has been investigating semiparametric models for longitudinal data with measurement errors and missing data.

Computational challenges for generalized linear mixed effects models and nonlinear mixed effects models still require further investigation. The likelihood method is a standard estimation approach for generalized linear and nonlinear mixed effects models, but the likelihood functions typically involve multi-dimensional and intractable integrations. Numerical or Monte-Carlo methods may be computationally intensive and may even offer computational difficulties if the dimensions of random effects are not small. Computation may become a major challenge in the presence of missing data and measurement errors. Approximated methods, such as that based on Taylor approximations or Laplace approximations, have been proposed and widely used. However, Lin and Breslow (1996) showed that these approximate methods may be biased for generalized linear mixed models with binary responses. More recently, Joe (2008) showed that the accuracy of these approximate methods may be poor for mixed effects models with binary or count responses, such as logistic regression models with random effects. Lee, Nelder, and Pawitan (2006) have proposed higher order Laplace approximations for generalized linear mixed models. A comprehensive evaluation of these approximate methods is still required. The performance of the foregoing methods for mixed effects models with missing data and measurement errors remains to be investigated.

Generalized estimation equation (GEE) methods are another popular approach for the analysis of longitudinal data. GEE methods only assume the first two moments without specific distributional assumptions, so they are robust against distributional assumptions. For longitudinal data, a working correlation matrix is typically assumed in a GEE method. GEE methods enable one to estimate regression parameters consistently even when the correlation structure is misspecified. For generalized linear mixed effects models, however, Chaganty and Joe (2004) showed that the choices of valid working correlation matrices can be very limited, and inappropriate choices of the working correlation matrices may lead to misleading results. Moreover, under mis-specifications of working correlation matrices, the estimators of the regression parameters can be inefficient. Qu, Lindsay, and Li (2000) and a series of articles thereafter introduced a method of quadratic inference functions that does not involve direct estimation of the correlation parameters, and that remains optimal even if the working correlation structure is misspecified. The idea is to represent the inverse of the working correlation matrix by the linear combination of basis matrices. They showed that under misspecified working correlation assumptions these estimators are more efficient than GEE estimators. This method may be applied to a wide variety of problems, including longitudinal data with missing values and measurement errors, so much research remains to be done. Yi and Cook (2002) and a series of articles thereafter studied GEE methods for clustered longitudinal data with missing values in which the correlation within clusters is incorporated, in addition to correlation between longitudinal measurements.

Transitional models assume Markov structures for longitudinal dependence. Nathoo and Dean (2008) considered multistate transitional models in which at any time point individuals may be said to occupy one of a discrete set of states and interest centers on the transition process between states. They developed a hierarchical modeling framework in which the processes corresponding to different subjects may be correlated spatially over a region and continuous-time Markov chains incorporating spatially correlated random effects are introduced. Yi and Cook (2002), Cook et al. (2004), and their recent work considered marginal models for incomplete longitudinal data arising in clusters. They used odds ratio and Markov structures to model dependence between multivariate discrete longitudinal data, incorporating missing data. Modeling clustered or multivariate longitudinal data with missing values or measurement errors can be challenging, both mathematically and computationally.

Joint modelling longitudinal data and survival data has received much attention in recent years. Such joint models are required in survival models with measurement errors in time-dependent covariates, longitudinal models with dropouts or certain events of interest, and many other situations in longitudinal studies. Professor Joseph Ibrahim is working on diagnostic measures for assessing the influence of observations and model misspecification for joint models of longitudinal and survival data, in the presence of missing data. Professor Jeremy Taylor is studying individual predictions of future event times for censored subjects using joint models. Professor Bin Nan is investigating joint modeling of longitudinal and survival data when the event time is a covariate. Professors Charmaine Dean and Farouk Nathoo are considering longitudinal studies in forestry where trees are subject to recurrent infection and the hazard of infection depends on tree growth over time. They have developed a joint model for infection and growth where a mixed non-homogeneous Poisson process is linked with a spatially dynamic nonlinear model representing the underlying height growth trajectories. Much work remains to be done for joint models with missing data and measurement errors. In particular, during the workshop many people emphasized the importance of software developments for joint models so that these models may be more widely used in practice.

A characteristic of longitudinal studies is that there are often missing data, dropouts, and measurement errors. Thus, in practice when modeling longitudinal data one often also needs to address missing data, dropouts, and measurement errors. Formal approaches for addressing missing data may require modeling of the missing data or dropout processes. As pointed out by Professor Roderick Little at the University of Michigan, it is important to check model assumptions and provide model justifications. When the missing data is nonignorable in the sense that the missingness may depend on the missing values, a missing data mechanism must be assumed and be incorporated in likelihood inference. However, such assumed missing data models are not testable based on observed data. Thus, sensitivity analyses under different missing data mechanisms is required. Professor James Carpenter and Mike Kenward at the University of London organized and led a discussion session in the workshop on sensitivity analysis for missing data problems. There have also been substantial developments in measurement error problems, as reviewed in Carroll, et al. (2006). Professor Xihong Lin has been working on measurement error problems in semiparametric models. Recently, there are interests in jointly modeling missing data and measurement errors. Wang (2004) proposed moment-based methods for nonlinear models with Berkson measurement errors, where only the first two moments are required for estimation and inference without distributional assumptions.

Presentation Highlights

Monday, August 17

The workshop began on Monday, August 17. Professor Charmaine Dean from Simon Fraser University chaired the sessions on Monday morning, and Professor Grace Yi from University of Waterloo chaired the sessions on Monday afternoon. The focus on Monday's talks is on missing data and measurement error problems in longitudinal studies.

The workshop began with an excellent presentation by Professor Roderick Little from University of Michigan. He provided an overview of missing data problems in longitudinal studies. Missing data are very common in longitudinal studies because of attrition, missed visits, dropouts, and other problems. Professor Little emphasized the importance of model assumptions and model justifications, pointed out that clever estimation methods will not help, and one should keep models simple and carefully design the studies to avoid missing data. He discussed the pros and cons of different forms of likelihood inference, Bayes and multiple imputation, robust estimation, GEE methods, the complete-case method, and other ad-hoc methods. He also reviewed selection models and pattern-mixture models for missing data problems, as well as sensitivity analysis. Following Little's presentation, Professor Joan Hu from Simon Fraser University presented a talk on Cox proportional hazards models with non-random missing covariates using a likelihood-based estimation method. Her research is motivated by a study for disease control. Following Hu's talk, Professor Annie Qu from University of Illinois at Urbana-Champaign gave a presentation on analysis of longitudinal data with data missing at random using an estimating function approach. Their approach differs from inverse weighted estimating equations and the imputation methods in that it does not require estimating the probability of missing data or imputing the missing data based on assumed models, and it is based on an aggregate unbiased estimating function which does not require the likelihood function.

Professor Mike Kenward from University of London discussed double robust estimators based on a multiple imputation method for missing data. Their method is based on Bang and Robins (2005) who showed how doubly robust estimators for monotone incomplete data problems can be obtained through a sequence of regressions, and they showed how reformulation of Bang and Robins (2005) approach in a multiple imputation framework leads to very convenient route for calculating doubly robust estimators, while at the same time providing an explicit and easily calculable variance estimator. They also extend the method to non-monotone missing value settings. Following Kenward's talk, Professor James Carpenter from University of London considered a class of models for multivariate mixtures of Gaussian, ordered or unordered categorical responses and continuous distributions that are not Gaussian, each of which can be defined at any level of a multilevel data hierarchy. He described a MCMC algorithm for fitting such models, and shows how this approach can be used to implement multilevel multiple imputation (assuming data are missing at random) and extended to allow imputation of missing data that is congenial/consistent with a complex multilevel model.

Professor Richard Cook from University of Waterloo presented marginal models for estimating treatment effects in cluster randomized longitudinal studies with incomplete responses and non-compliance. He proposed inverse weighted generalized estimating equation methods to address incomplete compliance data in a model for the compliance process and used a mean-scored approach to deal with the missing compliance data in the response model. Following Cook's talk, Dr. Baojiang Chen from University of Washington discussed models for longitudinal data where both the response and the covariates may be missing. A method based on inverse probability weighted generalized estimating equations was proposed, which incorporates the association between the missing data process and the response process.

Professor James Carpenter and Mike Kenward organized and led a discussion session on sensitivity analysis for missing data models. The speakers in the discussion session include Professor Andrea Rotnitky from Harvard University, Professor Joe Ibrahim from the University of North Carolina at Chapel Hill, and Professor Ray Carroll from Texas A & M University. Rotnitky discussed some issues of sensitivity analyses for inference in causal models. Ibrahim discussed Bayesian sensitivity analysis, proposed a perturbation model to simultaneously perturb the data, the prior, and the sampling distribution, and developed a Bayesian perturbation manifold to measure each perturbation in the perturbation model and applied the method to a wide variety of statistical models, allowing for missing data. Professor Raymond Carroll summarized the presentations and discussions in Monday's sessions and raised many interesting questions. For example, are we torturing investigators by doing fancy sensitivity analyses? What about just doing a few different approaches such as multiple imputations, augmented inverse probability weighting methods, and full model-based methods? He also pointed out that the last-observation-carried-forward method or the baseline-observation-carried-forward method may lead to severely biased results.

Tuesday, August 18

The presentations on Tuesday, August 18, focus on functional data, mixed effects models, and estimating equations. Professor Xihong Lin from Harvard University chaired the sessions on Tuesday morning, and Professor John Neuhaus from University of California at San Francisco chaired the sessions on Tuesday afternoons.

Tuesday's sessions began with a presentation by Professor Raymond Carroll from Texas A & M University. He discussed an efficient inference approach for additive models with repeated measures, where the additive model consists of a parametric component and an additive nonparametric component, in the presence of interactions. He derived efficient estimates using smooth backfitting and a Tukey-type 1-degree of freedom formulation, and derived a general profile-type score statistic which involves circumventing the need to solve an integral equation. He also proposed the "Carroll's law of nonparametric regression": "If things work out seamlessly for efficient kernel approaches, then they will work out for efficient spline methods. Thus, one can do theory for kernels and do practice for splines". Following Carroll's talk, Professor Hua Liang from University of Rochester discussed variable selections for semiparametric models with measurement errors. He explored variable selections for partially linear models when the covariates are measured with additive errors, and proposed two classes of variable selection procedures, penalized least squares and penalized quantile regressions, using the nonconvex penalized principle. He showed that the first procedure corrects the bias in the loss function caused by the measurement error by applying the so-called correction-for-attenuation approach, whereas the second procedure corrects the bias by using orthogonal residuals. Following Liang's talk, Professor Lu Wang from University of Michigan considered nonparametric regressions in longitudinal studies with dropout at random. She proposed inverse probability weighted (IPW) kernel generalized estimating equations (GEEs) and IPW seemingly unrelated (SUR) kernel estimating equations using either complete cases or all available cases, and showed that all these IPW kernel estimators are consistent when the probability of dropout is known by design or is estimated using a correctly specified parametric model. She also showed that the most efficient IPW kernel GEE estimator is obtained by ignoring the within-subject correlation while in contrast the most efficient IPW SUR kernel estimator is obtained by accounting for the within-subject correlation and is more efficient than the most efficient IPW kernel GEE counterpart.

Professor Naisyin Wang from University of Michigan presented functional latent feature regression models for data with longitudinal covariate process. She considered a joint model approach to study the association of nonparametric latent features of multiple longitudinal processes with a primary endpoint. She proposed estimation procedures and the corresponding supportive theory that allows one to perform investigation without making distributional assumptions of the latent features, and investigated the practical implications behind certain theoretical assumptions which aim at having a better understanding of where the estimation variation lies. Following Wang's talk, Professor Wenqing He from the University of Western Ontario discussed local linear regressions for clustered censored data. He presented a local linear regression method for the estimation of the relationship between censored response and covariates by considering a transformation of the censored response, and used simulation to assess the performance of the proposed method.

Professor Charles McCulloch from University of California at San Francisco discussed estimation efficiency problems in generalized linear mixed models under misspecified random effects distributions. Previous work has shown that incorrect specification of the random effect distribution typically produces little bias in estimates of covariate effects and very modest inaccuracy in predicted random effects, but few studies have assessed the effect of misspecification on standard errors and statistical tests. Professors McCulloch and Neuhaus examined the impact of a misspecified random effects distribution on estimation efficiency. They showed that linear mixed models are well-behaved in the sense that the random effects influence only the variance-covariance structure, not estimation of the fixed effects. For logistic regression models with random intercepts, they showed that (i) within cluster covariates show no loss of efficiency; (ii) between cluster covariates show loss of efficiency comparable to or better than a linear regression with assumed normal errors, so quite robust; (iii) fitting flexible distributional shapes is an easy way to check sensitivity of the results. Following McCulloch's talk, Mr. Daniel Li from the University of Manitoba presented a second-order least squares estimation method for mixed effects models. Li and Wang applied the second-order least squares method to estimate generalized linear mixed effects models where the distributions of the regression errors are nonparametric while those of random effects are parametric but not necessarily normal. They presented simulation studies of finite sample properties of the second-order least squares estimators and compared them with the maximum likelihood estimators.

Professor Peter Song from University of Michigan discussed analyzing unequally spaced longitudinal data with quadratic inference functions. Quadratic Inference Function (QIF) is getting increasingly popular, as an alternative to the well-known GEE method, to estimate parameters in the marginal models for longitudinal data. One limitation with the QIF is that it is currently only applicable for longitudinal data with equally spaced times. In his talk, Peter Song presented a generalized QIF method to relax this limitation. Following Song's talk, Professor Youngjo Lee from the Seoul National University discussed hierarchical generalized linear models (HGLMs) and variable selection. HGLMs provide a flexible and efficient framework for modeling non-Normal data when there are several sources of error variation, and they extend the familiar generalized linear models (GLMs) to include additional random terms in the linear predictor. Thus HGLMs bring a wide range of models together within a single framework and they also facilitate the joint modeling of mean and dispersion. He also showed how HGLM can be used for variable selection and showed how LASSO and its extension can be obtained via random-effect models.

Wednesday, August 19

The presentations on Wednesday, August 19, focus on joint models for longitudinal data and survival data. Professor Wei Liu from York University chaired the first session, and Professor Charmaine Dean from Simon Fraser University organized and chaired a discussion session on joint models. Wednesday afternoon is free, without formal scientific activities. In Wednesday evening Professor Andrea Rtonitzky from Harvard University offered a three-hour lecture on causal inference.

Professor Joseph Ibrahim from University of North Carolina at Chapel Hill began Wednesday's sessions with a presentation on local influence for joint models for longitudinal and survival data. He discussed diagnostic measures for assessing the influence of observations and model misspecification for longitudinal models and for joint models of longitudinal and survival data, in the presence of missing data. He proposed a local influence approach and examined various perturbation schemes for perturbing the models in this setting, and developed a perturbation manifold and various local influence measures to identify influential points and test model misspecification. Following Ibrahim's talk, Professor Jeremy Taylor from University of Michigan discussed using joint models for longitudinal and survival data to give individual predictions. He considered using a joint model to assist with individual prediction of future event times for censored subjects. The model and methods are developed in the context of a prostate cancer application where the longitudinal variable is PSA and the event time is recurrence of the cancer following treatment with radiation therapy. Estimates of the parameters in the model are obtained by MCMC techniques, and an efficient algorithm is developed to give individual predictions for subjects who were not part of the original data from which the model was developed. Many important statistical issues were discussed in his presentation. Following Taylor's talk, Professor Bin Nan from University of Michigan discussed joint modeling of longitudinal and survival data when the event time is a covariate. His research is motivated from estimation of the hormone profile, such as serum estradiol or follicle stimulating hormone, during menopausal transition. Due to limited follow up time, the age at the final menstrual period for many women in a study cohort is censored. He proposed a two-stage pseudo likelihood approach to estimate the hormone profile during menopausal transition using a nonparametric stochastic mixed model.

Professor Charmaine Dean from Simon Fraser University organized and chaired a discussion session on joint models of longitudinal data and survival data. Various important issues were raised and discussed, such as the current challenges in joint modeling, illustrations of various applications, and benefits and drawbacks of various approaches. Professor Farouk Nathoo from University of Victoria showed an interesting application of joint modeling in spatial statistics where the growth of trees is modelled using nonlinear mixed effects models. He also proposed various approaches for estimation and inference. An important issue that received much discussions is software developments for joint models. Currently, existing software for joint models is very limited. Developments of software, such as R packages, allow joint modelling methods to be more widely used in practice.

On Wednesday evening, Professor Andrea Rtonitzky from Harvard University offered a three-hour lecture on causal inference. The lecture was well received, with many live and interesting discussions.

Thursday, August 20

The presentations on Thursday, August 20, focus on important applications, binary, and count data. Professor Liqun Wang from University of Manitoba chaired the sessions on Thursday morning, and Professor Jiayang Sun from Case Western Reserve University chaired the sessions on Thursday afternoon.

Professor John Petkau from the University of British Columbia began Thursday's sessions with a presentation on an interesting application of neutralizing antibodies and the efficacy of interferon Beta-1b in multiple sclerosis clinical trials. He discussed the question of whether neutralizing antibodies (NAbs) impact on the efficacy of Type I interferons treatments, which is an unresolved scientific issue directly related to the question of how multiple sclerosis (MS) patients should be treated. He also described the initial analyses which raised the concern, and the analyses they have carried out to try to resolve this issue. A fascinating part of their project has been attempting to persuade the neurological community of the need for more detailed analyses of the clinical trial data than is customary in the field to fully address this issue. Following Petkau's talk, Professor Andrea Rtonitzky from Harvard University discussed estimation and extrapolation of optimal dynamic treatments and testing strategies from observational longitudinal data. They considered methods for using the data obtained from an observational database in one health care system to determine the optimal treatment regime for biologically similar subjects in a second health care system when, for cultural, logistical, and financial reasons, the two health care systems differ (and will continue to differ) in the frequency of, and reasons for, both laboratory tests and physician visits. Professor Tze Leung Lai from Stanford University discussed a dynamic empirical Bayes approach to econometric panel data via generalized linear mixed models. He first gave a brief review of the literature on credibility rate-making in insurance and default modeling of corporate loans in finance, particularly on the econometric models used to analyze the associated panel data. Then they proposed a new, unified class of dynamic empirical Bayes models for these longitudinal data and their subject-matter applications. The advantages of these models and their connections to generalized linear mixed models were also discussed and illustrated.

Dr. Taraneh Abarin from Samuel Lunenfeld Research Institute gave a talk on instrumental variable approach to covariate measurement errors in generalized linear models. They proposed a method of moments estimation for generalized linear measurement error models using the instrumental variable approach. They also proposed simulation-based estimators for the situation where the closed forms of the moments are not available. Following Abarin's talk, Mr. Zhijian Chen from University of Waterloo gave a talk on a marginal method for correlated binary data with misclassified responses. Much research in the literature has been directed to problems concerning error-prone covariates, and there is relatively little work on measurement error or misclassification in the response variable. They proposed a marginal analysis method to handle binary response which is subject to misclassification. Numerical studies were presented to assess the performance of the proposed methods.

Professor Renjun Ma from University of New Brunswick organized and chaired a discussion session on random effects modeling of longitudinal data with excessive zeros. He first described various applications of longitudinal data with excessive zeros in different subject areas and interesting datasets, and then he discussed different approaches to random effects modeling of longitudinal data with excessive zeros in the literature (models, estimation methods, etc.) and the relative advantages and limitations of these approaches. He also proposed new approaches to random effects modeling of data with excessive zeros.

Dr. Li Qin from Fred Hutchinson Cancer Center discussed a registration-based functional linear model for post-ART viral loads in patients with primary HIV infection. Traditionally, such data were analyzed by approximate parametric or dynamical models, but the parametric forms may be too restrictive while the dynamical models may be highly assumption dependent. The proposed model presents a trade-off between the parametric models and the flexible functional effects associated with the viral loads. L-splines are used to model the viral loads and account for the plausible monotonicity in the curves over time. Following Qin's talk, Professor Yang Zhao from University of Regina discussed likelihood methods for regression models with data missing at random. She extended the maximum likelihood methods to deal with missing data problems in longitudinal data analysis.

Friday, August 21

On Friday, August 21, Professors Xihong Lin and Grace Yi organized a discussion session on emerging issues of longitudinal data analysis. The session provided a brief summary of the presentations and discussions in the workshop, with discussions of some further issues. The formal session on Friday ended early since many participants needed to catch early shuttles and flights. Informal discussions continued until Friday afternoon. Professor Lang Wu from the University of British Columbia chaired the formal session on Friday.

Professor Xihong Lin from Harvard University discussed analysis of high-dimensional population-based "omics" data. Population-based "Omics" Studies are observational studies, including longitudinal studies, which contain a large number of subjects and massive high-throughout "omincs" data such as genomics, epigenomics, proteomics, and metabolomics. Genome-Wide Association Studies (GWAS) have recently become popular for identifying common gene variants for complex diseases, such as cancers. The goal is to identify genes or gene regions, gene-gene interactions that are associated with a disease or a phenotype. She discussed various approaches to analyze these data, including mixed effects models and GEE methods. Professor Grace Yi from the University of Waterloo provided a comprehensive overview of missing data problems and measurement error problems in longitudinal studies. She reviewed existing approaches, including likelihood methods, weighted GEE methods, selection models, pattern-mixture models, and shared-parameter models. She also discussed how to address measurement errors and missing data simultaneously and other important issues such as how to balance the complexity of modeling and interpretability of model parameters, model identifiability, model checking, sensitivity analysis, and computational issues. Professor James Carpenter from University of London provided a summary of the discussions on sensitivity analysis for longitudinal data with dropout. He argued that the most accessible way to frame discussion is to consider how post-withdrawal measures may differ from missing at random (MAR) predictions (i.e. a pattern mixture approach) and then estimation follows naturally by multiple imputations (MIs). He also summarized comments from Professors Andrea Rotnitsky and Joe Ibrahim on causal modelling ideas, inverse probability weighting methods, and local influence measures. The agreement is that sensitivity analyses should be accessible and relevant. Professor Jiayang Sun from Case Western Reserve University also provided a review on missing data and measurement error problems in longitudinal studies. She advocated alternative models and methods and considered mixed-effects selection and hybrid models, and presented two new non-Fourier density estimation procedures from data with measurement errors.

Outcome of the Meeting

All 42 participants attended the workshop. Among these 42 participants, there are 20 Canadians, 18 females, and 14 graduate students or junior researchers. The workshop also attracted a large number of well-known researchers. The workshop is a great success. Participants had extremely positive experiences with the workshop, and they were very satisfied with facilities, meals, and accommodation at the Banff Center. The workshop had a great impact on the graduate students and junior researchers with respect to their future career plannings. They find themselves greatly benefited from such a workshop. Senior researchers also find the workshop an excellent place for strengthening collaboration and communication. The organizers have received many very positive comments, e.g., “This is the best workshop I have ever been!”, “This is a wonderful workshop!”, “I really benefited a lot from the workshop”. The workshop provides an excellent opportunity for leading and young researchers in the field to discuss recent developments, emerging issues, and future directions in the analysis of longitudinal data.

One of the major goals of the workshop is to strengthen collaboration and communication among different research groups and consolidate existing ones. We have successfully achieved this goal. There were many interesting and active discussions throughout the 5-day workshop. Senior researchers offered their visions, suggestions, and guidance, and junior researchers learned many latest developments and exciting future research opportunities. Overall, the workshop is timely and provides a great platform for collaborative research and interactions between methodological and applied researchers. We find that BIRS is an ideal place for such communications, and we believe that we could not achieve the same results in a “classical” scientific meeting.

Finally, workshop participants have expressed great appreciation to BIRS and Banff Center staff members for the outstanding local arrangements and service. In particular, the workshop organizers would like to express their sincere appreciation to the BIRS Station Manager Brenda Williams and BIRS programme coordinator Wynne Fong for their extremely professional help. We understand that there is a large amount of work involved in the organization and local arrangements for the workshop. The wonderful BIRS staff team has made the workshop a very successful one!

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Chapter 17

New Mathematical Challenges from Molecular Biology and Genetics (09w5062)

Sep 6 - Sep 11, 2009

Organizer(s): Richard Durrett (Cornell University), Ed Perkins (University of British Columbia)

Introduction

This meeting brought together a broad spectrum of researchers across the continuum from mathematics to molecular biology. The coalescent and other genealogical or *dual* processes were a common tool for the study of the effects of natural selection, population subdivision, large family size, etc. on genetic diversity. The motivation for these investigations is, of course, to use various statistics to infer which forces have acted in the evolution of genetic loci. Rather than try to recount all of the many developments, we will highlight some.

Six Exciting Research Directions

Next generation sequencing methods produce a huge number of short DNA reads. Andy Clark described problems in the use of these methods to study gene conversion in tandem arrays of duplicated genes. Mathematical methods need to be developed if we are going to make optimal use of this type of data.

In some bacteria the genic content varies widely between individuals. Peter Pfaffelhuber discussed methods for inferring rates of evolution, which generalize the infinite alleles model but lead to a number of new mathematical and biological questions.

Much of genetics studies the evolution of neutral loci, which have no consequences for the reproduction of individuals. However, in many cases when the mean number of offspring is constant the variance of the number of offspring varies. Jay Taylor investigated the consequences of this fecundity variance polymorphism for genealogies. His model was biologically motivated but also had some fascinating mathematical properties. The models typically produce differences in the (backward) genealogies but not in the forward frequency diffusion models.

It is important to understand the mechanisms of regulatory complex assembly to understand how genes are described. Steve Evans showed that “simple models” with only a dozen states lead to serious mathematical complexities when one wants to symbolically compute formulas rather than just produce simulation curves.

Paul Joyce works with experimentalists studying the evolution of viruses in the laboratory. He showed how extreme value theory could give insights into the distribution of fitness improvements in this system. More specifically although past work assumes the fitness distribution belongs to the more standard Gumbel domain of attraction he considered two other domains from extreme value theory (Weibull and Frechet corresponding to truncated and fat tails, respectively). The lab results suggest that in some cases the theoretical results predicted by assuming it belongs to the Weibull domain of attraction provide a better fit. The combination of elegant mathematics with experimental results was particularly attractive.

Mueller's ratchet refers to the steady accumulation of deleterious mutations in systems, especially those without recombination. Anton Wakolbinger asked "When does the ratchet click rarely?" (in the large population limit). His conjectured answer (with Alison Etheridge and Peter Pfaffelhuber) in terms of the selection and mutation parameters is an interesting open problem which kept some of the participants busy through the meeting. The Fleming-Viot model used to model the ratchet in the large population limit is a discrete type version of a continuous branching model being studied by one of the students (Hardeep Gill) at the meeting.

Substantial progress on a recent topic

Six talks on the second day concerned the Λ-coalescent which occurs when individuals have widely variable number of offspring. Talks discussed how to use the model for inference, compute likelihoods for this model and its site frequency spectrum, investigate its properties when selection acts or individuals are distributed in space. At the biological end, Ori Sargsyan proposed a coalescent model (actually a mixture of Λ coalescents) to describe the multiple mergers of the genealogies of marine species such as the Pacific oyster. At the mathematical end, Jason Schweinsberg showed how this process and the Bolthausen-Sznitzman coalescent in particular arose in a branching Brownian motion with absorption. The latter is proposed as substitute for a fixed population model with selection proposed by Brunet et al who conjectured the Bolthausen-Sznitman coalescent should describe its genealogies.

This meeting allowed individuals who work on this topic to exchange ideas and to explain the workings and consequences of this model to biologists, and also allowed the biologists to present modeling situations where it may arise.

Scientific Progress Made

It is now three weeks since our meeting and it is unrealistic to point to immediate scientific breakthroughs which have been worked out since the meeting. However, a number of new insights were communicated at the meeting and new collaborations have been launched. The best evidence for the benefits of a meeting bringing together biologists and mathematicians is in the letters we have received from the participants (which include at least one "Eureka" after all):

Senior Scientists

Dear Ed and Rick,

Thanks so much for taking the time to set up such a successful week. The BIRS workshop will be helpful to me and my research group in a variety of ways.

- Rasmus Nielsen's work on probabilities of identity by descent and some of Jeff Jensen's approaches using summary statistics in an approximate Bayesian computation may very well be helpful to the researchers in Michael Hammer's lab

- Paul Joyce's small scale, laboratory based, evolutionary models is close in scientific perspective to Joanna Masel, an evolutionary biologist at the University of Arizona. I plan to present some of these results to the Masel group. Joanna and I share a doctoral student, Grant Peterson. Idaho looks like a good postdoctoral opportunity for Grant.

- Jay Taylor and Bob Griffiths have made advances in the work on ancestor selection graphs under a variety of backgrounds. This will be helpful in our group's work on the nature of modeling of genetic resistance to disease for our study area in the Indonesian archipelago.

- Steve Evans approach to regulatory complex assemblies is very similar to an approach used by some of my colleagues and me in the study of allosteric proteins in the presence of a heme. I plan to maintain contact with Steve as we try to add to the list of biochemical systems that will be amenable to this Marlov chain, pinch point methodology.

- Our group is presently working with Ryan Gutenkunst. However, it was a good opportunity to catch up.

Finally, I enjoyed the advances seen on work on the lambda coalescent and on sampling formulae. It was also very helpful to me to make personal connections with collaborators of my collaborators and delightful to have the opportunity to catch up with a few individuals that I have now known for more than a couple of decades.

Best, Joe Watkins, U. Arizona.

Rick and Ed, Great thanks to both of you for organizing a wonderful meeting in a fabulous setting. ... The mathematics of the lambda-coalescent is very interesting, but at this meeting I finally got some examples of marine organisms where this type of model might be useful. All of the talks were interesting, but a few stood out for me because they had a profound effect on my thinking. These talks either provided a new perspective on topics I am or have worked on, or they got me interested in topics that I had yet work on. Andy Clark's talk is a good example of the later. At Idaho, I am involved in a center grant from NIH where, as part of that grant, we invested in the relatively new 454 sequencing. So I have been trying to wrap my head around the implications of this next generation sequencing technology on population genetics. I had never thought about how these new short read sequence data could allow for a much more comprehensive study of gene conversion. It has been a long time since I have thought about this early work of Nagylaki and Ohta, so I am planning to go back and read those old papers as well as Andy's work. Rasmus Nielsen's talk was very insightful...I have been developing methods for detecting overdominance selection, where the primary data for my methods is the HLA region. However Rasmus's clever identity by descent approach shows that overdominance cannot explain the significant increase in IBD. His paper is now a must read for me. Yun Song's talk was really impressive and it made me wonder if the same logic used to get a large recombination approximation to the sampling distribution for the Ancestral Recombination Graph (ARG) could be used to get a similar approximation to the Ancestral Selection Graph (ASG) that was devised by Krone and Neuhauser in the late 90's. This could have a major impact on making ASG useable as an inference tool. I plan to be in contact with Yun to get his views on this problem. I always learn so much from these smaller conferences with a more specific agenda than the big conferences. So I again, thanks for putting this together. Also, BIRS and the Banff Center facilities were great.

Paul Joyce, U. Idaho

Dear Rick and Ed,

Thanks again for hosting the BIRS meeting on Mathematical Challenges from Molecular Biology. I learned a great deal at the meeting, and got an especially concrete take-home from the meeting. We have been working with a group that sequenced two genes in 15,000 people, with the goal to understand the nature of rare variation in humans. Rare variation is all the rage now in human genetics, because people think that it might be responsible for the "missing heritability" – the gap between estimated heritability and the variation explained by genome-wide association studies. Our sample of 15,000 actually exceeds the typical estimates of the human effective population size (10,000), violating assumptions of the Kingman coalescent. I did some scratching and simulating and saw that this should result in some multiple mergers, possibly inflating rare variants. Wakeley and Takahashi wrote a paper on the problem, but I had no idea that the lambda coalescent was generating so much excitement among mathematical population geneticists. I got a great deal from the talks and discussions with Bob Griffiths, Matthias Birkner, Ori Sarasayan and Nathaniel Berestycki. This is work that we are deeply engaged in now, so the timeliness of finding this literature and this gang of experts was perfect.

While I do a certain amount of work in theoretical population genetics, I am not a mathematician, and most of the mathematicians at the meeting could have given talks that would be totally incomprehensible to me. But they found a good compromise level that kept all well engaged. I deeply appreciate this kind of stretch-your-boundaries meeting, and the way the BIRS series is run is just superb.

sincere thanks, Andy

Andrew G. Clark Molecular Biology and Genetics 227 Biotechnology Building Cornell University

Ed and Rick,

1) I had some discussions with Rick Durrett and John Mayberry about cancer modeling, which could help to guide the direction of my future research in this area.

2) I had a short discussion with Jay Taylor about the model that I studied in my talk, which might enable me to formulate more biologically realistic models for future related work.

3) I had a brief discussion with Nathanael Berestycki concerning how to finalize the write-up of our joint paper. We also briefly discussed possible follow-up work.

4) I learned about some intriguing open problems, including a problem about Muller's ratchet from Anton Wakolbinger's talk and a problem about characterizing the possible coalescent processes dual to a given diffusion from Jay Taylor's talk.

5) From the conference overall, I got the impression that coalescent processes with multiple mergers were being taken more seriously by biologists than I had previously thought.

Jason Schweinsberg, U. California, San Diego.

Junior Researchers

As a young researcher, I found the dialogue between mathematicians and biologists at this meeting a refreshing example of what cross disciplinary research can be like. It was a great opportunity to learn more about what type of questions researchers in population genetics and evolutionary biology really care about and I came away from the meeting feeling less timid about actually discussing my work with people in other disciplines. The last two papers I worked on claimed to have biological motivation, but were criticized by referees (and rightly so) for lacking biological relevance. I realize now that any future projects I work on related to the analysis of biological models could be very much improved by increased discussion with actual biologists and hopefully made a couple of contacts this trip that I can write to in the future with questions.

John Mayberry, Cornell U.

I have been working pretty hard on the project I spoke about in Banff for the whole summer, and was fairly happy with how it came together. Only one thing was missing: a theorem describing the behavior for larger distance matrices. Although it's clear that the level of understanding we have of the system for small cases won't be possible in the larger case, one should be able to prove something about it. I was very much hoping to prove such a theorem in the weeks leading up the conference, and had even left a "gap" in the slides for such a theorem. Unfortunately, no such theorem presented itself. However, in the afternoon after giving my talk I had the insight I needed to prove the missing theorem. In fact, the new theorem is quite a bit more general than I had hoped for, and will be one of the main results for this work. I'm not sure if it was a product of the discussions I had at BIRS, the quiet time away from the university, or the stimulating mountain air, but I'm definitely going home with a souvenir!

Thank you again for inviting me!

Erick Matsen, U. California, Berkeley.

Hi Ed and Rick,

Thanks very much for organising the workshop – I really enjoyed it.

This is still an area I'm learning about, so it was incredibly useful to me to have such an array of experts, and many excellent talks, to learn from.

More particularly:

I learnt from Jason's beautiful talk that the Bolthausen-Sznitman coalescent (which I have spent a lot of time studying) might have some biological relevance after all!

It looks like Nathanael, Alison and I might have got started on a collaboration based on a something which came up in discussion following Ori's talk.

Nathanael and I had some interesting conversations about exceptional times for coalescent processes. Likewise, there's work for us to do here!

Best wishes, Christina Goldschmidt, U. Warwick.

List of Participants

Berestycki, Nathanel (Cambridge University)
Betancourt, Andrea (University of Edinburgh)

Birkner, Matthias (Universitaet Mainz)
Clark, Andrew (Cornell University)
Durrett, Richard (Cornell University)
Eldon, Bjarki (Harvard University)
Etheridge, Allison (Oxford)
Evans, Steve (University of California, Berkeley)
Geng, Xin (University of British Columbia)
Gill, Hardeep (UBC)
Goldschmidt, Christina (University of Warwick)
Griffiths, Robert (University of Oxford)
Gutenkunst, Ryan (Los Alamos National Laboratory)
Huerta-Sanchez, Emilia (Berkeley)
Jensen, Jeff (UC Berkeley)
Joyce, Paul (University of Idaho)
Maruki, Takahiro (Arizona State)
Matsen, Frederick (Berkeley)
Mayberry, John (Cornell)
Moehle, Martin (University of Tuebingen)
Nielsen, Rasmus (UC-Berkeley)
Perkins, Ed (University of British Columbia)
Pfaffelhuber, Peter (Freiburg)
Popovic, Lea (Concordia University)
Sanz-Sol, Marta (Universitat de Barcelona)
Sargsyan, Ori (Harvard)
Schmidt, Deena (Ohio State University)
Schweinsberg, Jason (University of California at San Diego)
Song, Yun (UC Berkeley)
Tang, Hua (Stanford University)
Taylor, Jay (Arizona State University)
Wakolbinger, Anton (Goethe-University Frankfurt)
Watkins, Joe (University of Arizona)
Zhang, Nancy (Stanford)

Chapter 18

Linear Algebraic Groups and Related Structures (09w5026)

Sep 13 - Sep 18, 2009

Organizer(s): Alexander Merkurjev (University of California Los Angeles), Vladimir Chernousov (University of Alberta), Zinovy Reichstein (University of British Columbia), Ján Minác (University of Western Ontario)

A brief historical introduction

In the early 19th century a young French mathematician E. Galois laid the foundations of abstract algebra by using the symmetries of a polynomial equation to describe the properties of its roots. One of his discoveries was a new type of structure, formed by these symmetries. This structure, now called a "group", is central to much of modern mathematics. The groups that arise in the context of classical Galois theory are finite groups.

Galois died in a duel at the age of 20; his work was not understood or recognized during his lifetime. It took much of the rest of the 19th century for his ideas to be rediscovered, absorbed and applied in other contexts. In the context of differential equations, these ideas were advanced by E. Picard, who, following a suggestion of S. Lie, assigned a Galois group to an ordinary differential equation. This group is no longer finite. It naturally acts on the n-dimensional complex vector space V of holomorphic solutions to the equation. In modern language, the Galois groups that arose in Picard's theory are algebraic subgroup of $\mathrm{GL}(V)$.

This construction was developed into differential Galois theory by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley, J.-P. Serre, T. A. Springer, and J. Tits starting in the 1950s. From the modern point of view algebraic groups are algebraic varieties, with group operations given by algebraic morphisms. Linear algebraic groups can be embedded in GL_n for some n, but such an embedding is no longer a part of their intrinsic structure. Borel, Chevalley, Serre, Springer and Tits used algebraic geometry to establish basic structural results in the theory of algebraic groups, such as conjugacy of maximal tori and Borel subgroups, and the classification of simple linear algebraic groups over an algebraically closed field. Considerations in number theory, among others, require the study of algebraic groups over fields that are not necessarily algebraically closed. This more general setting was the primary focus for much of the work discussed in the workshop.

In the 1960s J. Tate and J.-P. Serre developed a theory of Galois cohomology. Serre published his influential lecture notes on this topic in 1964; they have been revised and reprinted several times since then. Galois cohomology can be viewed as an important special case of étale cohomology,

In the 1970s the work of H. Bass, J. Tate and Milnor, established connections among Milnor K-theory, Galois cohomology, and graded Witt rings of quadratic forms. In particular, Milnor asked whether (in modern language) Milnor K-theory modulo 2, is isomorphic to Galois cohomology with $\mathbb{F}_2$ coefficients. A more general question, with 2 replaced by an odd prime, was posed in subsequent work of Bloch and Kato and became known as the Bloch-Kato conjecture.

Since the 1980s there has been rapid progress in the theory of algebraic groups due to the introduction of powerful new methods from algebraic geometry and algebraic topology. This new phase began with the Merkurjev-Suslin theorem which settled a long-standing conjecture in the theory of central simple algebras, using a combination of techniques from algebraic geometry and K-theory. The Merkurjev-Suslin theorem was a starting point of the theory of motivic cohomology constructed by V. Voevodsky. Voevodsky developed a homotopy theory in algebraic geometry similar to that in algebraic topology. He defined a (stable) motivic homotopy category and used it to define new cohomology theories such as motivic cohomology, K-theory and algebraic cobordism. Voevodsky's use of these techniques resulted in the solution of the Milnor conjecture for which he was awarded a Fields Medal in 2002. For a discussion of the history of the Milnor conjecture and some applications, see [16]. The Bloch-Kato conjecture was recently proved by Rost and Voevodsky; see [51, 58, 59, 60, 61, 62, 63].

Recent Developments

Quadratic forms

In the last 20 years there has been a virtual revolution in the theory of quadratic forms. Using motivic methods and Brosnan's Steenrod operations on Chow groups, Merkurjev, Karpenko, Izhboldin, Rost, Vishik and others have made dramatic progress on a number of long-standing open problems in the field. In particular, the possible values of the u-invariant of a field have been shown to include all positive even numbers (by A. Merkurjev, disproving a conjecture of Kaplansky), 9 by O. Izhboldin, and every number of the form $2^n + 1, n \geq 3$ by A. Vishik. (Vishik's result was first announced at our 2006 BIRS workshop.) Another breakthrough was achieved by Karpenko, who described the possible dimensions of anisotropic forms in the nth power of the fundamental ideal I^n in the Witt ring, extending the classical theorem of Arason and Pfister.

In [45] R. Parimala and V. Suresh settled the open question of whether the u-invariant of function fields of p-adic curves is 8 affirmatively if the p-adic field is non-dyadic. Their work relies upon the previous work of D. Saltman on Galois cohomology and on the work of Kato on certain unramified cohomology groups. In a completely different way using patching methods in Galois theory, D. Harbater, J. Hartmann, and D. Krashen reproved this result in [21]. Recently R. Heath-Brown used analytical methods to obtain sufficient conditions for common zeros of systems of quadratic forms over p-adic fields and this result was used by D. Leep to show in particular that the u-invariant of $\mathbb{Q}_p(t_1, \ldots, t_n)$ is 2^{n+2}. This extends the work of [45] and [21] in two significant ways: the transcendence degree need not be 1, and the prime p can be 2. Leep's work is not yet available in the preprint form.

Algebraic surfaces

An important development in the theory of central simple algebras is the proof by A. J. de Jong, of the long standing period-index conjecture; see [9]. This conjecture asserts that the index of a central simple algebra defined over the function field of a complex surface coincides with its exponent. Previously this was only known in the case where the index of a central simple algebra had the form $2^n \cdot 3^m$ (this earlier result is due to M. Artin and J. Tate). In a subsequent paper de Jong and J. Starr found a new striking solution of the period-index problem by constructing rational points on families of Grassmannians. Yet another geometric approach for the index-period problem was developed by M. Lieblich. Lieblich's approach is based on constructing compactified moduli stacks of Azumaya algebras and studying their properties. Using his geometric methods, M. Lieblich in particular was able to prove a variant of the period-index conjecture for a Brauer group of a field of transcendence degree 2 over $\mathbb{F}_p$. (See [35].)

Similar methods were used by A. J. de Jong, X. He, and J. Starr to establish Serre's conjecture II in the geometric case by showing that every G-torsor over the function field of a complex surface is split. (Here the linear algebraic group G is assumed to be connected and simply connected.) For details, see [15].

The methods they used and their refinements are likely to play an important role in future research on currently open problems in the theory of algebraic groups.

Cohomological invariants

Many fundamental questions in algebra and number theory are related to the problem of classifying G-torsors and in particular of computing the Galois cohomology set $H^1(k,G)$ of an algebraic group defined over an arbitrary field k. In general the Galois cohomology set $H^1(k,G)$ does not have a group structure. For this reason it is often convenient to have a well-defined functorial map from this set to an abelian group. Such maps, called cohomological invariants have been introduced and studied by J-P. Serre, M. Rost and A. Merkurjev. Among them, the Rost invariant plays a particularly important role. This invariant has been used by researchers in the field for over a decade but the details of its definition and basic properties have not appeared in print until the recent publication of the book [11] by S. Garibaldi, A. Merkurjev and J.-P. Serre.

This book, together with the previous book of M. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol ([29]) have become standard reference sources for current research in algebraic groups.

Galois theory

Let F be a field containing a primitive p-th root of 1. D. Benson, N. Lemire, J. Mináč and J. Swallow recently gave a complete classification of the non-trivial pro-p-groups G with a maximal closed subgroup which is abelian and of exponent p which are realizable as $G_F/G_E^p[G_E,G_E]$ where G_F is an absolute Galois group and G_E is a subgroup of index p in G_F, was obtained (see [6]).

They also used the Bloch-Kato conjecture to produce new examples of pro-p-groups which cannot be realized as absolute Galois groups.

Consider the p-descending central series $G_F = G_F^{(1)} \supset G_F^{(2)} \supset G_F^{(3)} \supset \dots$, where $G_F^{[i+1]} = (G_F^{(i)})^p$ $[G_F, G_F^{(i)}]$, and set $G_F^{[i]} = G_F/G_F^{(i)}$.

In the recent paper [11] it is shown that $G_F^{[3]}$ is a Galois-theoretic analogue of Galois cohomology. This group controls Galois cohomology (as a subring of its cohomology ring generated by one-dimensional classes) and $G_F^{[3]}$ can be constructed using Galois cohomology and Bockstein elements in $H^2(G_F^{[2]},\mathbb{F}_p)$. This is used in obtaining examples of interesting families of pro-p-groups which cannot be realized as absolute Galois groups. The group $G_F^{[3]}$ is interesting. On the one hand, it controls important arithmetic information about the field F, including all non-trivial valuations and orderings. On the other hand, the structure of this pro-p-group appears to be fairly accessible and should be studied further.

Essential dimension

Essential dimension is a numerical invariant of an algebraic group G, which, informally speaking, measures the complexity of G-torsors over fields. It is is usually denoted by $\mathrm{ed}(G)$.

For finite groups the notion of essential dimension was introduced in 1997 by Buhler and Reichstein in [8, 9] as a natural byproduct of their study of classical questions about simplifying polynomials by Tschirnhaus transformations and algebraic variants of Hilbert's 13th problem. There is also an interesting connection with generic polynomials and inverse Galois theory; see [8], [24, Section 8].

Essential dimension was then defined and studied for (possibly infinite) algebraic groups by Reichstein [49] and Reichstein–Youssin [50]. In this context the theory of essential dimension is a natural extension of the theory of "special groups" initiated by J.-P. Serre in [56]. Over an algebraically closed field k special groups are precisely those of essential dimension 0; these groups were classified by A. Grothendieck [20]. The essential dimension may thus be viewed as a numerical measure of how far a given algebraic group G is from being special. Another such measure is the related invariant of the canonical dimension of G; see [4, 28, 64].

Between 2000 and 2007 the essential dimension has been computed for a number of algebraic groups, using a variety of techniques. One interesting connection is with the notion of cohomological invariant, previously studied by Rost, Serre and others (see Section 2.3): if G has a cohomological invariant of degree d then $\mathrm{ed}(G) \geq d$. Another highly fruitful connection is with the existence of non-toral finite abelian subgroups in G; every such subgroup gives a lower bound on the essential dimension of G; see [50] and [19].

Initially these results were obtained over an algebraically closed base field of characteristic 0, many were then proved under milder assumptions on k; see [3, 13]. On the other hand, even over the field of complex numbers, for many groups G, the problem of computing the essential dimension of G remains wide open. For example, for all but finitely many values of n the projective linear group PGL_n, or the symmetric group S_n are in this category; in this cases the problem of computing $\mathrm{ed}(G)$ is closely related to classical questions in Galois theory and the theory of central simple algebras, respectively. Even for finite cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$ viewed as algebraic groups over the field of rational numbers, the exact value of $\mathrm{ed}(G)$ is not known for most n.

Merkurjev [39] and Berhuy–Favi [3] have further extended the notion of essential dimension to a covariant functor. In this setting the essential dimension of an algebraic group is recovered from its Galois cohomology functor $H^1(*, G)$.

Important developments in this subject have occurred over the past 3 years. The first breakthrough was due to Florence [16] who computed the essential dimension of cyclic p-groups $\mathbb{Z}/p^r\mathbb{Z}$ over a field containing a primitive pth root of unity.

Next came a key idea, due to Brosnan, to study essential dimension in the context of algebraic stacks. To a stack X defined over a field k one associates the functor

$$K \mapsto \text{isomorphism classes of } K\text{-points of } X$$

for any field extension K/k. The essential dimension of X is then defined as the essential dimension of this functor. The class of functors of this form turns out to be broad enough to include virtually all interesting examples, yet geometric enough to be studied by algebro-geometric techniques. There are many important stacks in algebraic geometry, e.g., the moduli stacks of smooth (or stable) curves of genus g or moduli stacks of principly polarized abelian varieties, and it is natural to ask what essential dimensions of these stacks are. These questions are answered in [7].

What is perhaps, more surprising is that stack-theoretic methods have led to strong new lower bounds in the "classical" situation, for some algebraic groups G. Note that in the language of stacks the essential dimension of an algebraic group G is the essential dimension of the classifying stack $\mathrm{B}G$. A key role in establishing this connection is played by the above-mentioned notion of canonical dimension and an incompressibility theorem of Karpenko for p-primary Brauer-Severi varieties [25]. Brosnan, Reichstein and Vistoli [5, 6] recovered Florence's results from this point of view and computed the essential dimension of the spinor group Spin_n for most values of n. Surprisingly, $\mathrm{ed}(\mathrm{Spin}_n)$ increases exponentially in n, while previous lower bounds were linear in n.

Karpenko and Merkurjev [28] refined the techniques of [5] and combined them with new results on Brauer-Severi varieties to give a simple formula for the essential dimension of any finite p-group G over a field containing a primitive pth root of unity. This is a far-reaching extension of the work of Florence [16]. A key ingredient of the proof is an extension of Karpenko's incompressibility theorem to products of p-primary Brauer-Severi varieties.

The Karpenko-Merkurjev theorem and its methods of proof have greatly influenced the research in the area over the past two years. In particular, it led to the solution of several previously open questions about essential dimension; see [42]. There has also been much work on extending Karpenko's Incompressibility Theorem to other classes of varieties, e.g., Hermitian spaces [55] or generalized Brauer-Severi varieties [26]. In [38] the techniques used in the proof of the Karpenko-Merkurjev theorem are further refined to give a general formula for the essential dimension of a larger class of groups, which include twisted p-groups and algebraic tori.

The latter formula was recently used by Merkurjev, in combination with the techniques developed in [40], to give striking new lower bounds on the essential dimension of PGL_n, where $n = p^r$ is a prime power. He shows that $\mathrm{ed}(\mathrm{PGL}_n) \geq (r-1)p^r + 1$. For $r = 2$ this was shown in [40] (and for $r = p = 2$ in [52]). For $r \geq 3$ the best previously known bound was $\mathrm{ed}(\mathrm{PGL}_n) \geq 2r$.

Lectures delivered at the workshop

For the purpose of this report we have grouped the 27 lectures presented in the workshop into seven sections as follows. Note that work of the participants is quite interlocked, and some of the talks relate to more than one of these topics.

1. Quadratic forms,

2. Algebraic surfaces,
3. Galois theory and Galois cohomology,
4. Essential dimension,
5. K-theory, Chow groups and Brauer-Severi varieties,
6. Structure of algebraic groups,
7. Representation theory of algebraic groups.

We will now briefly report on the content of each lecture.

Quadratic forms

Asher Auel: "A Clifford invariant for line bundle-valued quadratic forms".

Line bundle-valued quadratic forms on schemes were first implicitly considered in the early 1970s by Geyer, Harder, Knebusch, and Scharlau to study residue theorems, and by Mumford to study theta characteristics. Motivated by the triangular Witt and Grothendieck-Witt groups introduced by Balmer and Walter, and by the investigation of Azumaya algebras with involution on schemes by Knus, Parimala, Sridharan, and Srinivas, the theory of line bundle-valued quadratic forms has only recently taken on its own significance.

A line bundle-valued quadratic form $(\mathcal{E}, q, \mathcal{L})$ on a scheme X (where 2 is invertible) is the data of a locally free $\mathcal{O}_X$-module (vector bundle) $\mathcal{E}$ of finite rank, an invertible $\mathcal{O}_X$-module (line bundle) $\mathcal{L}$, and a symmetric $\mathcal{O}_X$-module morphism $q : \mathcal{E} \otimes \mathcal{E} \to \mathcal{L}$. A classical quadratic form on X is a line bundle-valued quadratic form with values in the trivial line bundle $\mathcal{O}_X$. A line bundle-valued quadratic form may be thought of as a family, over the points of X, of vector spaces with a quadratic forms taking values in a one dimensional vector space without a fixed choice of basis. Important examples arise from the middle exterior powers of cotangent bundles of smooth varieties of dimension divisible by 4.

The first natural cohomological invariant of a quadratic form, the discriminant, generalizes to line-bundle valued quadratic forms of even rank by the work of Parimala and Sridharan. This current work concerns the construction of the second natural invariant, the Clifford invariant, to line bundle-valued quadratic forms. The classical construction of the Clifford invariant (of an even rank quadratic form) as the Brauer class of the full Clifford algebra does not generalize to line bundle-valued quadratic forms. By the work of Bichsel and Knus, there is no full Clifford algebra of a line bundle-valued form with values in a nonsquare line bundle. This can be interpreted as the nonexistence of a natural "spin" cover of the group of orthogonal similitudes. In its place we have constructed a natural four-fold cover of the group of proper orthogonal similitudes by the even Clifford group. This yields an étale cohomological invariant of line bundle-valued forms of trivial discriminant and rank divisible by 4. This invariant has the novel feature of residing in the 2nd étale cohomology group with μ_4-coefficients $H^2_{\text{ét}}(X, \mu_4)$ and interpolating between the classical Clifford invariant and the 1st Chern class modulo 2 of the value line bundle. In low dimensional cases, this invariant recaptures the classifications of line bundle-valued quadratic forms in terms of reduced norms and pfaffians.

The work of Parimala and Scharlau on the Witt groups of curves over local fields provides examples of 2-torsion Brauer classes that are not represented by the Clifford invariants of quadratic forms. This seems to contradict Merkurjev's theorem over schemes. To the contrary, we conjecture that in the case of curves over local fields, all 2-torsion Brauer classes are represented by Clifford invariants of line bundle-valued quadratic forms.

Eva Bayer-Fluckiger: "Hasse principle for automorphisms of lattices".

An integral lattice is a pair (L, b), where L is a free $\mathbb{Z}$-module of finite rank, and $b\colon L \times L \to \mathbb{Z}$ is a non-degenerate symmetric bilinear form. Over $\mathbb{R}$ we can write b in the diagonal form form $\langle 1, \ldots, 1, -1, \ldots, -1\rangle$. The signature of (L, b) is then defined as (r, s) where r is the number of 1's and s is the number of -1's. We say that b is definite if r or s is 0. Otherwise b is indefinite. (L, b) is called even if $b(x, x) \in 2\mathbb{Z}$, for all $x \in L$.

Fact: $(r - s)$ is divisible by 8.

Assume that $t \in SO(L, b)$ and $r + s = \mathrm{rank}(L)$ is even. Then the characteristic polynomial $f(x) \in \mathbb{Z}[x]$ of t is reciprocal, i.e., $f(x) = x^{\deg f} f(x^{-1})$. Conversely, given a reciprocal polynomial $f(x) \in \mathbb{Z}$, we define:

Definition *(L, b) is an f-lattice if (L, b) is even, unimodular, and there exists $t \in \mathrm{SO}(L, b)$ whose characteristic polynomial equals f.*

Questions. 1) *For which $f \in \mathbb{Z}[x]$ does there exist an f-lattice?* **2)** *For which $f \in \mathbb{Z}[x]$ does there exist an f-lattice with a prescribed signature (r, s)?*

These questions are solved in the definite case. In the indefinite case, D. Gross and C. McMullen provided the necessary conditions on f. These conditions are conjecturally also sufficient if f is irreducible. D. Gross and C. McMullen proved this conjecture if $|f(1)| = |f(-1)| = 1$.

Bayer-Flückiger's main result is the following Hasse Principle for Question 1) above.

Theorem. (Eva Bayer-Fluckiger) *There exists an f-lattice over $\mathbb{Z}$ iff there exists an f-lattice over $\mathbb{Z}_p$.*

Bayer-Flückiger also briefly discussed a similar but somewhat more complicated Hasse Principle for Question 2). She concluded her lecture with several examples.

Detlev Hoffmann: "Differential forms and bilinear forms under field extensions".

The behaviour of algebraic objects such as Galois cohomology groups, Milnor K-groups or quadratic forms under field extensions is an important problem in the study of these objects. For example, a crucial part in the proof of the Milnor conjecture by Orlov-Vishik-Voevodsky relating Milnor K-groups modulo 2 and the graded Witt ring was the determination of the kernel of the map $K_n^M(F)/2 \to K_n^M(E)/2$ between Milnor K-groups modulo 2, where $E = F(q)$ is the function field of a particular type of quadric (given by a certain Pfister neighbor) over a field F of characteristic not 2. In the proof of the Bloch-Kato conjecture, such kernels are again important for field extensions given by function fields of so-called norm varieties as defined by Rost. Another example that has been studied extensively is the behaviour of Witt rings (in characteristic not 2) under field extensions. In general, determining such kernels is very difficult, and only few results are known. For instance, in characteristic not 2, a complete determination of Witt kernels $W(E/F) = \ker(WF \to WE)$ for arbitrary algebraic extensions of degree $[E : F] = n$ is only known for n odd (where the kernel is trivial due to Springer's theorem), for $n = 2$ (easy and well known) and $n = 4$ (proved by Sivatski only in 2008).

Here, we consider the case of a field F of characteristic 2 and the Witt ring WF of symmetric bilinear forms over F. It turns out that in this situation, Witt kernels $W(E/F)$ can be determined explicitly for a large class of field extensions going far beyond what is known in the case of characteristic not 2. Let $X = (X_1, \ldots, X_n)$ be an n-tuple of variables ($n \geq 1$), and let $g(X) \in F[X]$ be irreducible. The function field $E = F(g)$ is defined to be the quotient field of the integral domain $F[X]/(g)$. If $n = 1$, E is nothing else but a simple algebraic extension. For $n \geq 2$, one obtains function fields of hypersurfaces. We derive a complete and explicit description of $W(E/F)$ in terms of the coefficients of the polynomial $g(X)$. The proof relies heavily on the use of differential forms. More precisely, let F now be a field of positive characteristic $p > 0$ and let $\Omega^n(F)$ denote the Kähler differentials in degree n over F (with respect to the prime field $\mathbb{F}_p$). We compute the kernel $\Omega^n(E/F)$ for function field extensions $E = F(g)$ for arbitrary irreducible $g(X) \in F[X]$. In the case $p = 2$, one can then use a famous theorem by Kato and results by Aravire-Baeza to compute the kernels $I^n/I^{n+1}(E/F)$ for the graded Witt ring, from which the result on $W(E/F)$ follows by some standard arguments.

Algebraic surfaces

Mark Blunk: "del Pezzo surfaces of degree 6 and derived categories".

M. Blunk's thesis focuses on an explicit description of certain geometrically rational surfaces, *del Pezzo surfaces of degree 6*. He relates del Pezzo surfaces of degree 6 over an arbitrary field F to the following algebraic information: a triple (B, Q, KL), consisting of a separable algebra B of constant rank 9 with center K étale quadratic, a separable algebra Q of constant rank 4 with center L étale cubic, such that B and Q contain $KL := K \otimes_F L$ as a subalgebra, and the corestrictions $\mathrm{cor}_{K/F}(B)$ and $\mathrm{cor}_{L/F}(Q)$ are *split*, i.e, isomorphic to matrix rings. The main result is:

Theorem 0.1. There are bijections, inverse to each other, between the following two sets: The set of isomorphism classes of del Pezzo surfaces of degree 6 over F, and the set of triples (B, Q, KL), modulo the relation: $(B, Q, KL) \sim (B', Q', K'L')$ if and only if there are F-algebra isomorphisms $\phi_B : B \to B'$ and $\phi_Q : Q \to Q'$ such that ϕ_B and ϕ_Q agree on their restriction to the subalgebra KL. This restriction is then an isomorphism of F-algebras from KL to $K'L'$.

B and Q can be realized as the global endomorphism rings of two vector bundles $\mathcal{I}$ and $\mathcal{J}$ on S. M. Blunk is able to use these vector bundles to give an explicit description of the K-theory of the surface S.

Theorem 0.2. $K_n(S) \cong K_n(F) \oplus K_n(B) \oplus K_n(Q)$, where K_n is the nth Quillen K-functor.

Similarly, the vector bundles $\mathcal{I}$ and $\mathcal{J}$ can be used to relate the derived category of coherent sheaves on S to the derived category of finitely generated modules over the ring $A = \mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus \mathcal{I} \oplus \mathcal{J})$, a finite dimensional F-algebra with semi-simple quotient $F \times B \times Q$. In particular, the functor $\mathrm{Hom}(\mathcal{T}, -) : \mathrm{Coh}(S) \to \mod A$ induces a natural equivalence $\mathrm{RHom}(\mathcal{T}, -) : D^b(\mathrm{Coh}(S)) \tilde{\to} D^b(\mod A)$.

Daniel Krashen: "Patching topologies and local global principles". (Joint work with D. Harbater and J. Hartmann.)

Patching methods were successfully used by D. Harbater in Galois theory. He proved in particular that every finite group is a Galois group of a regular extension of $\mathbb{Q}_p(t)$. Recently some other exciting results in patching theory and its applications to u-invariants in quadratic forms and Brauer groups were obtained by D. Krashen, D. Harbater and J. Hartman. This talk is a preliminary report on the further development of patching theory. Its aim is twofold: to pay a special attention to the relationship between factorization and local-global principles and second, to extend the basic factorization result to the case of retract rational groups, thereby answering a question posed by Colliot-Thélène.

Broadly speaking, for a given field F the patching method is a procedure for constructing new fields F_ξ which will be in certain ways simpler than F and to reduce problems concerning F to problems about various F_ξ. The focus of Krashen's talk was the function field F of a p-adic curve X and different kind of geometric objects associated to it. Using geometric methods Krashen introduced a kind of "completions" F_ξ of F and using patching technique he talked about local-global principles for Brauer groups, quadratic forms, homogeneous varieties and etc. The details, references and some examples are in [31].

Raman Parimala: "Degree three Galois cohomology of function fields of surfaces". (Joint work with V. Suresh.)

A few years ago Parimala and Suresh proved a long standing conjecture that the u-invariant of the function field of a curve over a p-adic field where $p \neq 2$ is 8. Their proof heavily depends on properties of degree three Galois cohomology of function fields of curves. In her talk Parimala discussed local-global principle for degree three Galois cohomology of function fields of surfaces.

Theorem. (Parimala and Suresh). Let X is a regular 2-dimensional, excellent integral scheme, $F = F(X)$, $l \in \mathcal{O}_x^*$, $\mu_l \in F$. Let Ω be the set of discrete valuations of F associated to the points of $x \in X^1$ of codimension 1. Suppose $H^3_{nr}(F(X), \mu_l) = 0$, and $H^2_{nr}(k(x), \mu_l) = 0, \forall x \in X^1$. Then an element $\xi \in H^3(F, \mu_l)$ is divisible by $\alpha = (a)(b) \in H^2(F, \mu_l)$ if and only if it is divisible locally for all $v \in \Omega$.

Parimala also explained several applications of this local-global principle in computing u-invariant, studying properties of a conic fibration $Y \to X$ where X is a smooth projective surface over a finite field and describing 0-cycles of varieties over global fields.

David Saltman: "Ramification in bad characteristic".

In the past, Saltman obtained important results on central simple algebras over function fields of p-adic curves, by carefully examining ramifications. These results were used by R. Parimala and V. Suresh in showing that a u-invariant over a non-dyadic p-adic function field, is 8, and they are also clearly of independent interest. One particularly interesting motivation is the long-standing problem of whether each division algebra of degree p is cyclic.

In his talk, D. Saltman examined the most difficult case of mixed characteristic. Let S be a nonsingular surface with a field of fractions $K = F(S)$. For every curve $C \subset S$ consider the stalk $\mathcal{O}_{s,c}$. Then $\mathrm{Br}(S) = \cap_{\,C\subset S}\ \mathrm{Br}\ (\mathcal{O}_{S,c}) \leq \mathrm{Br}(F(S))$.

The key problem is to describe ways to split a central simple algebra α over $F(S)$ where the order of α in the Brauer group is not a unit in the residue field. In order to focus on the main difficulty, the following case investigated by K. Kato, was discussed.

K = a fraction field of R, R is a discrete valuation ring, char $K = 0$, char $\bar{R} = p \neq 0, K$ is complete, $[\bar{R} : \bar{R}^p] = p, e = v(p)$ = ramification index, $N = \frac{ep}{p-1}$, K contains a primitive pth-root of unity. (Hence $(p-1)/e)\ \mathrm{br}(K)$ = elements in the Brauer group of K of order p.

The filtration on units induces filtration on $\mathrm{br}(K)$: $\mathrm{br}(K)_0 \supseteq \mathrm{br}(K)_1 \supseteq \cdots \supseteq \mathrm{br}(K)_{N+1} = \{0\}$. Kato proved: **a**) $\mathrm{br}(K)_0/\mathrm{br}(K)_1 = k^*/k^{*p}$ $(k = \bar{R} =$ residue field of $R)$, **b**) $\mathrm{br}(K)_i/\mathrm{br}(K)_{i+1} = \Omega_k$ if $p \nmid i$, **c)** $\mathrm{br}(K)_i/\mathrm{br}(K)_{i+1} = k^+/k^{+p}$ if $p \mid i$, **d)** $\mathrm{br}(K)_n \cong H^1(k, \mathbb{Q}/\mathbb{Z})$.

Moreover, every element in **a), b), c)** can be represented by a single symbol and can be split by a pth-root of some unit. Saltman discussed several ideas, conjectures and examples in this setting.

Jason Starr: "Rational simple connectedness and Serre's "Conjecture II" ".

Starr's lecture was devoted to the ideas surrounding his recent work with de Jong on the existence of rational sections to fibrations $X \to B$ over an algebraic surface B and its application to Serre's Conjecture II. Recall that this conjecture says that the Galois cohomology set $H^1(F, G) = \{1\}$ for any semisimple simply connected algebraic group G defined over a perfect field F of cohomological dimension at most 2. Equivalently, the question is whether every G-torsor over Spec (F) is trivial. For history and details we refer to the survey [18].

The proof of the geometric case of Serre's Conjecture II (i.e. when F is the function field of a surface over an algebraically closed field k) in [15] is an outgrowth of a project of finding an algebro-geometric analogues of the topological notion of "r-connectedness". The notion of 1-connectedness (also known as rational connectedness) is well understood; the existence of a rational section of $X \to B$ where B is a curve over k and fibers are geometrically connected varieties is a celebrated theorem of Graber, Harris and Starr. The definition of 2-connectedness (also known as rational simple connectedness) is considerably more complicated, but it also implies the existence of a rational section of $\phi : X \to B$ under some natural mild conditions on X, B and ϕ.

In his talk Starr explained how these results are used to complete the proof of Serre's Conjecture II over function fields using P. Gille's inductive strategy.

Galois Theory and Galois Cohomology

Sanghoon Baek: "Cohomological invariants of simple algebras".

Let $A : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor. J.-P. Serre defined an *invariant* of A with values in a cohomology theory H (viewed as a functor from Fields/F to Sets) to be a morphism of functors $A \to H$. All the invariants of A with values in H form a group $\mathrm{Inv}(A, H)$. When $A = H^1(-, G)$ for an algebraic group G, we simply write $\mathrm{Inv}(G, H)$ for the group $\mathrm{Inv}(A, H)$. In particular, the cases $A = H^1(-, \mathbf{PGL}_n)$ and $A = H^1(-, \mathbf{GL}_n/\boldsymbol{\mu}_m)$ with m dividing n, i.e., the problems of classifications of invariants of central simple algebras of degree n and central simple algebras of degree n and exponent dividing m, respectively, are still wide open.

Let D be a central simple algebra over a field F. Denote by q_D the quadratic form on D defined by $q_D(x) = \mathrm{Trd}(x^2)$ for $x \in D$, where Trd is the reduced trace form for D. Let $e_n : I^n(F) \to H^n(F)$ be the cohomological invariant for the quadratic form, where $H^n(F) := H^n(F, \mathbb{Z}/2\mathbb{Z})$. Recently, M. Rost, J.-P. Serre and J.-P. Tignol showed that q_D decomposes in the $W(F)$ as the sum of a 2-fold Pfister form q_2 and a 4-fold Pfister form q_4 for $D \in H^1(-, \mathbf{PGL}_4)$ over a base field F such that char$(F) \neq 2$ and $-1 \in F^{\times 2}$. This provides cohomological invariants e_2 and e_4 given by $D \mapsto e_2(q_2)$ and $D \mapsto e_2(q_4)$ respectively. Another type of cohomological invariants for central simple algebras is from the *divided power* operation: $\gamma_n : K_i(F)/p \to K_{ni}(F)/p$ defined by $\gamma_n(\sum_{j=1}^r \alpha_j) = \sum_{1\leq j_1 < \cdots < j_n \leq r} \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_n}$, where the α_j are symbols of degree i. In particular, for $p = 2$ and $i = 2$, we have $\gamma_n : \mathrm{Br}_2(F) \simeq k_2(F) \to k_{2n}(F) \simeq H^{2n}(F)$. Restricting the divided powers on the subfunctor $H^1(-, \mathbf{GL}_{2^k}/\boldsymbol{\mu}_2) \subset \mathrm{Br}_2$ we view the γ_n as invariants of $\mathbf{GL}_{2^k}/\boldsymbol{\mu}_2$. Baek discussed his joint work with A. Merkurjev on the invariants of $\mathbf{GL}_{2^k}/\boldsymbol{\mu}_2$ for $1 \leq k \leq 3$. They proved that $\mathrm{Inv}(\mathbf{GL}_n/\boldsymbol{\mu}_2, H)$ is free $H(F)$-module with basis $\{1 = \gamma_0, \gamma_1, \ldots, \gamma_k\}$ for $n = 2^k$ and $1 \leq k \leq 3$. Furthermore, for any $D \in H^1(-, \mathbf{GL}_8/\boldsymbol{\mu}_2)$, the form q_D is a 6-fold Pfister form such that $e_6(q_D) = \gamma_6(D)$. As a consequence, we get $6 \leq \mathrm{ed}(\mathbf{GL}_8/\boldsymbol{\mu}_2) \leq 8$. At the end of the lecture Baek showed that similar result holds for an upper bound of $\mathrm{ed}(\mathbf{GL}_8/\boldsymbol{\mu}_2)$ if the base field F is of characteristic 2.

Skip Garibaldi: "Applications of the degree 5 invariant of E_8".

Recently Nikita Semenov discovered a new degree 5 cohomological invariant for E_8-torsors. (Invariants of torsors appeared also in the talk by Sanghoon Baek.) The construction of this invariant used motives (the technology underlying the proof of the Bloch-Kato conjecture), and unfortunately this does not give an explicit formula for the invariant. S. Garibaldi spoke on his joint work with Semenov where they produce a formula for the invariant for those torsors that appear in Tits construction and gave several applications. Specifically, they constructed new cohomological invariants for curtain groups of type E_7; constructed new examples of anisotropic groups of E_8; constructed new cohomological invariants of Spin_{16}-torsors; computed the essential dimension of the kernel of the Rost invariant on Spin_{16} (connecting his talk with other talks on essential dimension by A. Meyer, R. Lötscher, and M. MacDonald), and used the invariant of E_8-torsors to give concrete criteria for embedding certain finite simple groups in the split form of E_8, filling in a question mark from a 1998 note by Serre.

Arturo Pianzola: "Applications of Galois cohomology to infinite dimensional Lie theory". (based on joint projects with B. Allison, S. Berman, P. Gille, V. Kac, and M. Lau.)

Pianzola's talk focused on surprising connections between of non-abelian Galois cohomology of Laurent polynomial rings and extended affine Lie algebras (a class of infinite dimensional Lie algebras which, as rough approximations, can be thought off as higher nullity analogues of the affine Kac-Moody Lie algebras).

Though the algebras in question are in general infinite dimensional over the given base field (say the complex numbers), they can be thought as being finite *provided that the base field is now replaced by a ring* (in this case the centroid of the algebras, which turns out to be a Laurent polynomial ring). This leads us to the theory of reductive group schemes as developed by M. Demazure and A. Grothendieck. Once this point of view is taken, the language of torsors arise naturally. This novel geometrical approach has lead to unexpected interplays between infinite dimensional Lie theory and the theory of algebraic groups, such as the work of Raghunathan and Ramanathan on torsors over the affine space, isotriviality questions for Laurent polynomial rings, Azumaya algebras, and Serre's Conjecture I and II.

This new language is so flexible and powerful that can be adapted also to the study of Differential Conformal Superalgebras. This involves, at the very least, rewriting the descent formalism for the case when a base scheme is replaced by a *differential* scheme. Concrete application have already been found that relate to the classification of the "affine" N-conformal superalgebras, and work of Schwimmer and Seiberg.

Andrew Schultz: "The first Galois cohomology group as a $\mathrm{Gal}(E/F)$-module, and applications". (Joint with Ján Mináč and John Swallow.)

The talks of A. Schultz and J. Swallow are surveys of recent results on the Galois module structure of Galois cohomology and their applications to Galois theory. A. Schultz began by considering how certain

Galois embedding problems related to Kummer theory could be interpreted in terms of the Galois structure of $E^\times/E^{\times p}$, where $E^\times$ represents the multiplicative group of E. Investigations into the structure of this module began with work of Borevič and Faddeev in the case that E is a local field. Schultz presented the following result for extensions satisfying $\mathrm{Gal}(E/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$; in the following result, E_i is the extension of degree p^i of F within E/F.

Theorem. If $p > 2$ and $\xi_p \in F$, and if $\mathrm{Gal}(E/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$, then $E^\times/E^{\times p} \simeq X \oplus Y_0 \oplus Y_1 \oplus \cdots \oplus Y_n$ where each Y_i is a free $\mathbb{F}_p[\mathrm{Gal}(E_i/F)]$-module, and X is cyclic module of dimension $p^{i(E/F)} + 1$. The invariant $i(E/F)$ comes from the set $\{-\infty, 0, 1, \cdots, n-1\}$, where $p^{-\infty}$ is defined to be 0.

One can interpret $i(E/F)$ in terms of embedding problems: $i(E/F) = -\infty$ if E/F can be embedded in a cyclic, $\mathbb{Z}/p^{n+1}\mathbb{Z}$ extension E'/F, and otherwise $i(E/F)$ represents the smallest number i such that E/E_{i+1} can be embedded in a cyclic $\mathbb{Z}/p^{n-i}\mathbb{Z}$-extension E'/F.

This result has analogues in the cases $p = 2$ as well as when $\xi_p \notin E$, but they weren't discussed for expository reasons. The full results are in [44].

Schultz explained how this theorem could be used to show that the appearance of certain Galois groups over F can force the appearance of other Galois groups over F, corollaries in the vein of so-called automatic realization results. The expectation is that the Galois structure of $E^\times/E^{\times p}$ will be used in arithmetic and geometric constructions beyond Galois theory, much in the same way that the structure of $E^\times/E^{\times 2}$ can be used to understand quadratic forms when E is a quadratic extension of F.

John Swallow: "Galois cohomology groups as Galois modules, and applications". (Joint work with D. Benson, J. Labute, N. Lemire and J. Mináč.)

Let p be prime and ξ_p a primitive pth root of unity. Let k_mF denote the reduced Milnor K-theory of the field F modulo p, and let $H^m(F)$ denote the cohomology group $H^m(G_F, \mathbb{F}_p)$. The Bloch-Kato conjecture (now the Rost-Voevodsky theorem) tells us that the norm residue map $k_mF \to H^mF$ is an isomorphism. The purpose of this talk was to explicitly interpret this powerful theorem in terms of structural properties of absolute Galois groups.

To begin, Swallow showed how this theorem forced a stratification in the Galois module structure of certain Galois cohomology groups. Let U be an open normal subgroup of index p in G_F. We write $G := G_F/U$, with E the fixed field of U. Kummer theory shows that $E = F(\sqrt[p]{a})$ for some $a \in F^\times$. We then have the following

Theorem. [LeMS] When viewed as a $\mathbb{F}_p[G]$-module, H^mE is a direct sum of indecomposable submodules of dimensions 1, 2 and p.

Indeed, one can be quite explicit in this decomposition. For instance, one can give the multiplicities of each summand type in terms of arithmetic information related to $(a), (\xi_p)$ and quotients of the filtration $H^{m-1}F \supseteq \mathrm{ann}\{a, \xi_p\} \supseteq \mathrm{ann}(a)$, where $\mathrm{ann}(\cdot)$ denotes the annihilator of the given cohomology class.

The power of this result is exhibited by its applications. For instance, one can use this result to give certain "hereditary" properties of Galois cohomology.

The Bloch-Kato conjecture also allows us to translate certain questions about pro-p groups to the context of Galois cohomology; indeed, the inflation map gives an isomorphism $\inf : H^i(G_F(p), \mathbb{F}_p) \to H^i(G_F, \mathbb{F}_p)$ for all $i \in \mathbb{N}$, where $G_F(p)$ is the maximal pro-p quotient of G_F. One can then ask how standard cohomological properties are translated in terms of these Galois modules. The computed module structure then gives

Theorem. [LLMS] The cohomological dimension of $G_F(p)$ is at most n if and only if $\mathrm{cor} : H^nE \to H^nF$ is surjective for all E/F cyclic of degree p.

One can also give an interesting generalization of Schreier's formula using the Galois module structure of Galois cohomology. Recall that Schreier's formula tells us that if the cohomological dimension of a pro-p group G is 1, then for all open subgroups H in G, $h_1(H) = 1 + [G : H](h_1(G) - 1)$, where $\dim_{\mathbb{F}_p} H^i(H, \mathbb{F}_p) = h_i(H)$. Using the stratified decomposition of Galois cohomology in our case, we have

Theorem. Suppose $h_n(G_F) < \infty$ and that $\mathrm{cor} : H^nE \to H^nF$ is surjective. Then $h_n(G_E) = a_{n-1}(E/F) + p(h_nG_F - a_{n-1}(E/F))$, where $a_{n-1}(E/F) = \dim_{\mathbb{F}_p} \frac{H^{n-1}F}{\mathrm{ann}(a)}$.

One can further develop a formula for the partial Euler-Poincaré characteristic and classify certain small quotients of absolute Galois groups. For details see [14] and [6].

Essential Dimension

Alexander Duncan: "Groups of essential dimension 2".

Let G be a finite group and $\mathbb{C}$ be the field of complex numbers. A theorem of Buhler and Reichstein asserts that $\mathrm{ed}_{\mathbb{C}}(G) = 1$ if and only if G is cyclic or dihedral. The proof is based on the fact that the only rational complex curve is $\mathbb{P}^1$.

Duncan spoke on his recent classification of finite groups of essential dimenson 2 over $\mathbb{C}$. Here the underlying geometry is considerably more difficult. The minimal rational surfaces with the action of a finite group G were classified by F. Enriques, Yu. Manin, and V. A. Iskovskikh, but this classification is rather involved, and it is not always clear which surfaces occur for a given G.

The starting point of Duncan's work was a recent classification of finite subgroups of the 2-dimensional Cremona group by I. Dolgachev and Iskovskikh, and the following recent results on the essential dimension of finite groups.

- (H.-P. Kraft, R. Lötscher and G. W. Schwarz) Let G be a finite group whose center is non-trivial. Then $\mathrm{ed}(G) = 2$ if and only if G embeds into $\mathrm{GL}_2(\mathbb{C})$.
- (N. Karpenko and A. Merkuriev) Let G be a finite p-group. Then $\mathrm{ed}_{\mathbb{C}}(G)$ is the minimal value of $\dim(\rho)$, where ρ ranges over the faithful complex linear representations of G.

Duncan's main result is the following theorem.

Theorem Let G be a finite group. Then $\mathrm{ed}_{\mathbb{C}}(G) \leq 2$ if and only if G is a subgroup of one of the following groups:

1) $T \rtimes D_{12}$ and $|G \cap T|$ is not divisible by 2 or 3,

2) $T \rtimes D_8$ and $|G \cap T|$ not divisible by 2,

3) & 4) $T \rtimes S_3$ and $|G \cap T|$ is not divisible by 3, (there are two such group up to isomorphism),

5) The general linear group $\mathrm{GL}_2(\mathbb{C})$,

6) The finite projective linear group $\mathrm{PSL}_2(\mathbb{F}_7)$;

7) The symmetric group S_5.

The most intricate parts of Duncan's proof are based on the results he obtained about the Cox ring of a toric variety with a finite group action. These intermediate results are of independent interest.

Roland Lötscher: "A multihomogenization technique for the study of essential dimension of algebraic groups".

Let k be a field, and G be a finite group. A rational covariant of G is the G-equivariant map $\varphi : V \dashrightarrow W$, where V and W are G-modules. φ is called *generically free* if $\overline{\varphi(\mathbb{V})}$ is generically free. $\dim\varphi :=$ dimension of $\overline{\varphi(\mathbb{V})}$. The essential dimension $\mathrm{ed}_k G$ can be expressed in terms of rational covariants: $\mathrm{ed}_k(G) = \min\{\dim\varphi \mid \varphi$ is a generically free covariant of G over $k\} - \dim G$. The related notion of *covariant dimension* $\mathrm{covdim}_k(G)$ defined in a similar manner, using regular, rather than rational covariants. It is easy to see that

$$\mathrm{ed}_k(G) \leq \mathrm{covdim}_k(G) \leq \mathrm{ed}_k(G) + 1\,.$$

Reichstein asked for which groups $\mathrm{ed}_k(G) = \mathrm{covdim}_k(G)$.

Lötscher, H. Kraft, and G. W. Schwarz gave a complete answer to this question. Their main result is the following theorem.

Theorem: Let G be a non-trivial finite group. Then $\mathrm{ed}_{\mathbb{C}}(G) = \mathrm{covdim}_{\mathbb{C}}(G)$ if and only if G has a non-trivial center.

The proof relies on a multihomogenization technique pioneered by Florence [16] and further developed by Lötscher, H. Kraft, and G. W. Schwarz. The idea is to replace a faithful covariant $\varphi : V \to W$ by a homogeneous (and more generally, a multihomogeneous) faithful covariant $\varphi_h : V \to W$ such that $\dim(\varphi) \geq \dim(\varphi_h)$.

Lötscher has found other applications of this technique. In particular, it can be used to simplify the proof of the theorem of Karpenko and Merkurjev [28] on the essential dimension of a finite p-group.

Mark MacDonald: "Essential p-dimension of algebraic tori".

MacDonald spoke on his recent joint work with Lötscher, Meyer and Reichstein. The starting point of this project is the following theorem, due to Karpenko and Merkurjev.

Theorem 0: Let G be a finite p-group and k be a field containing a primitive pth root of unity. Then $\mathrm{ed}_k(G;p) = \mathrm{ed}_k(G) = \min\ \dim(V)$, where the minimum is taken over all faithful k-representations $G \hookrightarrow \mathrm{GL}(V)$.

MacDonald and his collaborators proved similar formulas for for a broader class of algebraic groups G, which includes all twisted p-groups and all algebraic tori. Their main result is as follows.

Theorem 1: Let k be a p-closed field of characteristic $\neq p$. Suppose there exists an exact sequence $1 \to T \to G \to F \to 1$ of algebraic groups over k, where T is a torus and F is a twisted finite p-group. Then: **(a)** $\mathrm{ed}_k(G;p) \geq \min \dim(\rho) - \dim G$, where the minimum is taken over all p-faithful linear representations ρ of G_k over k. **(b)** If G is the direct product of T and F then equality holds in (a). Moreover, $\mathrm{ed}(G) = \mathrm{ed}_k(G;p)$.

Note that for the purpose of computing $\mathrm{ed}(G;p)$, the assumption that k is p-closed is harmless; the value of $\mathrm{ed}(G;p)$ does not change if k is replaced by its p-closure.

If G a direct product of a torus and an abelian p-group, the value of $\mathrm{ed}_k(G;p)$ given in part (b) can be rewritten in terms of the character module $X(G)$. This often renders it computable by standard methods of integral representation theory. In the case of a torus, this results in the following simple formula.

Theorem 2: Let T be an algebraic torus defined over a p-closed field k of characteristic $\neq p$. Suppose the absolute Galois group $\Gamma = \mathrm{Gal}(k)$ acts on the character lattice $X(T)$ via a finite quotient $\overline{\Gamma}$. Then $\mathrm{ed}_k(T) = \mathrm{ed}_k(T;p) = \min \mathrm{rank}(L)$, where the minimum is taken over all exact sequences of $\mathbb{Z}_{(p)}\overline{\Gamma}$-lattices of the form $(0) \to L \to P \to X(T)_{(p)} \to (0)$. with P permutation. Here $X(T)_{(p)}$ stands for $X(T) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

MacDonald outlined a proof Theorems 1 and 2 and discussed several applications. For details, see [38]. Other applications were suggested by workshop participants during the question period.

Aurel Meyer: "A bound on the essential dimension of central simple algebras".

Given a central simple algebra A over a field K, one can ask whether A can be written as $A = A_0 \otimes_{K_0} K$ where A_0 is a central simple algebra over some subfield K_0 of K. In that situation we say that A *descends* to K_0. Let us assume that K contains a base field k, which is assumed to be fixed throughout. The *essential dimension* of A, denoted $\mathrm{ed}(A)$, is the minimal transcendence degree over k of a field $k \subset K_0 \subset K$ such that A descends to K_0. It can be thought of as "the minimal number of independent parameters" required to define A.

For a prime number p, the related notion of essential dimension at p of an algebra A/K is defined as $\mathrm{ed}(A;p) = \min\ \mathrm{ed}(A_{K'})$, where K'/K runs over all finite field extensions of degree prime to p. We also define $\mathrm{ed}(\mathrm{PGL}_n) := \textbf{max}\,\{\ \mathrm{ed}(A)\ \}$, and $\mathrm{ed}(\mathrm{PGL}_n;p) := \textbf{max}\,\{\ \mathrm{ed}(A;p)\ \}$, where the maximum is taken over all fields K/k and over all central simple K-algebras A of degree n. The appearance of PGL_n in the symbols $\mathrm{ed}(\mathrm{PGL}_n)$ and $\mathrm{ed}(\mathrm{PGL}_n;p)$ has to do with the fact that central simple algebras of degree n are in a natural bijective correspondence with PGL_n-torsors.

The problem of computing $\mathrm{ed}(\mathrm{PGL}_n)$ was first raised by C. Procesi in the 1960s in the context of his (and S. Amitsur's) pioneering work on universal division algebras. Procesi showed (using different terminology) that in fact, $\mathrm{ed}(\mathrm{PGL}_n) \leq n^2$; see [48, Theorem 2.1].

Meyer talked about the following new upper bounds on the essential p-dimension of the projective linear group PGL_{p^r}: $\mathrm{ed}(\mathrm{PGL}_n; p) \leq 2\frac{n^2}{p^2} - n + 1$. L. H. Rowen and D. J. Saltman [53] showed that if $s \geq 2$ then there is a finite field extension K'/K of degree prime to p, such that $A' := A \otimes_K K'$ contains a field F, Galois over K' with $\mathrm{Gal}(F/K') \simeq \mathbb{Z}/p \times \mathbb{Z}/p$. The above bound is thus a consequence of the following theorem.

Theorem: Let A/K be a central simple algebra of degree n. Suppose A contains a field F, Galois over K and $\mathrm{Gal}(F/K)$ can be generated by $r \geq 1$ elements. If $[F : K] = n$ then we further assume that $r \geq 2$. Then $\mathrm{ed}(A) \leq r\frac{n^2}{[F:K]} - n + 1$.

Meyer explained how to prove this theorem. The construction of a suitable subalgebra A_0 is based on the theory of $\mathrm{Gal}(F/K)$-lattices. For details, see [43].

K-theory, Chow Groups and Brauer-Severi Varieties

Mikhail Borovoi: "Extended Picard complexes and homogeneous spaces". (Joint work with Joost van Hamel.)

Inspired by a result of Kottwitz, for a smooth algebraic variety X over a field k of characteristic 0, Borovoi and van Hamel introduce a certain complex of Galois modules $\mathrm{UPic}(X)$, which they call the extended Picard complex of X. From $\mathrm{UPic}(X)$ one can compute the Picard group $\mathrm{Pic}(X)$ and the algebraic Brauer group $\mathrm{Br}_a(X)$. Borovoi and van Hamel compute $\mathrm{UPic}(G)$ (up to an isomorphism in the derived category), where G is a connected linear algebraic group over k. Moreover, they compute $\mathrm{UPic}(X)$ (again up to an isomorphism in the derived category) where X is a homogeneous space of a linear algebraic group over k (they do not assume that X has a k-point). This permits them to compute $\mathrm{Br}_a(X)$ for such X. In the course of the proof they consider the equivariant Picard group $\mathrm{Pic}_G(X)$, where now k is an algebraically closed field of characteristic 0 and X is any integral variety over k with any action of a connected k-group G. They compute $\mathrm{Pic}_G(X)$ in terms of divisors and rational functions (on X and on $X \times_k G$).

Baptiste Calmès: "Invariants, torsion indices and oriented cohomologies of flag varieties". (Joint work with Viktor Petrov and Kirill Zainoulline.)

After the work of M. Levine and F. Morel on algebraic cobordism, it is a natural program to try and lift the calculations from specifically-oriented cohomology theories such as Chow groups, the Grothendieck group K, and connective K-theory, to any oriented cohomology h in the sense of M. Levine and F. Morel. In joint work with V. Petrov and K. Zainouilline, B. Calmès succeeded in adapting Demazure's 1973 calculation of the Chow ring of G/B, where G is a semisimple, simply connected linear algebraic group G over a field k, and B is its Borel subgroup to such a calculation of $h^*(G/B)$ where h^* is any oriented cohomology.

As an application, they prove a generalization to all oriented cohomology theories, Borel's description of the singular cohomology of complete flags of type A_n in terms of symmetric polynomials. Also they provide an algorithm to compute the ring structure of the algebraic cobordism of G/B.

Nikita Karpenko: "Incompressibility of quadratic Weil transfer of Severi-Brauer varieties".

Recall that if X is a smooth complete irreducible variety X/F, then X is incompressible if any rational map $X \dashrightarrow X$ is dominant. Equivalently, canonical dimension of $X = \dim X$. Let K/F be a separable quadratic extension, and let $\mathcal{D}/K$ be a 2-primary division algebra such that $N(\mathcal{D})$ = a corestriction of $\mathcal{D}$ from K down to F is Brauer-trivial. Let $\mathcal{SB}(\mathcal{D})$ be the Severi-Brauer variety of $\mathcal{D}$ and $R(\mathcal{SB}(\mathcal{D}))$ be its Weil transfer.

Theorem. *Then the variety $R(\mathcal{SB}(\mathcal{D}))$ is 2-indecomposable (hence 2-incompressible).*

One can consider generalized Severi-Brauer varieties $\mathcal{SB}_{2^i}(\mathcal{D}), i = 0, 1, \ldots, n$ (where the degree of $\mathcal{D}$ is 2^n). One can still prove that $R\mathcal{SB}_{2^i}(\mathcal{D})$ is 2-incompressible. The proof uses some very interesting motivic decompositions of the motives of these varieties.

It is known that a non-hyperbolic orthogonal involution on a central simple algebra A remains non-hyperbolic after passing to the function field of SB(A). J.-P. Tignol recently observed that the same is true for unitary involutions on algebras of exponent 2. Karpenko's work was motivated by trying to extend this observation to unitary involutions on algebras of arbitrary exponent.

Max-Albert Knus: "Severi-Brauer varieties over the field with one element". (Joint work with Jean-Pierre Tignol.)

A field $\mathbb{F}_1$ with one element may look humorous, but in fact it has recently attracted considerable attention and inspiration.

The idea of a field $\mathbb{F}_1$ first showed up in a paper published by M. J. Tits in 1957. In that paper Tits associated geometries to Dynkin diagrams. Let D be a Dynkin diagram. Let $G_F(D)$ be a Chevalley group over a field F attached to D and let $W(D)$ be a corresponding Weyl group. Tits showed that there exist unique geometries $\Gamma_F(D)$ and $\Gamma_w(D)$ such that the automorphism groups of the geometries are resp. $G_F(D)$ and $W(D)$. Tits called the geometries $\Gamma_w(D)$ attached to Weyl groups, *geometries over the field* $\mathbb{F}_1$ *of characteristic* 1.

Example. Geometry of type A_{n-1} over $\mathbb{F}_1$. $\mathbb{P}^{n-1}\mathbb{F}_1 \overset{\text{def}}{=}$ an n-element set X. The projective geometry of dimension $n-1$ over $\mathbb{F}_1$ is $A = S_n$. Observe that $|\mathbb{P}^{n-1}(\mathbb{F}_q)| = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. Hence if $q = 1 \Rightarrow |\mathbb{P}^{n-1}(\mathbb{F}_1)| = n$. This explains the "$\mathbb{F}_1$ terminology."

Many properties of usual central simple algebras and central simple algebras with involutions in relation with classical groups have direct analogues over $\mathbb{F}_1$. In particular, one can define exterior powers, Clifford algebras and discriminants in this setting. For example if Γ is an absolute Galois group over F, the étale algebras of dimension n correspond to Γ-projective spaces over $\mathbb{F}_1$ of dimension $n-1$. Some interesting connections between triality and étale algebras were discussed. For details see [30].

Alexander Vishik: "Rationality of integral cycles".

Let k be a field of characteristic 0, Y is a smooth quasiprojective variety over k, F/k is a field extension. Let $\mathrm{Ch}^m(Y) \to \mathrm{Ch}^m(Y_F)$ be the natural map of mth Chow groups of Y and $Y \otimes_k F$. Elements in this image are called k*-rational*. The motivation for this is the calculation of discrete invariants which lead to the construction of fields with a u-invariant equal to $2^s + 1, s \geq 3$.

mod 2 **case.** Q is a smooth projective quadric.

Theorem. *Assume* $\bar{Y} \in \mathrm{Ch}^m(Y_{\bar{k}})/2, m < \frac{\dim Q}{2}$*. Then* $\bar{Y}$ *is* k*-rational* $\Leftrightarrow$ $\bar{Y}$ *is* $k(Q)$*-rational.* (Also it is true in some special cases for $m \geq \frac{\dim Q}{2}$.)

In this talk, A. Vishik discussed the proof of the following theorem.

Theorem. *Assume* $\bar{y} \in \mathrm{Ch}^m(Y_{\bar{k}})$ *and* **(1)** $m < \frac{\dim Q}{2}$, and **(2)** *The first Witt invariant* $i_1(Q) > 1$*. Then* $\bar{y}$ *is* k*-rational* $\Leftrightarrow$ *it is* $k(Q)$*-rational.*

The overall structure of the proof is similar to the mod 2 case, but there are some additional significant additional complications. In particular one uses algebraic cobordism Ω^*, constructions by Levine and Morel, and symmetric cohomological operations on Ω^* introduced by A. Vishik.

Structure of Algebraic Groups

Philippe Gille: "Algebraic groups with few subgroups". (Joint work with S. Garibaldi.)

If G is a reductive algebraic group G over $\mathbb{C}$, using Dynkin's work one can list all connected reductive subgroups of G. One can also do it over local or global fields. But over "general fields" the situation is significantly more difficult.

In the early 1990s, in his lectures at Collège de France J. Tits showed that "generic" groups of type E_8 have no other connected subgroups than maximal tori. P. Gille's talk was a variation on a theme of

Tits' lectures (Gille attended Tits' lectures in the early 1990s being a graduate student). In his talk he gave an alternative proof of the Tits' result based on Totaro's computation of the torsion index of E_8. He also discussed the case of other exceptional groups, in particular the trialitarian case.

Note that in general case the problem of describing reductive subgroups of exceptional groups is still open. Conjecturally all "generic" simple groups of exceptional type have no proper semisimple subgroups.

Alex Ondrus: "Minimal anisotropic groups of higher real rank".

Motivation for A. Ondrus's work is provided by E. Ghys's conjecture which says that if G is a connected, semisimple real Lie group with finite center, rank $G \geq 2$ and Γ is any irreducible lattice in $G(\mathbb{R})$, then Γ has a non-trivial orientation-preserving action on $\mathbb{R}$. The statement is equivalent to saying that Γ has no total order $\leq$ stable by left multiplication. If Γ has such an order then any subgroup also has such an order. Thus to prove Ghys conjecture it suffices to consider almost minimal lattices of higher rank. By the Margulis arithmeticity theorem every such lattice is isomorphic to the group of integer points of a minimal $\mathbb{Q}$-simple algebraic group of higher real rank; hence we arrive to necessity of classification of such groups.

In the isotropic case the classification of such minimal G up to isogeny, was achieved by V. Chernousov, L. Lifschitz and D. W. Morris. They succeeded to do so over any algebraic number field F of higher real rank. A. Ondrus obtained such a classification for anisotropic groups, as follows.

Theorem. *If G is an absolutely simple, minimal anisotropic group over an algebraic number field F, then G is isomorphic to one of the following groups (up to isogeny):*

1) $\mathcal{SU}_3(L, f)$ *for L/F quadratic, f hermitian on L^3 with at least one real place v such that* $L \otimes F_v \cong F_v \times F_v,$ *or*

2) $\mathcal{SU}_1(D, \tau)$ *a central division algebra of degree $p \geq 3$ over L/F quadratic with involution of the second kind τ such that either*

A) $L \otimes F_v \cong F_v \times F_v$ *for some real place $v,$ or*

B) $\tau \otimes 1$ *corresponds to a hermitian form of index ≥ 2 over $M_p(\mathbb{C})$ for some real place v.*

3) $SL_1(D), D$ *is a division algebra with $\deg(D) = p$ odd.*

Vladimir L. Popov: "Cross-sections, quotients, and representation rings of semisimple algebraic groups".

Let $G \neq \{1\}$ be a connected complex semisimple algebraic group. In 1965 Steinberg proved that if G is simply connected, then there exists a closed irreducible cross-section S of the set of closures of regular conjugacy classes. That is, every such orbit closure intersects S in exactly one point. Equivalently, there exists a regular section of the categorical quotient map $\pi\colon G \to G/\!/G$. This section played an important role in Steinberg's celebrated solution of Serre's Conjecture I.

In a letter to J.-P. Serre, dated January 15, 1969, A. Grothendieck asked whether there exists such a section of π if G is not simply connected. He also asked for which G π has a rational section.

Both problems were solved within the last year. Popov showed that π has a regular section if and only if G is simply connected, and J.-L. Colliot-Thélène, B. Kunyavskiĭ, Popov, and Reichstein, showed that a rational section exists for any G. Moreover, Popov obtained similar results for groups defined over an algebraically closed field of any characteristic. Here, once again, a rational section always exists and a regular section exists if and only if the universal covering isogeny $J : \hat{G} \longrightarrow G$ is bijective on k-points.

Popov also discussed other related questions, such as: What is a minimal generating set of $k[G]^G$? What are the singularities of $G/\!/G$? What is a minimal generating set of the representation ring of G? For details and further references, see [47].

Representation theory of algebraic groups

Sunil Chebolu: "Freyd's generating hypothesis and the Bloch-Kato conjecture". (Joint work with Jon Carlson, Ido Efrat, and Ján Mináč.)

The generating hypothesis (GH) is a famous conjecture in homotopy theory due to Peter Freyd. It states that a map $\phi\colon X \to Y$ between finite spectra that induces the zero map on stable homotopy groups is null-homotopic. Motivated by this long-standing unsolved problem, the authors formulate and solve its analogue in the stable module category $\mathrm{stmod}(kG)$ of a finite group. It is assumed that characteristic of k is p and p divides $|G|$. Consider the thick subcategory thick (k) generated by k which is the smallest subcategory of $\mathrm{stmod}(kG)$ that is closed under exact triangles and retractions. The main theorem states that the Tate cohomology functor $\widehat{H}^*(G, -)\colon$ thick $(k) \to$ graded k-vector spaces is faithful if and only if the Sylow p-subgroup of G is either C_2 or C_3. Motivated by the general failure of the generating hypothesis for the stable module category, the authors define the ghost number of kG (for a p-group G) to be the smallest non-negative integer l such that the composition of any l ghosts between finite-dimensional kG-modules is trivial in $\mathrm{stmod}(kG)$. They obtain various bounds on these new invariants and compute them in specific groups.

A closely related question is the finite generation problem for Tate cohomology. For which finitely generated kG-modules M is the Tate cohomology $\widehat{H}^*(G, M)$ finitely generated as a module over the Tate cohomology ring $\widehat{H}^*(G, k)$? Motivated by many partial results they proved on finite generation for Tate cohomology, the authors conjecture that if $\widehat{H}^*(G, M)$ is finitely generated over $\widehat{H}^*(G, k)$ then the support variety $V_G(M)$ of M is equal to the entire maximal ideal spectrum $V_G(k)$ of the group cohomology ring.

There was no time to cover the small Galois pro-p-groups which determine entire Galois cohomology and their applications for investigating arithmetic of fields and structure of Galois groups of maximal p-extensions of fields. For details see [11].

Eric Friedlander: "Restrictions to $G(\mathbb{F}_p)$ and $G_{(r)}$ of rational G-modules". (Joint work with J. Carlson, J. Pevtsova and A. Suslin.)

Standard modular representation theory considers as representation spaces, vector spaces over an algebraically closed field k of $\mathrm{char}(k) = p > 0, p\big|\,|G|$. Let G be a finite group scheme, $\mathcal{G}$ - a connected reductive algebraic group defined over $\mathbb{F}_p$. $\mathcal{G}(\mathbb{F}_p)$ are points over $\mathbb{F}_p$. Consider rational $\mathcal{G}$-modules M (finite dimensional vector spaces over k).

Frobenius kernel of $\mathcal{G}$: Let $F : \mathcal{G} \longrightarrow \mathcal{G}$ be the Frobenius map. Then set $\mathrm{Ker}\{F^r\} = \mathcal{G}_{(r)} \hookrightarrow \mathcal{G}$. Every rational $\mathcal{G}$-module restricts to give a $\mathcal{G}_{(r)}$-module.

Basic Question. *Relate invariants of $\mathcal{G}(\mathbb{F}_p)$ and $\mathcal{G}_{(r)}$ for various r.*

If M is a rational $\mathcal{G}$-module we can consider $\Phi_x^* M$ as a $\mathbb{G}_a$-module. (Here $x^p = 1$ and $\Phi_x : \mathbb{G}_a \longrightarrow \mathcal{G}$ is a "1-parameter subgroup" such that $\Phi_x(1) = x$ and some further restrictions on the image of Φ. (This is work of G. Seitz.)) Hence we obtain a map $\Phi_x^* M \longrightarrow \Phi_x^* M \otimes k[t]$.

$s(M)$ is an important invariant, the least integer such that certain operators indexed by integers $\geq s(M)$ act trivially on M.

Let G be a group scheme and consider Spec $H^\bullet(G, k)$, where $\bullet = *$ if $p = 2$ and $\bullet$ ranges over the even non-negative numbers if $p > 2$. (Note that in both cases $H^\bullet(G, k)$ is commutative.) A well-known theorem of Quillen says that Spec $H^*(\mathcal{G}(\mathbb{F}_r), k) = \mathrm{colim}\, E \otimes k,\ E < \mathcal{G}(\mathbb{F}_r)$, where $E \otimes k$ is an affine space of rank t ($E \cong (\mathbb{Z}/p\mathbb{Z})^t$). A. Suslin, E. Friedlander and C. Bendel showed that Spec $H^*(\mathcal{G}_{(r)}, k) \approx V(\mathcal{G}_r)$ where k-points are the 1-parameter subgroups of $\mathcal{G}_{(r)}$. E. Friedlander and J. Pevtsova further found a description of Proj $H^\bullet(G, k)$ using certain equivalence relations on some functions $\alpha : k[t]/t^p \longrightarrow kG$. For a given M one can define the support variety of M. One way to do so is to set $(\Pi\, G)_M = \{[\alpha] :\ \alpha^* M$ is not free$\}$.

Theorem (J. Carlson, Z. Lin, D. Nakano) *For a large enough prime p (depending on $\mathcal{G}$) there exists an embedding $\Pi\,\mathcal{G}(\mathbb{F}_r) \hookrightarrow \Pi\,\mathcal{G}_{(r)}/\mathcal{G}(\mathbb{F}_r)$ for any $r \geq 1$ and if $p^r \geq S_{\mathbb{F}_r}(M)$, then $(\Pi(\mathcal{G}(\mathbb{F}_r))_M \cong \left(\Pi\,\mathcal{G}_{(r)}\right)_M / \mathcal{G}(\mathbb{F}_r) \cap \Pi\,\mathcal{G}(\mathbb{F}_r)$.*

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Chapter 19

Complex Monge-Ampère Equation (09w5049)

Oct 18 - Oct 23, 2009

Organizer(s): Pengfei Guan (McGill University), Zbigniew Blocki (Jagiellonian University), Duong Phong (Columbia University)

Overview of the field

Complex Monge-Ampère equations occupy a position of central importance in complex geometry and the theory of non-linear partial differential equations. On the geometric side, it has been known since the works of Calabi, Chern, Nirenberg, Yau, Kohn, Fefferman, Siu, and others that the various problems of finding a representative of a cohomology class with given volume form; of finding Kähler-Einstein metrics; and of determining the Bergman kernel and the boundary behavior of holomorphic functions, can all be reduced to the study of a complex Monge-Ampère equation. On the analytic side, complex Monge-Ampère equations are prime examples of fully non-linear elliptic or degenerate-elliptic partial differential equations. They have been extensively investigated, starting with the foundational works of Yau (1978) on a priori estimates on compact Kähler manifolds, of Bedford-Taylor (1978) on generalized solutions, and of Caffarelli-Kohn-Nirenberg-Spruck on boundary regularity for strongly pseudoconvex domains in $\mathbb{C}^n$ (1984).

The last decade has witnessed an explosive growth in the subject, which has opened up entire new venues for investigation. This growth is due on one hand to the unexpected appearance of the Monge-Ampère equation in several new geometric problems (such as the problem of geodesics in the space of Kähler potentials), as well as the need, in both geometry and physics, for considering new related equations (such as the Kähler-Ricci flow, singular Kähler-Einstein metrics, the equation for Kähler metrics of constant scalar curvature, the Strominger system, the Einstein-Sasaki system, and symplectic or even general Hermitian manifolds). On the other hand, there has also been spectacular progress in the development of methods for solving these equations. In particular, powerful new techniques of pluripotential theory, of geometric flows, of variational methods, and of constructions of solutions by algebraic approximations have been introduced. Some of this is described in greater detail below.

The Workshop

The purpose of the workshop was to bring together leading experts in order to discuss these developments and outline open problems. The workshop was timely, since the bulk of the progress described above actually took place in the last three or four years. At the same time, it is perhaps remarkable that major contributions came from researchers from all over the world (Canada, China, France, Great Britain, Poland, Sweden, USA, etc.). Thus the workshop also provided a unique opportunity for interaction between different groups who would normally reside in several distinct continents. With a view towards the long-term vitality of the field, a high proportion of young post-doctoral researchers and graduate students was also invited to participate.

In order to leave a maximum amount of time for discussions and foster close interactions between the participants, the number of formal presentations was kept at 2 per session. The essential role of the formal presentations was to initiate topics of discussions, which can then be taken up at greater length in the more informal sessions outside of the talks.

Recent developments discussed at the workshop

The workshop provided an in-depth coverage of the developments described in the overview. These developments are all closely inter-related, but for the purpose of listing them, it is convenient to give the following rough classification. As in the previous discussion, we divide them into groups by either underlying geometric problems, or by method of solution. To highlight the contributions of the participants of the workshop, we have given their names in italics.

(A) Geometric problems

• *Kähler-Einstein metrics:* Perhaps the most famous problem in Kähler geometry is the problem of finding when Kähler-Einstein metrics exist on Fano manifolds. A well-known conjecture of Yau states that the existence of such metrics should be equivalent to the stability of the manifold in the sense of geometric invariant theory. Since Kähler-Einstein metrics can also be viewed as the stationary points of the Kähler-Ricci flow, this problem is the same as the one of convergence of this flow. Its relation with the Monge-Ampère equation is particularly strong, since the Kähler-Ricci flow is just a parabolic version of the Monge-Ampère equation.

Several approaches to this problem were discussed at the workshop. In particular, in their talks, *Szekelyhidi* discussed the relation of the convergence of the Kähler-Ricci flow to stability conditions such as K-stability, *Berman* discussed a new variational approach to the corresponding Monge-Ampère equation, and *Lu* discussed the notion of K-stability for hypersurfaces.

• *Kähler metrics of constant scalar curvature:* The Kähler-Einstein problem can be viewed as a special case of the more general problem of determining when a positive integral class $c_1(L)$ admits a metric of constant scalar curvature (the Kähler-Einstein case corresponds then to L being the anti-canonical bundle). The conjecture of Yau can be generalized to this case, and the prime candidate for the appropriate notion of stability is K-stability, different versions of which had been defined by Tian and Donaldson. In his talk, *Apostolov* described the construction of constant scalar curvature metrics for manifolds which can be realized as certain fibrations of toric varieties. In his, *Lu* provided an explicit analysis of versions of the notion of K-stability for hypersurfaces.

• *Donaldson's infinite-dimensional geometric invariant theory (GIT):* In the late 1990's, Donaldson proposed another condition for the existence of Kähler-Einstein metrics. This condition can be viewed as an infinite-dimensional version of stability in geometric invariant theory, where geodesics in the space of Kähler potentials play the role of one-parameter subgroups. These geodesics can be interpreted as solutions of a Dirichlet problem for a completely degenerate Monge-Ampère equation, and they are partly responsible for the great recent interest in the solution and properties of such equations. In his talk, *Sturm* described the construction of geodesics as limits of one-parameter subgroups and their resulting regularity. Such results are expected to play a major role in an eventual link of infinite-dimensional GIT with finite-dimensional GIT, as well as in Donaldson's program for the existence of constant scalar curvature metrics.

• *Analytic minimal model program:* When the first Chern class of the manifold is not definite, a smooth Kähler-Einstein metric cannot exist. It is then of great interest to determine which canonical metrics can exist instead, of which singular Kähler-Einstein metrics are prime examples. There has been considerable progress on this question, thanks in particular to works of Eyssidieux-*Guedj*-Zeriahi, *Demailly*-Pali, Tian-Zhang, *Song*-Tian, and others, making use of the C^0 estimates of *Kołodziej* for the Monge-Ampère equation with L^p right hand sides for $p > 1$. Closely related to such questions are the singularities of the Kähler-Ricci flow on such manifolds. In his talk, *Song* described a complete analysis of the Kähler-Ricci flow on Hirzebruch surfaces. He also outlined what may be viewed as an analytic, Kähler-Ricci flow version of the minimal model program. In his talk, *LaNave* discussed relations of the Kähler-Ricci flow with the moment map, and constructions of test configurations with smooth total space.

• *Extensions of Kähler metrics:* The original Monge-Ampère equation $(\omega + \frac{i}{2}\partial\bar{\partial}\phi)^n = F\,\omega^n$ in Kähler geometry involved a background form ω which is a Kähler form (i.e., closed and positive definite). Applications in modern complex geometry require generalizations in several different directions: when the class of ω is just "big" (*Berman, Demailly*), when the form ω is just a Hermitian form (*B. Guan-Q. Li*, *Tosatti*-Weinkove, *Dinew-Kołodziej*), or when the underlying almost-complex structure is no longer integrable (Donaldson, *Tosatti*-Weinkove-*Yau*). In their talks, *Q. Li* and *Tosatti* described the solution of the Monge-Ampère equation with Hermitian backgrounds and several geometric applications.

• *Extensions of Calabi-Yau manifolds and Hermitian-Einstein bundles:* Compactifications of superstrings preserving supersymmetry are of great importance in string theory and related areas of theoretical physics. The best-known examples of such compactifications are Calabi-Yau manifolds, the study of which has had a great influence on mathematics in the last 25 years. However, more general compactifications allowing torsion are also of great interest in physics and geometry. They satisfy the so-called Strominger system of equations, which can be viewed as an extension of the Calabi-Yau equation, incorporating torsion as well as the Hermitian-Einstein equation. In his talk, *Fu* described recent joint work of his with Yau, providing a first non-perturbative solution of the Strominger system. Extensions of equations from Kähler geometry to the more general context of balanced metrics were also discussed in his talk as well as in *Yau*'s talk.

• *Sasaki-Einstein manifolds:* Sasaki-Einstein manifolds can be viewed as odd-dimensional analogues of Kähler-Einstein manifolds. They also arise in compactifications of superstring theory. Although their mathematical theory can be traced farther back, they are still not well-understood. In his talk, *Yau* raised many questions about their existence and properties, and in particular with the formulation of suitable stability conditions.

• *Asymptotic volumes:* The notion of volume is one of the most important geometric invariants associated with a holomorphic line bundle. It is defined by the asymptotics of the dimension of the space of holomorphic sections of the bundle. As such, it admits natural extensions by considering higher cohomology groups. In his talk, *Demailly* described what is presently known about volumes for pseudo-effective bundles, and formulated some precise conjectures. Some key tools in the approaches which he described are approximations of plurisubharmonic functions, the regularity of envelopes of big cohomology classes (joint work of his with *Berman*), and holomorphic Morse inequalities.

• It is well-known, from e.g. the works of Kerzman, Kohn, and Nirenberg in the mid 1970's, that the Monge-Ampère equation can provide some deep information on the boundary behavior of holomorphic functions. In his talk, *S.Y. Li* described its relations with CR geometry and applications. In his talk, *B. Guan* provided a survey of several recent major advances, including the use of subsolutions for the Dirichlet problem instead of pseudoconvexity conditions, the solution of *P.F. Guan* of the Chern-Levine-Nirenberg conjecture, and works of his and *Błocki* on the pluri-Green's function.

(B) Analytic methods

Closely intertwined with the above geometric problems are an array of significant progresses on the analytic methods for solving the Monge-Ampère equation.

• *Pluripotential theory:* In 1998, *Kołodziej* had obtained C^0 a priori estimates for the Calabi-Yau equation, assuming only some integral conditions on the right hand side (for example, L^p integrability for any $p > 1$ is sufficient). Recently, it was shown by *Guedj-Kołodziej*-Zeriahi that the solution is actually Hölder continuous. In his talk, *Dinew* described very recent work of his and co-authors with an explicit and improved Hölder exponent.

• *Approximations by polynomials:* As had been mentioned above, geodesics in the space of Kähler potentials are solutions of a completely degenerate complex Monge-Ampère equation. For the ultimate purpose of relating them to geometric invariant theory, it is of particular interest not just to construct them, but also to approximate them by geodesics in the finite-dimensional space of Bergman metrics. Such results were described by *Sturm* in his talk. Some key tools are the Tian-*Yau*-Zelditch approximation theorem, Bedford-*Taylor* pluripotential theory, and C^1 estimates of *Błocki, P. Guan, B. Guan-Q. Li*. A precise analysis of the special case of toric varieties has been carried out by *Song* and Zelditch. Here the methods are those of semi-classical analysis and large deviation theory.

• *Variational methods:* Much recent progress on the complex Monge-Ampère equation has been through either a priori estimates and/or pluripotential theory. In his talk, *Berman* described very recent joint works of his with Boucksom-*Guedj*-Zeriahi, where a variational method is developed. Besides providing new proofs of some classical results, it also allowed to extend the theory to big cohomology classes. Interestingly, the method makes use of the sharp form of a Moser-Trudinger inequality established a few years ago by *Phong-Song-Sturm*-Weinkove.

• *Parabolic Monge-Ampère equations:* Another approach to solving the Monge-Ampère equation is through the Kähler-Ricci flow. Here the problem becomes that of determining the time of existence, the singularities which may form, the continuation through the singularities, and the convergence of the flow. In his talk, *Szekelyhidi* considered the Fano case, where the flow exists smoothly for all time and he showed, under various additional assumptions, how K-stability can be used to show the convergence of the flow. Even with the additional assumptions, these results are of particular interest since there are few, if any, results in the direction of the sufficiency of K-stability. In his talk, *Song* described the Kähler-Ricci flow from rough initial data, and how such results can be used to continue the flow through singularities when the first Chern class is not definite.

(C) Open Problems

There was a strong emphasis on open problems at the workshop. Some specific ones which were discussed extensively were the following:

• The talk of *Yau* was devoted almost entirely to the description of open problems ranging from the mid 1970's to present day. They span the entire breadth of Kähler geometry, complex Monge-Ampère equations, and their extensions and applications to string theory and theoretical physics. They include affine Monge-Ampère equations, the Strominger-Yau-Zaslow conjecture, the Strominger system, Sasaki-Einstein metrics, and balanced metrics. This talk was one of the two which were video-taped at the workshop.

• The second talk which was video-taped was that of *Demailly*. Here a range of important problems around the key notions of volume and higher cohomology analogues for pseudo-effective line bundles was described. Some precise conjectures for these notions in terms of holomorphic Morse inequalities and eigenvalue distribution were formulated.

• An analytic minimal model program, based on the Kähler-Ricci flow with singularities and formulated by *Song* and Tian, was described by *Song*. One particular problem is to determine the Gromov-Hausdorff convergence of the flow.

• An important issue for the theory of complex Monge-Ampère equations is the issue of regularity. Specific questions are: the optimal Hölder regularity for solutions with L^p right-hand sides; the smoothness of solutions, assuming that they are bounded, and satisfy the equation in the sense of pluripotential theory; and the regularity and rank of the Hessian of the solution, when the equation is degenerate.

Soon after the workshop, *Demailly, Dinew and Kołodziej* managed to prove that the Hölder exponent of the solution of the equation with the right hand-side in L^p (this Hölder regularity had been proved by *Kołodziej*) is independent of the geometry of the manifold (it depends only on p and the dimension). More recently, *Székelyhidi and Tosatti* posted a proof of smoothness for a broad class of right hand sides, assuming that the solution is in L^∞. In a different direction, *Błocki* obtained an alternative proof of C^0 estimates for the Calabi-Yau equation for general Hermitian background forms.

• Balanced metrics are a generalization of Kähler metrics which can still provide a deep probe of the complex geometry of the underlying manifold. This appears to be an important direction for further investigations, from both geometric and analytic viewpoints. Many remarkable results and open problems of this nature were described by *Fu* in his talk.

Talks

- Vestislav Apostolov *Extremal Kähler metrics on projective bundles over a curve*
  - Robert Berman *Complex Monge-Ampère equations and balanced metrics*
  - Jean-Pierre Demailly *Asymptotic cohomology and holomorphic Morse inequalities*
  - Sławomir Dinew *Hölder continuity of solutions of Monge-Ampère equations with right hand side in* L^p
  - Jixiang Fu *On balanced metrics*
  - Bo Guan *Some special Dirichlet problems for the complex Monge-Ampère equation*
  - Gabriele La Nave *On the Kähler-Ricci flow and the V-soliton equation*
  - Qun Li *Complex Monge-Ampère equations and totally real submanifolds*
  - Song-Ying Li *On the rigidity theorems and problems associated to degenerate elliptic operators*
  - Zhiqin Lu *Remarks on hypersurface K-stability*
  - Jian Song *The Kähler-Ricci flow through singularities*
  - Jacob Sturm *Regularity of geodesic rays*
  - Gábor Székelyhidi *On convergence of the Kähler-Ricci flow*
  - Valentino Tosatti *Complex Monge-Ampère equations on symplectic and Hermitian manifolds*
  - Shing-Tung Yau *Canonical metrics and Monge-Ampère equations*

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Chapter 20

Mathematical Methods in Emerging Modalities of Medical Imaging (09w5017)

Oct 25 - Oct 30, 2009

Organizer(s): Peter Kuchment (Texas A&M University), David Finch (Oregon State University), John Schotland (University of Pennsylvania), Yuan Xu (Ryerson University)

Overview of the Field

Computerized tomography (CT) is a major method of biomedical imaging, as well as of industrial non-destructive testing, geophysics, and other areas [35, 36, 20]. Various modalities have been developed since 1960s, including for instance the standard by now X-ray clinical "CAT scan", MRI, Optical, Ultrasound, and Electrical Impedance Tomography [25]. All of these techniques have advantages and deficiencies in terms of resolution, cost, safety, sensitivity, specificity, as well as the physiological and metabolic features they can detect. Thus a quest continues for new medical (as well as industrial) imaging modalities [25]. In recent years, a variety of novel methods have been emerging, which has lead to the need of developing the corresponding analytic and numerical tools (e.g., [3, 43, 44, 45, 46]). The mathematics of CT has always been known for challenging and beautiful problems, and the interest of mathematicians (and surely medical professionals and engineers) in CT has been grown steadily.

The workshop assembled an impressive group of 36 mathematicians, physicists, engineers, and medical researchers, in order to assess the novel developments in CT and discuss the outstanding mathematical challenges. Among the participants there were 4 graduate students and 2 postdoctoral associates.

Among the novel techniques addressed were, for instance, elastography [28, 27, 26], electron microscope tomography [13, 40, 41], as well as several emerging so called "hybrid" modalities [1, 3, 4, 16, 21, 22, 23, 39, 47, 43, 44, 45, 46]. In the latter, radiation/waves of different physical nature are combined in such a way that their individual deficiencies cancel out, while advantages combine.

Recent Developments and Open Problems

Computed tomography, as a medical diagnostic technique, is a mature field. However, in the last decade it has experienced fast and major new developments. On one hand, the older CT modalities (X-ray CT, PET, SPECT, MRI, Ultrasound) have been going through improvements, due to technological and mathematical progress [36]. On the other hand, brand new techniques were being developed. The reasons for this advance are manifold. For instance, new physiological and metabolic parameters of biological tissues, e.g. stiffness, electrical conductivity, or hemoglobin oxygenation are attempted to be imaged. Besides, some previously addressed optical and electric parameters (e.g., optical absorption, or electric conductivity) could not be stably imaged by already existing techniques, such as Optical Tomography (OT) or Electrical Impedance Tomography (EIT). Thus, a variety of novel imaging modalities are being developed.

A variety of the so called "hybrid methods" are being introduced and studied. In such techniques, two or more types of physical waves (in most cases, ultrasound and electromagnetic) are involved, in order to overcome the individual deficiencies of each of them and to combine their strengths.

Probably the most developed, both experimentally and mathematically, among these is the so called Thermoacoustic Tomography (TAT), also known as Opto- or Photo-acoustic Tomography (PAT)

[1, 5, 6, 7, 14, 15, 21, 22, 23, 29, 30, 39, 42, 43, 44, 45, 46, 47]. This technique attempts to use the high contrast between cancerous and healthy tissues when irradiated by a radiofrequency electromagnetic wave or a laser beam. In TAT, a brief broad homogeneous microwave pulse irradiates the object. As the result, small portions of the EM energy are absorbed throughout the tissue. The absorption coefficient, and thus amount of energy absorbed, is known to be several times higher in cancerous areas than in the healthy ones, which leads to a wonderful contrast. However, the waves used are too long to allow for high resolution. They are used only to create energy absorption and thus minute heating of the tissues.

In PAT, the same heating is achieved by irradiating by a broadened short laser pulse. However, light is also not suitable for imaging, since at the depth of several centimeters photons enter diffusion regime and the resolution is lost. Imaging in TAT/PAT is achieved by using the thermo-acoustic effect: local heating generates a propagating pressure wave, which can be detected by ultrasound transducers placed around the object of interest. These pressure measurements over a period of time allow one to recover the initial pressure distribution, directly linked to (in a crude approximation proportional to) energy absorbed. The experimental work on TAT/PAT has been going on for about 15 years, resulting in some devices industrially manufactured. However, the sorely needed mathematics of this technique has started being developed in earnest in the last 5-6 years. Major work has been done on describing the forward operator, resolving uniqueness issues, devising inversion algorithms, obtaining stability estimates and range descriptions, and considering reconstructions from incomplete data and related deterioration of images. In spite of these achievements, several important issues remain not completely resolved. One of them is recovering the actual optical properties of the tissues, rather than the initial pressure activated by heating. Another is accounting for and eliminating effect of the ultrasound attenuation. Still another is recovery of the unknown acoustic properties of the medium (in most initial studies the medium is assumed acoustically homogeneous, which might be an acceptable approximation for breast imaging, but not for imaging through a skull).

A lot of attention has been paid recently to improving methods of imaging of the internal conductivity of tissues, which provides medically important information. The standard EIT [8, 9, 11, 18]

is known to suffer from intrinsic instabilities and the related low resolution [31, 12]. These deficiencies are impossible to eliminate, unless additional *apriori* information is incorporated, or the imaging process is modified. In the first case, a successful (albeit expensive) approach was recently designed that combines EIT with MR (magnetic resonance) measurements [33]. The early indications have been that incorporation of MR data stabilizes the reconstruction. However, the mathematics of this technique is rather involved and requires further development.

Another approach (or rather a variety of approaches) to stabilizing EIT is to combine it with ultrasound irradiation, which modifies the electric properties of the medium [4, 23, 30, 29, 47]. Although existence of such a (weak) electro-acoustic effect has been established [19, 24, 47], even the experimental work on this technique is in its infancy, and the mathematics (including the basic modeling) is just starting to been developed. Early indications are that the technique (called AET - electro-acoustic tomography) has a high potential for high resolution reconstruction of conductivity, which is well beyond the EIT possibilities.

Yet another approach is to notice that the currents generated in EIT trigger local heating throughout the object of interest, which in turn creates the same thermoacoustic effect that is used in TAT, and thus TAT reconstruction techniques can be used [16].

An extremely valuable Optical Tomography suffers from the same low resolution and instability ailments as EIT. An approach to OT similar to AET has been extensively studied experimentally in the last decade [43, 44, 46]. Namely, scanning the tissues with a focused ultrasound modifies locally optical properties of the tissues and thus influences the boundary optical measurements. This additional "internal" information is believed to be able to stabilize the optical tomography procedure. So far, most of the experimental set-ups work only at a shallow depth and are of the direct measurement, rather than reconstruction, type. Going deeper (a few centimeters) inside the tissue, as is needed in breast imaging, requires reconstruction techniques. The mathematics needed here is in its infancy [2, 34, 5], and even the mathematical models are being debated (as they were at this BIRS workshop).

Another recently developing medical imaging modality is elastography, which attempts to image mechanical properties (e.g., stiffness) of tissues, which are known to provide valuable medical information. Although this field is in early stages of both experimental and mathematical development, the initial experimental and mathematical studies show a high potential for medical applications.

There are manifold other novel techniques, such as for instance Electron Microscope Tomography (ET) and Cryo-Imaging, which are actively being developed for small (nano-)scale imaging of biological samples, including protein imaging. ET still faces manifold technological, and even more mathematical challenges and is being actively developed.

Presentation Highlights

Lectures at the workshop were given by S. Arridge (University College London), G. Bal (Columbia University),

W. Bangerth (Texas A&M University), P. Burgholzer (Upper Austrian Research), S. Carney (University of Illinois Urbana-Champaign), D. Finch (Oregon State University), D. Isaacson (Rensselaer Polytechnic Institute), A. Katsevich (University of Central Florida), P. Kuchment (Texas A&M University), L. Kunyansky (University or Arizona), A. Lawrence (National Center for Microscopy and Imaging Research, UC San Diego), C. Li (Washington University, St Louis), A. Manduca (Mayo Clinic), V. A. Markel (University of Pennsylvania), J. McLaughlin (Rensselaer Polytechnic Institute), S. Moskow (Drexel University), A. I. Nachman (University of Toronto), F. Natterer (University of Münster), V. Palamodov (Tel Aviv University), E. T. Quinto (Tufts University), E. Ritman (Mayo Clinic College of Medicine),

O. Scherzer (University of Vienna), J. Schotland (University of Pennsylvania), P. Stefanov (Purdue University), Y. Xu (Ryerson University, Toronto).

The presentations were, as much as possible, clustered according to the topics. Three active panel discussions were held after dinner.

We present below the outlines of the lectures.

Several talks were devoted to the actively being developed thermoacoustic/photoacoutic tomography (TAT/PAT).

A survey of main mathematical results and open problems of thermoacoustic tomography was presented in the joint talk "Can one hear the heat of a body? Survey of the mathematics of Thermo- and Photo-acoustic tomography" by D. Finch, P. Kuchment, and L. Kunyansky. Issues of mathematical modeling, uniqueness of reconstruction, inversion algorithms, stability, incomplete data problems, and others were addressed.

P. Stefanov, in a joint talk with G. Uhlmann "Thermoacoustic tomography with a variable sound speed", considered the mathematical model of thermoacoustic tomography in a medium with a variable speed for a fixed observation time interval $[0, T]$, such that all signals issued from the domain reach its boundary by time T . In case of measurements on the whole boundary, a solution in terms of a Neumann series expansion was described. Conditions close to necessary and sufficient were given for uniqueness and stability of reconstruction when the measurements are taken on a part of the boundary.

P. Burgholzer, in his talk "Image Reconstruction in Photoacoustic Tomography taking acoustic attenuation into account" addressed the important, and still not completely resolved [10, 32], issue of accounting and compensating for ultrasound attenuation in TAT. Optical detectors can provide a high bandwidth up to several 100 MHz, but the resolution is often limited by the acoustic attenuation in the sample itself, because attenuation increases with higher frequencies. Compensation for this frequency-dependent attenuation is an ill-posed problem and is limited by the thermodynamic fluctuation of the measured pressure around its mean value. These fluctuations are closely related to the dissipation caused by acoustic attenuation (fluctuation dissipation theorem) and therefore a theoretical resolution limit for the maximal compensation of photoacoustic attenuation can be estimated.

While most works on PAT are devoted to recovery of the initial pressure, which is considered the sought for tomogram, in fact the underlying optical parameters of the tissues are of interest. This barely touched problem was addressed in the talks by S. Arridge and by G. Bal.

S. Arridge, in the joint talk “Quantitative Reconstruction in PhotoAcoustic Tomography” with Ben Cox and Paul Beard,

addressed the problem of quantifying the optical properties underlying the sound generation, which requires one to consider coupled models for optical and acoustic propagation. The talk presented some recent work on this problem, which utilizes a non-linear algorithm for recovering optical absorption coefficient.

The talk “Inverse transport and inverse diffusion theories for photoacoustics” by G. Bal also addressed the issue of reconstructing the optical properties from the deposited radiation, which is assumed to be obtained on the first, “standard” PAT step. It was shown what can or cannot be reconstructed in the optical parameters for two regimes of propagation of radiation: transport (joint work with A. Jollivet and V. Jugnon) and diffusion (joint work with G. Uhlmann) [7, 6].

Experimental approaches to and problems arising in PAT were considered in the Thursday presentation by C. Li (joint work with L. Wang and others).

Two talks were devoted to various versions of the recently being actively developing elastography and its medical applications.

J. McLaughlin, in her talk “Biomechanical Imaging: Viscoelastic Models, Algorithms, Reconstructions; Application to Breast, Prostate and Brain” gave an overview of the recent work of her group devoted to mathematics of bio-mechanical imaging and its applications. Biomechanical imaging was described as a promising new technology that enables monitoring of and predicting disease progression and the identification of cancerous and fibrotic tissue. The dynamic data, which was the input for the work described, were movies of propagating or harmonic waves. These movies were created from sets of MR or ultrasound data that was acquired while the tissue was moving in response to a pulse or an oscillating force. The main characteristics of these movies are: either (1) there is a wave propagating with a front; or (2) there is a traveling wave created by two sources oscillating at different but nearly the same frequencies; or (3) there is multifrequency harmonic oscillation. It was shown how data with the characteristics (1) or (2) above could be applied for cancer identification. The remaining part of the talk concentrated on the mathematical model, algorithms and reconstructions for movie data acquired when the tissue is undergoing response to a single or multifrequency harmonic oscillation. Viscoelastic and elastic models were discussed, as well as approximations to the mathematical model, with the related error estimates and algorithms. Stability and accuracy of algorithms were addressed. It was also discussed, why some biomechanical parameters cannot be reliably recovered. Images were presented that were reconstructed from synthetic, in vivo, and in vitro data.

The talk by A. Manduca “Magnetic Resonance Elastography: Overview and Open Problems” described the work on MRE done by a Mayo Clinic group. An overview of MRE and inversion techniques was presented, along with a discussion of open problems and issues. Preliminary studies indicate that MRE has substantial potential as a diagnostic tool, since it can quantitatively and non-invasively measure full $3D$ vector displacement data from propagating acoustic waves in vivo. From these data, inversion algorithms can calculate biomechanical tissue properties such as stiffness, viscosity, and anisotropy.

As it has already been mentioned, the standard Electrical Impedance Tomography (EIT) suffers from instabilities and low resolution. Several talks addressed recent attempts of creating hybrid imaging modalities that involve electrical impedance measurements as a part and aim to overcome the EIT deficiencies.

One of such modalities is the current density impedance imaging (CDI), an emerging method that combines magnetic resonance and electrical impedance measurements. Using an MR imaging device allows one to find current density inside the object to be imaged. This additional information leads to the possibility of stable reconstruction of the conductivity of the tissues. A tutorial on CDI was given by A. Nachman in his talk “Current Density

Impedance Imaging”, where he described the results of joint work with several mathematicians and engineers on theoretical and experimental development and implementation of this technique.

It was reviewed how the current density can be determined inside an object using a Magnetic Resonance Imager (a technique invented by M.Joy’s group). If two currents are available, an analytic formula for calculating the conductivity was described, and experimental validation of the method (joint work with K. Hasanov, W. Ma, and M. Joy) was demonstrated. Much of the talk addressed the problem of reconstructing the conductivity from knowledge of just the magnitude of one current in the interior (joint work with A. Tamasan and A. Timonov). The corresponding equipotential surfaces happen to be minimal surfaces in a conformal metric determined by the given data. Proof of identifiability was described and examples of numerical reconstructions given. The discovery that it suffices to measure just the magnitude of one current may lead to novel physical approaches to obtaining this data directly.

The lecture was videotaped and is available at the BIRS site

http://www.birs.ca/birspages.php?task=eventvideos&event_id=09w5017

Other hybrid techniques that allow one to overcome instabilities of the EIT were described by L. Kunyansky, O. Scherzer, and Y. Xu.

L. Kunyansky, in his joint talk "Synthetic focusing in Acousto-Electric Impedance Tomography" with P. Kuchment,

discussed the so called acousto-electric tomography (AET). In AET, a focused ultrasound beam is scanned throughout the object of interest, thus modifying locally the electric conductivity, which in turn influences the boundary impedance measurements. It was shown on numerical tests that availability of this interior information stabilizes the problem of conductivity reconstruction and allows for sharp images to be recovered. A mathematical model, based on the smallness of the electro-acoustic effect, was introduced and a reconstruction algorithm discussed. Then the underlying assumption of the possibility of "perfect" focusing was addressed. Since in fact such focusing is not practical, a "synthetic focusing" approach was suggested [23] and shown to work in mathematical experiments. In this technique, a set of unfocused waves is used, and the would-be response to the impractical focused illuminations is derived mathematically.

A different way of combining electrical impedance and ultrasound measurements for ultrasound purposes was introduced by O. Scherzer (joint work with B. Gebauer) [16]. Here the starting point is that electric currents create small amount of heating throughout the object, and thus thermo-elastic expansion, similar to the one that is the basis of TAT/PAT. Then the resulting acoustic signal is picked up at the boundary and is used to recover the initial pressure and thus a tomographic image related to the unknown conductivity.

Yuan Xu delivered the lecture "Ultrasound mediated imaging methods for electrical properties of biological tissues."

He described his experimental work on three different methods of combining ultrasound with electromagnetic waves for imaging purposes [30, 29, 47], as well as the underlying physics and the signal strength. The first one utilizes the changes in the ultrasound echoes from biological tissues induced by an external electrical field. The second category is devoted to detecting the electrical potential difference in biological tissues caused by applying an ultrasound field to the tissues. The third type addresses detection of the ultrasound waves emitted by biological tissues, caused by applying an electrical field to the tissues.

Yet another hybrid method under discussion, Acousto-Optical, also called Ultrasound Modulated Optical Tomography (AOT or UOT), involves a combination of optical tomography (OT) with ultrasound illumination. Here the body is irradiated by a laser beam, and the outgoing optical signals are measured at the boundary. Since the aim is to image objects of several centimeters size, one deals with multiple scattering of photons, and thus the diffusion approximation is appropriate. Simultaneous scanning with focused or unfocused ultrasound produces small, but measurable responses in the boundary measurements, which are used to reconstruct the unknown attenuation coefficient inside the body.

The talk by J. Schotland (joint with G. Bal) contained the development of a mathematical model for the case when incoherent light and standing ultrasound waves are used in AOT. An iterative algorithm is described for the cases of recovery the attenuation, or both the attenuation and the diffusion coefficient. It is expected that the reconstruction is well posed (stable), although numerical implementation and rigorous results are still to be developed.

W. Bangerth, who presented in his talk "Reconstructions in ultrasound modulated optical tomography" the joint work with M. Allmaras and P. Kuchment [2], addressed the case when coherence of the light is assumed (as it is in current experimental approaches) and the response at the ultrasound frequency of a boundary correlation function is used as the data. A mathematical model (which triggered an active discussion and some objections) was described and shown to work stably in numerical experiments, reconstructing rather sharp images, impossible in the regular OT. Although main uniqueness and stability questions are still open, a stability result for the linearized problem was described.

Another block of two lectures was devoted to the actively being developed Electron Microscope Tomography (ET) [13, 40, 41] in talks by A. Lawrence and by E. T. Quinto with co-authors.

A. Lawrence's talk "Advances in Large Field and High Resolution Electron Tomography", reflecting on the work done by his group at the Center for Research in Biological Systems of University of California, San Diego, provided an overview of the field and its challenges. The ET is used for constructing 3D views of sectioned biological samples, which are rotated around an axis and images are acquired for each "tilt" angle. It enables high resolution views of cellular and neuronal structures. ET 3D reconstruction encounters numerous challenges, including a low signal-to-noise ratio, curvilinear electron path, sample deformation (warping), scattering, magnetic lens aberrations, etc.

The talk "Electron Microscope Tomography" by E. T. Quinto (reflecting the results of his joint work with O. Öktem, U. Skoglund, and H. Rullgard)

discussed ET from a mathematical perspective. Algorithms being developed and the microlocal analysis behind them were described. For small field ET, one can assume that the electrons travel over lines, and reconstructions that use this model were shown. However, for larger field ET, the electrons travel over curvilinear paths. A new mathematical model for this case was presented that involves a Radon transform over curvilinear paths. Microlocal analysis of this transform and reconstructions from real data, as well as theoretical underpinning of algorithms were presented.

Several lectures were devoted to other tomographic techniques or challenges.

S. Carney presented the talk "Deconstructing the Born series." A new method is proposed for directly obtaining from experiment the separate orders of the scattered fields from the exact scattered field. The approach applies to any system for which a solution may be cast as a Liouville-Neumann series. The method was simulated and shown to reduce multiple-scatter artifacts in linearized inverse scattering. It has potential to also be useful in so-called super-resolution problems.

The lecture "Problems in the diagnosis and treatment of breast cancer" was delivered by D. Isaacson.

He discussed, in particular, how the ordering of the administration of drugs used in the chemotherapy of breast cancer can make a significant difference in the outcome.

A. Katsevich gave the lecture "An accurate approximate algorithm for motion compensation in two-dimensional tomography," in which he proposed an approximate inversion formula for motion compensation in tomography. The formula can be easily implemented numerically. Results of numerical experiments in the fan-beam case demonstrate good image quality even when motion is relatively strong.

V. Markel in his lecture "Inverse radiative transport with the method of rotated reference frames" presented the recently developed method of rotated reference frames. It allows one to obtain the plane-wave decomposition of the Green's function for the radiative transport equation in the slab geometry, which can be efficiently used to solve the linearized inverse problem for the RTE. Mathematical formulation of the method and its application to inverse problems and imaging were discussed and examples of image reconstruction with simulated and experimental data were presented.

S. Moskow delivered the lecture "The Inverse Born Series for Diffuse Waves" (joint work with J. Schotland).

The inverse scattering problem for diffuse waves was considered. The results on convergence, stability and approximation error of the solution derived from inversion of the Born series, as well numerical simulations, were presented. The method has potential for being useful for other problems such as propagating scalar waves and Electrical Impedance Tomography.

In his lecture "Possibilities and limitations of ultrasound tomography," F. Natterer discussed the results of his recent work on ultrasound transmission and reflection imaging (e.g., [37, 38]). Mathematical issues, algorithms, and results of reconstructions were presented. It was concluded that the ultrasound tomography is currently close to clinical use, with reconstruction algorithms available for resolution down to 2mm. Reflection imaging was noted as a possible future development.

V. Palamodov's talk "Reconstruction in Doppler tomography" addressed the problem of recovering a vector field from some integral measurements. Such data arise in tomographic imaging of liquid or gas flows, tumor detection, optics and plasma physics, and other areas. The longitudinal line integrals of a vector field (or a 1-differential form f) form provide what is called Doppler transform. The reconstruction problem in Doppler tomography has some peculiarities, due to the object of reconstruction being more complex than a function. It was mentioned that the transform is gauge-invariant, i.e. all line integrals vanish for any exact form (irrotational part of the field). Thus, only reconstruction of the differential df is possible. The solenoidal part of f can be uniquely reconstructed from df. In $2D$, the problem reduces to Radon transform inversion, while in $3D$ the complete data is redundant. In the talk, some known results were discussed and it was shown that df can be recovered using acquisition geometry similar to the one used for recovering scalar functions.

The method is extended to differential forms of arbitrary degree.

The talk "Micro-Tomographic Imaging of Coherent X-ray Scatter using a Polycapillary X-Ray Optic Imaging System and Multi-Energy X-Ray Detection" was delivered by E. Ritman. It was noticed that coherent scatter of X-rays can convey information about the chemical bonds in the material being irradiated, when the length of chemical bonds is of the order of an X-ray wave length. Consequently, the scatter can be used to discriminate certain chemical compounds (with similar atomic content) that have very similar X-ray attenuation coefficients. The scatter intensity can either be recorded as a function of angle (θ) from an illuminating, monochromatic, x-ray beam, or can be recorded by an energy-binning detector at one angle to a bremsstrahlung x-ray beam. Both methods provide a scatter intensity profile that is characteristic for the illuminated material, but its intensity must be corrected for attenuation of the illuminating beam, as well as of the scattered beam. Thus, a conventional CT image is needed to provide the attenuation map.

Several modes of illumination and scatter recording that trade off signal-to-noise, speed of the data collection and need for Radon-type tomographic reconstruction from line integrals of the scatter were considered. If a collimator is used then line or sheet illumination can provide scatter data from known points within the 3D structure, so that no tomographic reconstruction is needed. If a non collimated detector array is used, or a volume is illuminated and scatter recorded via a collimator, then line integrals are recorded and the object has to be rotated in order to provide the data needed for Radon-type tomographic reconstruction. Polycapillary x-ray optics and energy selective imaging were used to generate tomographic image data that can discriminate polymeric materials with very similar attenuation coefficients.

Outcome of the Meeting

The meeting satisfied the need that existed for addressing the pressing issues of novel tomographic techniques. During the presentations and discussions, the current state of mathematical development of several emerging medical imaging modalities was surveyed and recent results and outstanding challenges were delineated.

The participants of the meeting were extremely actively involved into scientific discussions, especially during the evening panels, which have worked very efficiently.

This has led to strengthening existing and forging new collaborations. There is no doubt that the meeting will facilitate the much needed progress in this practically important and mathematically challenging area.

Many presentation files are posted on the BIRS' Web site
at
http://temple.birs.ca/˜09w5017/

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Chapter 21

Interdisciplinary Workshop on Fixed-Point Algorithms for Inverse Problems in Science and Engineering (09w5006)

Nov 1 - 6, 2009

Organizer(s): Heinz Bauschke (University of British Columbia Okanagan), Russell Luke (University of Delaware), Henry Wolkowicz (University of Waterloo), Regina Burachik (University of South Australia), Patrick Combettes (Universite Pierre et Marie Curie - Paris VI) Veit Elser (Cornell University)

Summary

The objective of this workshop was to bring together researchers with a strong interest in projection and first-order fixed-point algorithms, both from mathematics and from the applied sciences, in order to survey the state-of-the-art of theory and practice, to identify emerging problems driven by applications, and to discuss new approaches for solving these problems.

Various monographs and conference proceedings on projection methods and their applications have been published recently. The participants have not met before and it is very unlikely they will meet again at ordinary optimization conferences. We expect this workshop to be the base for new innovative research and collaborations by its unique mix of experts whose areas of applications are broad, ranging from variational analysis, numerical linear algebra, machine learning, computational physics and crystallography.

Overview of the Field and Relationships with the Workshop

In this section, we highlight some of the recent developments and open problems discussed at the workshop. In particular, we focus on recent scientific progress as well as contributions of participants to the workshop. The topics are grouped into four distinct areas, but common themes that arose throughout the conference are the potential of first-order methods for solving large-scale and/or nonconvex problems, and the need for a theoretical foundation to explain their success. A remarkable aspect of the talks was the role that experimental mathematics has played in the development of theoretical intuition. The use of experimental results on benchmark problems has long been standard practice in research on numerical algorithms, however the use of mathematical software to test theoretical hypotheses is a relatively recent phenomenon. The development of this practice has been well-documented in the recent books by Bailey, Borwein and collaborators [5, 4, 6]. The prevalence of computer-aided mathematical discovery in the presentations at this conference indicates that this methodology has matured to an established practice.

Douglas-Rachford / Difference Map Algorithms

The Douglas-Rachford algorithm [38], which is a linear implicit iterative method, was originally developed in 1956 for solving partial differential equations. In 1979, Lions and Mercier [56] extended the Douglas-Rachford algorithm to an operator splitting method for finding a zero of the sum of two maximal monotone operators (see [31] for an historical account and theoretical details).

The Douglas-Rachford algorithm was discussed in several talks and from different viewpoints [32, 44, 57, 63, 81, 97]. When applied to normal cone operators of two nonempty closed convex sets A and B, with associated projectors P_A and P_B as well as reflectors $R_A = 2P_A - \mathrm{Id}$ and $R_B = 2P_B - \mathrm{Id}$, the governing iteration takes the form

$$x_0 \in X, \qquad (\forall n \in \mathbb{N})\ \ x_{n+1} = \frac{\mathrm{Id} + R_B R_A}{2} x_n, \tag{21.1}$$

where Id denotes the identity operator of the Hilbert space X. Under appropriate assumptions, the so-generated sequence $(x_n)_{n\in\mathbb{N}}$ has the remarkable property that $(P_A x_n)_{n\in\mathbb{N}}$ converges to a solution of the underlying feasibility problem, i.e., to a point in $A \cap B$.

As is the case with good algorithms and ideas in science, this method was rediscovered by different people working in different disciplines. Noteworthy is the application of the Douglas-Rachford algorithm in phase retrieval with a support constraint (as opposed to support and nonnegativity), where it is known as the *hybrid input-output (HIO)* algorithm, pioneered by Fienup [47] in 1982. (See also [8] for a view from convex optimization.) A very interesting development originates with Elser [42], who has recently very successfully applied the Douglas-Rachford algorithm to various continuous and discrete, *nonconvex* problems [43, 50]. In the physics community, the algorithm is now known as the *difference map algorithm* and its product space version à la Pierra [77] as *divide and concur*. Novel applications were given in his talk [44], which is available in video format [45]. The constraint sets that arise in the non-convex settings studied by physicists — e.g. spaces of orthogonal or low-rank matrices — often have projections that can be computed efficiently and yet are outside the scope of conventional, linear programming based methods. By including non-convex constraints in the general formalism even NP-complete problems are open to these solution methods. In such applications, where the iterates behave chaotically, the question of convergence shifts to mathematical themes more closely linked to dynamical systems and ergodicity. Macklem [65] illustrated how the software package *Cinderella* [55] is a visual aid in refuting conjectures and building intuition for the Douglas-Rachford algorithm in low dimensions. The flexibility of the projection-based method in crystallographic applications [58, 8, 89] was illustrated with the protein envelope reconstructions reported in the talk by Lo [57]. Finally, Yedidia [97] reported on a recent modification of the belief propagation algorithm based on the difference map algorithm, which led to a new decoder that is currently state-of-the-art.

Other Projection-type Algorithms

Ben-Israel [18] presented his very recent work [19] on the *inverse* of the classical Newton iteration, which leads to a geometric interpretation of iterations and chaos. Cegielski [28] described general frameworks for projection methods as well as his recent generalization [29] of the classical Opial Theorem, which is of fundamental importance in algorithmic fixed point theory [73]. De Pierro [37] described his recent work on gradient and subgradient methods [53] and provided applications to SPECT (Single Photon Emission Computed Tomography). Ideas of self similarity were presented by Ebrahimi [39], who considered Banach contraction-based techniques [40]. Based on a statistical multiscale criterion, Marnitz [66] proposed an algorithm for solving linear ill-posed equations. His algorithm also employs Dykstra's method for finding the best approximation to the intersection of convex sets. Another new application of projection type methods was reported by Mostafa Nasri [71]. Nasri combines an augmented Lagrangian scheme with projections, for solving equilibrium problems whose feasible sets are defined by convex inequalities. This method finds first an approximate solution of an unconstrained equilibrium problem, and then, either an extragradient-type step or a projection onto a suitable hyperplane is performed.

Inverse Problems, Convex Analysis, Optimization

In the inverse problems community, a lot of work is currently focused on the development of efficient numerical techniques for solving minimization problems under sparsity-promoting constraints, e.g. [33, 36, 88, 91, 27], as well as rank reducing constraints, e.g. [46, 75]. Plemmons [78] opened the workshop with a presentation on spectral image analysis. He showed the importance of identifying and quantifying the materials present in the object or scene being imaged. He described a variational fuzzy segmentation model coupled with a denoising/deblurring model based on fast total variation regularized computations, [98]. Beck [15] presented developments in the spirit of his recent work in image recovery [16, 17] that aim at improving acceleration techniques originally proposed by Nesterov [72]. Luke [63] described a dual-space method developed with Jonathan Borwein in which the regularized dual problem is solved via a subgradient descent method with exact line search [20]. The selection of the "best" subgradient is formulated as a best approximation problem, to which a relaxed Douglas Rachford algorithm [64] is applied. The problem of *sensor network localization* was addressed by Henry Wolkowicz [92]. This problem can be modeled as a rank constrained semidefinite programming problem. However, the special structure of the problem allows one to take advantage of the NP-hard rank constraint and solve huge problems (of the order of a million sensors) in reasonable time to machine precision, [54]. A new first order algorithm for a class of smooth constrained minimization problems, called *the moving balls approximation method*, was presented by Marc Teboulle [86], see also [2]. This relies on a simple geometric idea that approximates the constraint set by a sequence of balls, and combines this with a fixed point approach. Another approach to nonsmooth optimization problems arising in signal processing was proposed by Yamada [93], who employed the Moreau envelope to smooth the original problem and used a fixed point model to represent the constraints in the spirit of [94, 95].

Another interesting development in the use of modern optimization tools in signal processing was proposed by Modersitzki [68] in the context of regularized variational image registration (see also [69]).

Convex combinations of resolvents and the underlying potential (the "proximal average") were considered by Wang [90], with particular emphasis on applications in linear algebra, and by Moffat [70] for non-quadratic kernels. (See [9, 11] for underlying theory.) Nonconvex variations were explored by Hare [52]. Bauschke [7] described results on Chebyshev and Klee sets with respect to Bregman distances induced by Legendre functions [13, 14, 10]. Lucet [61] described his implementation of graph-calculus for computational convex analysis, based on a calculus introduced by Goebel [51]. Bot [21] surveyed recent work on the stability of Fenchel duality [22, 23], which provide an answer to a problem posed by Simons. Corvellec [34] presented new results [3, 35] on the *error bound principle*. Yao [96] provided two linear maximal monotone operators that he used to show that the answer to a recent question by B.F. Svaiter [85] is negative [12]. Zinchenko [99] reported on a cost-effective usage of GP-GPUs to accelerate linear-algebraic computations needed to solve large scale optimization problems arising in intensity modulated radiation therapy treatment planning [30]. An application of augmented Lagrangian schemes for nonconvex and nonsmooth problems was presented by Burachik [24]. She described the recently devised Inexact Modified Subgradient algorithm for solving the (convex) dual of a nonconvex optimization problem. Even though the original problem is nonconvex, the method presented enjoys both primal and dual convergence.

In honor of Rudolf Kalman being awarded the US National Medal of Science, Jim Burke presented an interior point algorithm for computing Kalman-Bucy smoothers with constraints [25, 26]. The method obtains the same decomposition that is normally obtained for the unconstrained Kalman-Bucy smoother, hence the resulting number of operations grows linearly with the number of time points.

Scherzer [82] presented some theoretical results establishing the relationship between lower semicontinuity and separate convexity for non-local functionals that have attracted attention in image denoising. Extending his techniques further, Scherzer outlined novel characteristics of Sobolev spaces to derive approximations of the total variation energy regularization and hence to recover existing numerical schemes for total variation minimization in addition to novel numerical schemes.

Monotone Operator Theory

Algorithm (21.1) is the iteration of a firmly nonexpansive mapping. Eckstein in his thesis [41] noted that even though the projection operators P_A and P_B are proximal mappings[1], the operator iterated in (21.1)

[1] Schaad provided a simpler example where A and B are the x-axis and the diagonal in $\mathbb{R}^2$ [81], respectively.

need itself *not* be a proximal mapping, i.e., it may be the resolvent of a maximal monotone operator that is *not* a subdifferential of a convex function. As such, *monotone operator theory* appears to be critical for a proper understanding of the Douglas-Rachford / difference map algorithm. The talks by Revalski [79] and by Simons [83] go in this direction: Revalski [48, 49, 76, 80] surveyed extended and variational sums of monotone operators, which are believed to play a role in the analysis of algorithms featuring resolvents when the sum of the underlying maximal monotone operators is not maximal, and Simons considered generalizations of maximal monotonicity to more abstract settings [84], nowadays called Simons or SSD spaces. López [60] and Martín-Márquez [67] explored monotone operator theory on Hadamard manifolds, including convergence results for a proximal point algorithm. Another important aspect of monotone operators is duality [1, 74]. Combettes [32] examined composite monotone inclusions in duality and proposed primal-dual proximal splitting algorithms to solve them.

Outcome of the Meeting

The organizers will edit a Conference Proceedings volume entitled *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, part of the Springer-Verlag series "Optimization and Its Applications". A good number of the participants will contribute to this volume. In addition, several researchers who were unable to attend the workshop have committed manuscripts as well, including: J. Borwein (Newcastle), Y. Censor (Haifa), G.T. Herman (New York), and S. Reich (Technion).

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Chapter 22

Statistical Mechanics on Random Structures (09w5055)

Nov 15 - 20, 2009

Organizer(s): Pierluigi Contucci (University of Bologna), Anton Bovier (Rheinische Friedrich-Wilhelms-Universität Bonn), Frank den Hollander (University of Leiden and EURANDOM), Cristian Giardina (TU Eindhoven and EURANDOM)

Overview of the Field

The theme of the workshop has been equilibrium and non-equilibrium statistical mechanics in a random spatial setting. Put differently, the question was what happens when the world of interacting particle systems is put together with the world of disordered media. This area of research is lively and thriving, with a constant flow of new ideas and exciting developments, in the best of the tradition of mathematical physics.

Spin glasses were at the core of the program, but in a broad sense. Spin glass theory has found applications in a wide range of areas, including information theory, coding theory, algorithmics, complexity, random networks, population genetics, epidemiology and finance. This opens up many new challenges to mathematics.

Recent Developments and Open Problems

The workshop brought together researchers whose interest lies at the intersection of disordered statistical mechanics and random graph theory, with a clear emphasis on applications. The multidisciplinary nature of the proposed topics has attracted research groups with different backgrounds and thus provided exchange of ideas with cross-fertilisation. As an example, we mention two problems on which we focused during the workshop.

The first problem has its origin in the many fundamental issues that are still open in the theory of spin glasses. Even tough today we have a rigorous proof, in the context of mean-field models, of the solution for the free energy first proposed by G. Parisi, certain relevant properties of this solution (e.g. ultrametricity) still lack a firm mathematical understanding. Moreover, when considering spin-glass models on the lattice with short-range interaction, the mean-field picture that predicts the existence of many different equilibrium states is challenged by the droplet scenario, where only a few relevant equilibrium states are predicted (related via the spin-flip symmetry).

The second problem concerns the application of ideas and methods from statistical mechanics to the social sciences. Dichotomic social issues (yes or no matters) are known to manifest sudden changes, very much like phase transitions. An approach that has appeared recently is to use mean-field models (i.e., non-local interactions) to describe peer-to-peer relations, possibly extended to social interaction networks carrying the interesting small-world and scale-free features of random graphs. The final aim of this investigation is to establish conditions for opinion spreading, nucleation, cluster formation, and other observed social phenomena, especially within the enormous migratory fluxes nowadays present in developed countries. The use of statistical mechanics for social purposes will also lead to the necessity of statistical parameter estimation via polls, a newly emerging field with several applications.

The workshop took advantage of an introductory meeting on the same topic organized at EURANDOM, Eindhoven, The Netherlands, in March 2008. Some of the students and postdocs who attended YEP-V (the fifth in a successful series aiming at Young European Probabilists) had the occasion to meet again in Banff, together with leading scholars in the field.

Presentation Highlights

Here we summarize the themes that were discussed during the workshop.

• Edwards-Anderson model

Monday was largely devoted to the Edwards-Anderson (EA) model, which describes short-range spin glasses.

In a series of four lectures, presented by *Chuck Newman*, *Dan Stein*, *Louis-Pierre Arguin* and *Michael Damron*, the notion of "metastate" was introduced, and the question was addressed whether or not the EA-model (with a continuous, bounded and symmetric disorder distribution) has one or more pairs of ground states. These two options distinguish between the "droplet picture" and the "mean-field picture". It was shown that for the two-dimensional EA-model in the half-plane there is only one pair of ground states. The proof combines ideas from two-dimensional dependent percolation with a detailed study of how ground states are affected by local spin-flips.

Claudio Giberti presented results of numerical simulations on the three-dimensional EA model which indicates that overlap is a good order parameter, in the sense that states conditioned to a given value of the overlap are clustering. He also discussed analytically the effect of flipping the interaction in a subvolume of the lattice: from bounds on the fluctuation, new integral identities involving the overlap quenched expectation are derived.

Hidetoshi Nishimori spoke about the EA-model in the setting of the replica method, and showed that on self-dual lattices a recursion formula for the replicated partition function combined with a gauge argument yields a zero-temperature phase transition.

• Disordered statistical mechanics model on graphs

Tuesday was largely devoted to statistical mechanics on sparse random graphs.

Andrea Montanari gave an introduction to the Ising model on graphs that are locally tree-like, showing that interesting behavior occurs at the critical line. He proved that in the low-temperature phase the Gibbs measure converges locally to a symmetric linear combination of the 'plus' and 'minus' state.

Amir Dembo pulled the theory into a more general context and reviewed a number of results for statistical mechanical models away from criticality.

Sander Dommers spoke about the Ising model on a sparse random network obtained by randomly connecting vertices, called the "configuration model". He argued that the critical behavior is sensitive to the exponent in the tail of the distribution of the number of connected neighbors of a typical point.

Lenka Zdeborova spoke about the random field Ising model on sparse random graphs, and also about some correlation inequalities ruling out earlier conjectured phases.

Guilhem Semerjian explained how to construct quantum spin models on sparse random graphs using the cavity method in a non-cummutative setting. The results are of interest e.g. for quantum computing and quantum annealing.

• Disordered systems

Various topics in the realm of disordered systems were addressed.

Michael Aizenman spoke about "Imry-Ma" (volume/surface) versus "Parisi-Sourlas" (dimensional reduction) for the random field Ising model, showing that in two dimensions disorder destroys the phase transition no matter how weak. He subsequently argued that for quantum systems the same type of result holds, requiring however different techniques.

Gerard Ben Arous spoke about complexity of the spherical p-spin Hamiltonian, showing that the number of critical points of a given degree (minima, saddle points, etc.) at a given height can be counted in terms of the spectrum of large random matrices drawn from the Gaussian Orthogonal Ensemble. This provides an interesting link between two major research areas.

Silvio Franz provided a renormalization scheme for p-spin models on a Dyson hierarchical lattice, where the interaction between the spins depends on their hierarchical distance and decays with a given exponent. This model interpolates between the mean-field model (infinite range) and the random energy model (zero range). The renormalization scheme is hard to control mathematically, but numerical computations suggest the presence of a phase transition similar to that in the random energy model.

Balint Virag reviewed a number of topics from the theory of random Schrödinger operators, in particular, the decay of the probability of large gaps in the spectrum of associated large random matrices, indicating a strong form of repulsion between the eigenvalues.

• Mean-field spin glasses and neural networks

Various results for mean-field spin glasses and neural networks were presented.

Shannon Starr explained the theory of thinning of random partition structures, which serve as a caricature for the description of the organization of phases in mean-field spin glasses. He outlined the central role that is played by the Poisson-Kingman partition structure.

Anton Klimovsky considered the SK-model with vector-valued spins. Examples are the Potts spin glass and the Heisenberg spin glass. A Parisi-type solution was found for the case where the a priori distribution of the spins is Gaussian. For an alternative model, based on isotropic Gaussian processes, a full description of the replica symmetric solution and the solution with partial and full replica symmetry breaking can be obtained. The behavior is similar to that of the random energy model.

Alessandra Bianchi derived sharp estimates on metastable exit times for the Curie-Weiss random field Ising model subject to Glauber spin-flip dynamics. For a particular starting distribution called the last-exit biased distribution, the exit time can be computed with the help of potential theory. By using coupling methods, it can subsequently be shown that the exit time is largely independent of the starting configuration, allowing for a proof that the normalized exit time has an exponential distribution.

Christof Külske spoke about metastates for mean-field disordered systems, such as the Curie-Weiss random field Ising or Potts model with binary disorder. He showed that the metastates are convex combinations of a restricted set of pure states, called the "visible" phases, while other pure states never occur, called the "invisible" phases. He also explained how the weights in the convex combination can be computed.

Adriano Barra spoke about the Hopfield model with Gaussian random patterns. With the help of a new interpolation method he was able to find the replica symmetric solution of the model. The model can be applied to describe the phenomenon of migration integration in social networks.

• Graphs, random graphs and probabilistic structures

A number of related topics on graphs were presented.

Allon Percus addressed the problem of bisection of the Erdös-Renyi random graph. Of particular interest is the regime where the giant component takes up more than half of the vertices. He showed that, just past the critical threshold for this regime, the bisection solutions occur in clusters. This result requires a detailed understanding of the geometry of the giant component.

Malwina Luczak considered the problem of generating a countably infinite partially ordered random set with an order-invariance property, i.e., its probability distribution is exchangeable. It turns out the generation can be done with the help of a Markovian growth process that adds on elements sequentially. She showed how these growth processes can be classified.

Raffaela Burioni spoke about statistical mechanics on "physical" graphs, i.e., infinite graphs with certain large scale regularity properties that are common in applications. She explained why the spectral dimension of such graphs plays a central role in the behavior of a wide range of random processes on the graph, including random walks, random polymers and Ising models.

• **Applications**

Several applications of spin-glass theory were presented.

Matteo Marsili told the participants about models from statistical mechanics aimed at describing the recent financial crisis. These are based on dynamic random networks where the sites represent banks, the edges represent trading between banks, and parameters are chosen in accordance with a priori insights. Such models may exhibit sharp transitions between ”non-crisis” and ”crisis” states, corresponding to phase transitions of the network. This allows for some control of the network.

Nicolas Macris presented a theory for error correcting codes based on sparse random graphs, such as the “low density parity check” code. The code words to be transmitted along the noisy channel are the ground states of a spin Hamiltonian with random interactions. The spin glass phase of the Hamiltonian allows for many ground states and therefore for an efficient transmission. The theory uses cluster expansion techniques.

Nicolas Kistler talked about Branching Brownian Motion and its relation to Derrida’s REM and GREM models. This process is relevant in application of spin glasses to evolution and population models, where a recent class of models having mutation and selection are described by the FKPP-equation.

Elena Agliari discussed the use of statistical mechanics on random structures to model the immune system. She introduced a random bond Ising ferromagnet embedded in a random diluted network, which provides good accuracy in explaining the autopoietic nature of the immune system.

David Sherrington closed the workshop with a talk that presented many possible new applications of spin-glass models to range-free networks. For instance, he described a model having a phase transition from a regime of Poisson-degree distribution to a regime of power-law degree distribution.

Abstracts of the talks

(in alphabetic order by speaker surname)

- *Speaker*: Elena Agliari (University of Parma).

  Title: The autopoietic immune network: a statistical physics perspective.

  Abstract: A systematic approach to the immune system has been argued already three decades ago, yet, due to lack of a paved mathematical backbone, this was not investigated exhaustively. Here we develop a minimal model, which takes into account the reciprocal affinities among immunoglobulins, giving rise to a random-bond Ising ferromagnet embedded in a diluted network. We first discuss the topology of the emerging underlying graph and the statistical mechanics approach for its study. Then, we derive its thermodynamics and analyse both the equilibrium and the linear response regimes by means of mathematical modeling and extensive numerical simulations. Our results are consistent with experimental data and strongly support the autopoietic nature of the immune system.

- *Speaker*: Michael Aizenman (Princeton University).

  Title: Effects of quenched disorder on 1st-order quantum phase transitions (QPT_1).

  Abstract: The talk will present the recent proof that the addition of an arbitrarily small random perturbation to a quantum spin system rounds a first-order phase transition in the conjugate order parameter in $d > 2$ dimensions, or for cases involving the breaking of a continuous symmetry in $d > 4$. This establishes rigorously for quantum systems the existence of the Imry-Ma phenomenon, which for classical systems was proven by Aizenman and Wehr. The extension was enabled by an improvement in the argument, in which the role of the classical metastate (covariant Gibbs equilibrium measure) was de-emphasized and replaced by a more direct analysis of the free energy. This is joint work with R.L. Greenblatt and J.L. Lebowitz.

- *Speaker*: Louis-Pierre Arguin (New York University).

  Title: Uniqueness of the ground state for the EA model in the half-plane II.

Abstract: We consider the EA model on the half-plane with Gaussian (or other) couplings, zero external field and periodic boundary conditions in the horizontal coordinate and free boundary conditions in the vertical coordinate. We show that, for almost every realization of the couplings, the distribution on Ground State Pairs (in the metastate sense) is supported on a single pair. This talk, the third of a four-part series, presents results that are joint work of L.-P. Arguin, M. Damron, C. Newman and D. Stein.

- *Speaker*: Gerard Ben Arous (New York University).

  Title: Critical points of random Morse functions on the sphere.

  Abstract: How many critical points does a random (Gaussian) smooth Morse function have on a large dimensional sphere? How many below a given level and of a given index? In physics this question comes under the name of Complexity of Spherical Spin Glasses. In a joint work with J. Cerny and A. Auffinger we solve it using Random Matrix Theory. I will show that the answer presents a surprising structure for the low-lying critical poins of Spherical Spin Glasses.

- *Speaker*: Adriano Barra (Roma University).

  Title: The replica symmetric scenario in the analogical neural network.

  Abstract: We present some recent progress in our understanding of neural networks whose patterns are stored continuously (Gaussian weighted) on the real line. Mapping this model onto an equivalent bipartite spin-glass, we use a new interpolating scheme to obtain the free energy explicitly in the replica symmetric Ansatz. Then we study the rescaled order parameter fluctuations to identify, via their divergences, the critical line defining ergodicity breaking, which indeed matches earlier results by Amit and coworkers in the dichotomic counterpart. This is joint work with Francesco Guerra.

- *Speaker*: Alessandra Bianchi (Bologna University).

  Title: Coupling in potential wells: pointwise estimates and exponential laws in metastable systems.

  Abstract: In many situations of interest, the potential-theoretic approach to metastability allows to derive sharp estimates for quantities characterizing the metastable behavior of a given system. In this framework, the average metastable times can be expressed through the capacity of corresponding metastable sets, and capacities can be estimated with the application of two different variational principles. After recalling these basic concepts and techniques, we will describe a new method to couple the dynamics inside potentials wells. Under some general hypothesis, we will show that this yields pointwise estimates and exponential laws on metastable times. Our key example will be the random field Curie-Weiss model.

- *Speaker*: Raffaella Burioni (Parma University).

  Title: Statistical mechanics on physical graphs and spectral properties.

  Abstract: The spectral dimension of an infinite physical graph, defined according to the asymptotic behavior of the infrared singularities of the Gaussian model, appears to be the right generalization of the Euclidean dimension d of lattices to non-translational invariant networks when dealing with dynamical and thermodynamical properties. The spectral dimension exactly replaces d in most physical laws where dimensional dependence explicitly appears: the spectrum of harmonic oscillations, the average autocorrelation function of random walks, the generalized Mermin-Wagner theorem and the generalization of the Fröhlich-Simon-Spencer bound to graphs. We present the proof of the invariance properties of the spectral dimension under quasi-isometries (including coarse-graining and addition of finite-range couplings) and we discuss its relevance in phase transitions on graphs.

- *Speaker*: Michael Damron (Princeton University).

  Title: Uniqueness of the ground state for the EA model in the half-plane I.

  Abstract: We consider the EA model on the half-plane with Gaussian (or other) couplings, zero external field and periodic boundary conditions in the horizontal coordinate and free boundary conditions in the vertical coordinate. We show that, for almost every realization of the couplings, the distribution on Ground State Pairs (in the metastate sense) is supported on a single pair. This talk, the fourth of a four-part series, present results that are joint work of L.-P. Arguin, M. Damron, C. Newman and D. Stein.

- *Speaker*: Amir Dembo (Stanford University).

  Title: Unimodularity, random cluster models and Bethe states on sparse random graphs.

  Abstract: Theoretical models of disordered materials lead to challenging mathematical problems with applications to random combinatorial problems and coding theory. The underlying mathematical structure is that of many discrete variables that are strongly interacting according to a mean-field model determined by a random sparse graph. Focusing on random cluster measures on graphs that converge locally to trees, we review recent progress in validating the 'cavity' prediction for the limiting free energy per spin. This talk is based on collaborations with Andrea Montanari and with Nike Sun.

- *Speaker*: Sander Dommers (Eindhoven University)

  Title: Ising models on power-law random graphs.

  Abstract: In many real-world networks, such as the Internet and social networks, power-law degree sequences have been observed. This means that, when the graph is large, the proportion of vertices with degree k is asymptotically proportional to k^a, for some $a > 1$. Often, these networks have a degree distribution with finite mean, but infinite variance $(2 < a < 3)$. We will study a ferromagnetic Ising model on random graphs with a power-law degree distribution and compute the thermodynamic limit of the pressure when the mean degree is finite $(a > 2)$.

- *Speaker*: Silvio Franz (Universite Paris-Sud 11).

  Title: Hierarchical random energy models.

  Abstract: A long-standing problem in statistical physics of disordered systems is the possible existence of ideal glassy phases beyond mean-field theory. In this talk I will discuss convergent evidences for a finite-temperature condensation in a hierarchical REM coming from: (1) exact numerical solution; (2) high temperature series expansion; (3) stability arguments.

- *Speaker*: Claudio Giberti (Modena and Reggio Emilia University).

  Title: Rigorous and numerical results for the Edwards-Anderson model.

  Abstract: The first part of the talk will be devoted to a numerical study of conditional quenched measures in the Edwards-Anderson model. In particular two issues are discussed in the restricted ensemble: the relative fluctuations of different overlaps (related to the property of overlap equivalence) and the structure of overlap correlation (clustering). In the second part of the talk I will discuss the properties of fluctuations of free and internal energies of two spin-glass systems that differ by having some of the interactions flipped. From a bound on fluctuations new overlap identities for the equilibrium state are obtained.

- *Speaker*: Nicolas Kistler (Bonn University).

  Title: REM, GREM and Branching Brownian Motion.

  Abstract: Derrida's Random Energy Models (REM and GREM) have played a crucial role in our understanding of the Parisi Theory. It has however become clear that this class of models cannot fully encode the large-time properties of more realistic spin glasses of mean-field type. A natural extension of the GREM is the so-called hierarchical field, the continuous counterpart being Branching Brownian Motion (BBM): both models have an in-built hierarchical structure where the number of levels in the underlying tree grows with the size of the system. Contrary to the GREM, for which we have a remarkably accurate and rigorous understanding, the microscopic properties of BBM still remain rather mysterious, even at a heuristic level. I will try to give an account of what is known, and report on some modest progress from an ongoing project with L.-P. Arguin and A. Bovier.

- *Speaker*: Anton Klimovsky (Erlangen-Nurnberg University).

  Title: Around one-and-a-half Parisi-type formulae for the free energy.

Abstract: We start by describing our results on the Sherrington-Kirkpatrick model with multidimensional spins. We identify the candidates for the order parameters and for the Parisi-type functional. These candidates are related to the free energy through a saddle-point variational formula obtained by means of Guerra's interpolation. So far, we have not been able to prove the Parisi-type formula for the general a priori distributions of multidimensional spins, though we can do so in the case of the Gaussian distribution. The proof boils down to showing that Guerra's remainder term vanishes on the optimiser of the Parisi-type functional. In the second part of the talk, motivated by recent work of Fyodorov and Sommers concerning the Gibbsian random landscapes generated by isotropic Gaussian random processes indexed by high-dimensional Euclidean balls, we prove the Parisi-type formula for the free energy of a single particle in the random landscape. One of the main messages that can be extracted from our analysis is that the overlap-like order parameters (familiar in the context of disordered spin systems) are fundamental also in the context of continuous parameter Gaussian processes with isotropic stationary increments (at least, if the parameter space is high-dimensional). The Gaussian processes are allowed to have short- and long-range correlations (e.g., the multiparameter fractional Brownian motion). Depending on whether the Gaussian process has short- or long-range correlations, the order parameter is either a step distribution function with two jumps (one step of replica symmetry breaking) or a continuous distribution function (full replica symmetry breaking), respectively. Both proofs of the Parisi-type formulae are based on the techniques of the remainder term estimates due to Talagrand and exploit the abundantly available rotational symmetries.

- *Speaker*: Christof Külske (Bochum University).

  Title: The metastate approach to random statistical mechanics systems.

  Abstract: The metastate is a probabilistic concept to describe random symmetry breaking. It was introduced by Chuck Newman and Dan Stein to describe the behavior of random systems in situations when the Gibbs measure is not unique, by assigning probability weights to each Gibbs measure according to how frequently it appears in sequences of large volumes. We will discuss lattice and mean-field systems, including a new geometric characterization of visible and invisible phases in the mean-field setup. This is joint work with Giulio Iacobelli.

- *Speaker*: Malwina Luczak (London School of Economics).

  Title: Order-invariant measures on causal sets.

  Abstract: A causal set is a partially ordered set on a countably infinite ground-set such that each element is above finitely many others. A natural extension of a causal set is an enumeration of its elements that respects the order. We bring together two different classes of random processes. In one class, we are given a fixed causal set, and we consider random natural extensions of this causal set: we think of the random enumeration as being generated one point at a time. In the other class of processes, we generate a random causal set, again working from the bottom up, adding one new maximal element at each stage. Processes of both types can exhibit a property called order-invariance: if we stop the process after some fixed number of steps, then, conditioned on the structure of the causal set, every possible order of generation of its elements is equally likely. We develop a framework for the study of order-invariance that includes both types of example: order-invariance is then a property of probability measures on a certain space. Our main result is a description of the extremal order-invariant measures. This is joint work with Graham Brightwell.

- *Speaker*: Nicolas Macris (Ecole Polytechnique Lausanne).

  Title: Correlations in sparse graph error correcting codes.

  Abstract: The subject of the talk will be transmission over noisy channels using error correcting codes based on sparse graphs. The optimal decoder based on the posterior measure over the code bits, and its relationship to the sub-optimal belief propagation decoder, will be discussed. A proof will be outlined of the exponential decay of correlations between code bits in suitable noise regimes. A consequence is the equality of performance curves for the optimal and belief propagation decoders. These systems can be interpreted as a special class of spin glasses and the analysis proves that the replica predictions are exact.

- *Speaker*: Matteo Marsili (ICTP Trieste).

  Title: Stability and complexity in financial markets (spin-glass techniques for understanding economic equilibria).

  Abstract: Trust lies at the foundation of market economies, as starkly remarked by the recent financial crisis. Important progress has been made in understanding, from the game-theoretic perspective, the mechanisms by which trust can break down, relating it to strategic uncertainty. The aim of this talk is to scale these insights to the system level by analyzing a simple model of a large population of individuals engaged in credit relationships. This economy can converge to a "good" equilibrium, where a dense network of credit relations exists and the risk of a run, and subsequent default, is negligible. However, a "bad" equilibrium is also possible: here, the credit network is sparse because investors are more nervous and are prone to prematurely foreclose their credit relationships, thereby precipitating counterparty default and contagion. The transition between the two equilibria is sharp and both states exhibit a degree of resilience; once a credit crisis tips the system into the sparse state state, the restoration of a dense credit network requires a shift of the parameters well beyond the turning point. At the same time, when the system reverts to the good state, this is robust even to deteriorating conditions.

- *Speaker*: Andrea Montanari (Stanford University).

  Title: A local limit theorem for Ising models on locally tree-like graphs.

  Abstract: Sequences of graphs that converge locally to trees are of interest for a number of reasons. Among others, sequences of sparse random graphs fall in this class for several graph distributions. In this talk we consider Ising models on locally tree-like graphs and prove a complete characterization of the limiting measure when the graphs are regular. In particular, we establish a coexistence phenomenon that was understood so far only in the case of finite-dimensional lattices. This talk is based on joint work with Elchanan Mossel and Allan Sly.

- *Speaker*: Chuck Newman (New York University).

  Title: Introduction to metastates for Edwards-Anderson models.

  Abstract: We introduce metastates as probability measures on the space of infinite-volume Gibbs states that may be used to study disordered systems such as the Edwards-Anderson (EA) model with potentially many 'competing' states. Extensions to metastates and excitation metastates for competing ground states are also discussed. This talk will be the first of a four-part series, which gives some of the background needed for the second part by Stein, concerning domain walls for the two-dimensional EA model (in the full plane), and for the third and fourth parts by Arguin and Damron, presenting a new result on uniqueness of ground states in the half plane.

- *Speaker*: Hidetoshi Nishimori (Tokyo Institute of Technology).

  Title: Absence of a spin glass transition in two dimensions.

  Abstract: Although numerical evidence is overwhelming for the absence of spin glass transitions in two dimensions, analytical studies are still rare. We have developed a theory to show the absence of finite-temperature spin glass transitions for the Ising spin glass on self-dual lattices. The analysis is performed by an application of duality relations, the replica method and gauge symmetry. I will discuss how and when the predictions of this theory can be exact.

- *Speaker*: Allon Percus (Claremont Graduate University).

  Title: The peculiar phase structure of random graph bisection.

  Abstract: The phase structure of mincut graph bisection displays certain familiar properties when considered over sparse random graphs, but also some surprises. It is known that when the mean degree is below the critical value of $2 \log 2$, the cutsize is zero with high probability. We study how the minimum cutsize increases with mean degree above this critical threshold, finding a new analytical upper bound that improves considerably upon previous bounds. Combined with recent results on expander graphs, our bound suggests the unusual scenario that random graph bisection is replica symmetric up to and beyond the critical threshold, with a replica symmetry breaking transition possibly taking place above the threshold. An intriguing algorithmic consequence is that although the problem is NP-hard, we can conceivably find near-optimal cutsizes (whose ratio to the optimal value approaches 1 asymptotically) in polynomial time for typical instances near the phase transition. This is joint work with Gabriel Istrate, Bruno Goncalves, Robert Sumi and Stefan Boettcher.

- *Speaker*: Guilhem Semerjian (Ecole Normale Paris).

  Title: Quantum spin models on sparse random graphs.

  Abstract: Classical spin models defined on random graphs have been the object of an intense research activity motivated, among others, by their relationship to random combinatorial optimization problems. The heuristic cavity method allowed to make several qualitative and quantitative predictions about the behaviour of such random systems in their large size limit, some of these predictions having been confirmed rigorously. In this talk I will discuss a more recent development of the heuristic cavity method towards quantum models defined on random graphs. These models can be constructed, for instance, by turning a classical energy function of N Ising spins into an operator acting on the Hilbert space spanned by the 2^N configurations, and adding to it a non-commutative operator as a transverse field. Such models can be represented through path integrals of imaginary time configurations. The cavity method can then be implemented at the quantum level by devising a sampling procedure of such spin trajectories. The case of frustrated classical spin models (for which finding the minimal energy states can be a difficult task) in a transverse field is of particular interest in view of application to quantum computing.

- *Speaker*: David Sherrington (University of Oxford).

  Title: Dynamics of information-driven and range-free networks: some results, some thoughts and some questions.

  Abstract: Range-free (or infinite-range) many-body problems are independent of spatial dimensionality and offer the opportunity for exact solution in the large N limit, through mapping to descriptions in terms of macroscopic variables. These mappings and their subsequent analysis are non-trivial when there is frustration between fast variables and either other control parameters or rules are microscopically quench-disordered or are themselves dynamical with slower characteristic timescales. In quenched cases with equilibrating fast-variable dynamics conventional Boltzmann statistical mechanics can be utilised, but still with subtle challenges for rigorous analysis. Without such equilibration often some progress can be made at the level of theoretical physics, but many challenges remain in complete analysis as well as in rigorous justification. Such range-free problems are first-pass reasonable models for a number of situations. They are also a natural effective mapping for problems driven by distance-independent information such as occurs for the internet, stock-market indices, news, etc. Hence, in many man-made social and economic scenarios there are important effective correlation effects with no separation or spatial dimension-dependence. Coupled with conflicting aims and global sum rules/constraints, one has frustration, typically also with a distribution of individual inclinations. Network growth is also range-free in certain cases, again typically those driven by internet interaction, simplifying analyses, but still leaving interesting issues of optimal algorithms, topological phase transitions, consequences of both uncontrolled and rule-controlled evolution and possible effects of frustration. In this talk I shall discuss some of these issues, giving some examples, but also posing questions for discussion and future study.

- *Speaker*: Shannon Starr (Rochester University).

  Title: Thinning partition structures.

  Abstract: A random partition structure is a random point process on $[0, 1]$ such that the points add up to 1. We consider a thinning dynamics: independently removing points, keeping each one with probability $p > 0$, and then rescaling all remaining points to make the sum 1. We assume infinitely many points initially. We prove that all partition structures that are infinitely-divisible with respect to this process are mixtures of Poisson-Kingman processes. The ones that are invariant for all p include the Poisson-Dirichlet structures, which are also the invariant measures for the uncorrelated cavity step dynamics, as proved by Aizenman and Ruzmaikina, and Arguin. But there are also others. This is joint work with Brigitta Vermesi.

- *Speaker*: Dan Stein (New York University).

  Title: Domain wall structure in the two-dimensional Edwards-Anderson model.

  Abstract: We review a result due to Newman and Stein, for the 2D EA model with Gaussian (or other) couplings and in zero field, which shows that if a coupling-independent boundary condition metastate is supported on incongruent (i.e., statistically dissimilar) ground states, then the symmetric difference between two such states must consist of a single positive-density simply-connected domain wall. This talk will be the second of a four-part series, using results on metastates presented by Newman in the first part and laying the groundwork for a new result, presented by Arguin and Damron in the third and fourth parts, on uniqueness of ground states in the half-plane.

- *Speaker*: Balint Virag (Toronto University).

  Title: One-dimensional random Schrödinger operators and random matrices.

  Abstract: Random Schrödinger operators can be thought of as Markov kernels for random walks in a random environment of obstacles. In the critical regime, the probability decay for large gaps between eigenvalues of these operators resembles those for random matrices. However, the eigenvalue repulsion is much stronger.

- *Speaker*: Lenka Zdeborova (Los Alamos National Laboratory).

  Title: Revisiting the random field Ising model.

  Abstract: Since the dynamical behavior of the random field Ising model (RFIM) has some glassy features many authors have discussed the existence of a putative spin-glass phase in this model. In this talk, I will first show how an elementary, yet rigorous, bound can be derived for the spin-glass susceptibility that allows to essentially answer the question. In the second part I will present an exact solution of the model on a random graph. I will discuss a method to compute the phase diagram at fixed values of the magnetization and its consequences.

List of Participants

Agliari, Elena (University of Parma)
Aizenman, Michael (Princeton University)
Aldous, David (University of California Berkeley)
Arguin, Louis-Pierre (New York University)
Barra, Adriano (Sapienza Universita' di Roma)
Ben Arous, Gerard (New York University)
Bianchi, Alessandra (Weierstrass Institute Berlin)
Bovier, Anton (Rheinische Friedrich-Wilhelms-Universität Bonn)
Burioni, Raffaella (Universita' di Parma)
Damron, Michael (Princeton University)
Dembo, Amir (Stanford University)
den Hollander, Frank (University of Leiden and EURANDOM)
Dommers, Sander (Eindhoven University of Technology)
Fedele, Micaela (Bologna University)
Franz, Silvio (Universite Paris-Sud 11)
Giardina, Cristian (TU Eindhoven and EURANDOM)
Giberti, Claudio (Universita' di Modena e Reggio Emilia)
Kistler, Nicolas (Bonn University)
Klimovsky, Anton (Universität Erlangen-Nürnberg)
Kuelske, Christof (Rijksuniversity Groningen)
Luczak, Malwina (London School of Economics)
Macris, Nicolas (Ecole Polytechnique Federale de Lausanne)
Marsili, Matteo (Abdus Salam ICTP Trieste)
Montanari, Andrea (Stanford University)

Newman, Charles (New York University)
Nishimori, Hidetoshi (Tokyo Institute of Technology)
Percus, Allon (Claremont Graduate University)
Semerjian, Guilhem (Ecole Normale Supérieure Paris)
Sherrington, David (University of Oxford)
Starr, Shannon (University of Rochester)
Stein, Daniel (New York University)
Virag, Balint (University of Toronto)
Zdeborova, Lenka (Los Alamos National Laboratory)

Chapter 23

Search in Constraint Programming (09w5125)

Nov 29 - Dec 4, 2009

Organizer(s): Gilles Pesant (École Polytechnique de Montréal), Meinolf Sellmann (Brown University)

Overview of the Field

Constraint Programming (CP) is a powerful technique to solve combinatorial problems. It applies sophisticated inference to reduce the search space and a combination of variable- and value-selection heuristics to guide the exploration of that search space. Like Integer Programming, one states a model of the problem at hand in mathematical language and also builds a search tree through problem decomposition. But there are mostly important differences, among them: CP works directly on discrete variables instead of relying mostly on a continuous relaxation of the model; the modeling language offers many high-level primitives representing common combinatorial substructures of a problem — often a few constraints are sufficient to express complex problems; each of these primitives, called *constraints*, may have its own specific algorithm to help solve the problem; one does not branch on fractional variables but instead on indeterminate variables, which currently can take several possible values (variables are not necessarily fixed to a particular value at a node of the search tree); even though CP can solve optimization problems, it is primarily designed to handle feasibility problems.

We believe that a more principled and autonomous approach for search efficiency has to be started in Constraint Programming. Our main global objective is to advance in a significant way research on search automatization in CP so that combinatorial problems can be solved in a much more robust way. By furthering the automation of search for combinatorial problem solving, this workshop should have a direct impact on the range and size of industrial problems that we will be able to solve with this approach.

Open Problems

Here are some of the open problems identified by the participants, some of which were investigated in smaller groups (see Section 23):

1. Can reinforcement learning applied to complete search lead to superior search performance?
   (a) What is a good reward function for CSP/COP?
   (b) Branching dependent on expected satisfiability? More generally, how can a statistical prediction of an instance's objective function and/or feasibility value be useful in a solver?

   (c) How can we measure progress in a complete search trajectory?
   (d) How should we evaluate the performance of (new) algorithms?

2. Can we make no-good learning work for CP?
3. Is there a correlation between trajectory-based intensification and diversification measures and the performance of Local Search solvers?
4. What well-defined, macroscopic properties of an algorithm's behaviour can effectively explain performance differences between (existing) solvers / on different types of landscapes?
5. How can we formulate the notions of intensification and diversification to achieve 3 and 4?
6. What is the conceptual difference/unification between local and "non-local" (systematic) search?
7. Can we achieve an improvement in the state of the art by using automated algorithm configuration in a large design space of CP solvers?
8. What are useful/practical design spaces for modelling in CP, MIP?

Recent Developments

In order to familiarize participants with the range of techniques currently considered to improve search, several hour-long background talks were given followed by a discussion period, taking place during the first two days of the workshop.

Statistical Measures for Local Search—Summary of Talk (M. Sellmann)

In the realm of local search, there exist a number of interesting techniques to boost expected search performance by learning. We argue that any heuristic search bias that is introduced for the solution of a particular problem must be justified by a rigorous statistical analysis of characteristic problem instances.

We reviewed two statistical measures that have been considered in the literature, fitness-distance correlation and so-called global valley structure. Fitness-distance correlation is defined by the correlation of solutions' objective function value and their ”distance” to the nearest global optimum. Fitness-distance correlation is low when on average the better the quality of a solution, the lower the distance to the nearest optimum. Distance can be measured in the number of local search steps (and therefore depends on the neighborhood used) or, more practically, as Hamming distance. Jones and Forrest found that, in general, low fitness-distance correlation correlates with good performance of genetic algorithms. We argued that low fitness-distance correlation gives a statistical justification for intensifying the search in the neighborhood of previously visited high-quality solutions.

The second measure that we reviewed is the so-called ”global valley structure.” The name conjures a questionable mental image. What it formally denotes is an anti-correlation of the quality of local minima and the average distance to other local minima. That is, we speak of a ”global valley structure” when local minima have the better objective values the closer they are, on average, to other local minima. Boese found such a ”global valley structure” on TSP instances by analyzing 2500 random local minima. We argued that ”global valley structure” can justify a bias towards searching for improving solutions between previously found local minima. This is, e.g., the case when performing path relinking.

Now, each search bias that is introduced needs to be balanced with a counter-force that ensures effective search space exploration. This in combination with the idea to intensify the search around high quality solutions and to search between local minima leads to a local search method which we introduced in 2009, Dialectic Search. In dialectic search, we employ a greedy improvement heuristic to find a local optimum (the ”thesis”). Then, we randomly perturbate the local minimum and locally improve the perturbation (the ”anti-thesis”). We then search the space between those two local minima (for example by performing path relinking). If the best solution found during this search (the ”synthesis”) is better than the thesis, we locally optimize it and continue from this solution. Otherwise, we perform a new random perturbation of the thesis to obtain a new anti-thesis. Dialectic search conservatively continues its search with the best found solution (the current thesis) until no improvement is achieved for some time. Only then does the method move to the anti-thesis and continues the search from there.

Experimental results on set covering problems, magic squares problems, and continuous optimization problems show that this meta-heuristic, which is by design conservative and combines ideas from other methods, such as iterated local search, path relinking, variable neighborhood search, and genetic algorithms, can effectively lead the design to highly efficient optimization algorithms that beat the existing state-of-the-art by one or more orders of magnitude in performance. What is currently missing is an evaluation of the performance of dialectic search and its components and the statistical measures mentioned in the beginning.

Lessons from MIP Search—Summary of Talk (J. N. Hooker)

Mixed integer programming (MIP) offers several ideas that can benefit search in general. Chief among these is the use of a strong relaxation, normally a continuous linear programming (LP) relaxation, to guide the search.

For more than 50 years, the solution method of choice for general-purpose MIP solvers has been *branch-and-bound* search . At any point in the search there is a partial search tree, in which some of the leaf nodes are open and some are closed. The LP relaxation is solved at a selected open node, whereupon the search closes the node or branches to creates two more open nodes—depending on whether the LP solution is integral or better than the incumbent integer solution. The search branches on a selected variable with a fractional value in the LP solution. *Branch-and-cut* methods add cutting planes at various nodes to strengthen the LP relaxation.

The two main search decisions are *variable selection* (which variable on which to branch next) and *node selection* (which open node to explore next). Two methods for variable selection use pseudo-costs and strong branching, both of which rely on the LP relaxation. *Pseudo-costs* are estimates of the effect on the objective function value of rounding a fractional variable. *Strong branching* uses the dual simplex method to calculate a bound on this effect. The two most common node selection heuristics are depth-first and best bound. Depth first uses minimal memory and allows fast LP updates, but the tree can explode. Best bound opens the node with the best LP relaxation value.

Feasibility heuristics find integer solutions. *Local branching* uses a strategic branch to create a large-neighborhood local search at the left branch. It uses MIP to search the large neighborhood of a known integer solution. The process repeats at the right branch. The *feasibility pump* alternately rounds the current LP solution and then finds the LP solution that is closest to the rounded solution. This is an LP problem because the L1 norm is used.

At least two lessons can be learned from MIP for search in general. One is the use of duality. The LP dual can be generalized as an *inference dual*. Lagrangean, surrogate, subadditive, and many other duals are inference duals. The solution of an inference dual is a proof. The premises that are active in the proof can be used to construct a *nogood* or Benders cut that directs the search away from poor solutions. This method appears in OR as Benders decomposition, in SAT solvers as clause learning, and in AI as (partial-order) dynamic backtracking. *Logic-based Benders decomposition*, a generalization, has been used in a number of problems areas, often with order of magnitude improvement in solution time. Recently, at least one MIP solver has incorporated nogoods, returning the idea full circle to OR.

A second lesson from MIP is that a richer and stronger relaxation makes it worthwhile to invest more processing time at each node of the search tree. By contrast, constraint programming (CP) solvers use a weak relaxation (the domain store) and consequently prefers to search very large trees with little processing at the nodes. *Binary* (or *multivalued*) *decision diagrams* (BDDs) are an alternative to the constraint store that propagates more information from one constraint to the next. They provide a relaxation whose strength depends on the specified maximum width (and becomes equivalent to the domain store for a max width of one). Constraint-specific BDD refinement algorithms subsume the filtering algorithms currently used in CP. Preliminary computational testing shows that BDDs can reduce the search tree for multiple all-different constraints from a million nodes to one node, with more than a hundredfold reduction in computation time. In testing with multiple among constraints in nurse scheduling problems, they reduce the search tree by a factor of 10,000 and the computation time by a factor of fifty.

MIP search methods are surveyed in [9] and cutting planes in [10]. Local branching is described in [3] and the feasibility pump in [3]. Inference duality and applications are described in [7, 8], and Benders cuts for MIP in [11]. An introduction to the use of BDDSs in constraint programming can be found in [1], with further applications in [4, 5, 6].

Impact-Based and Counting-Based Search—Summary of Talk (G. Pesant)

To solve difficult problems, successful constraint programming requires both effective inference and search. After many years of advances on inference, there has been a more recent focus on search heuristics. This talk presented two proposals from the literature that take a more global view than individual variable domains.

Impact-Based Search Inspired by *pseudo-costs* in Integer Programming, Impact-Based Search (IBS) [21] aims to provide an efficient general-purpose search strategy for Constraint Programming. It learns from the observation of search space reduction during search, approximated as the change in the domains of the variables. The impact of assigning value d to variable x is computed roughly as the ratio of the size of the search space after that assignment to its size beforehand. The impact of variable x sums the individual impacts of the values in its domain. IBS typically branches on the variable with the greatest impact and then on its value with the smallest impact.

Computing such impacts for every variable at each search tree node requires probing and is time consuming. Typically one computes them at the root of the search tree and then occasionally at other nodes on a small subset of the variables and to break ties among heuristic choices. Justification for this is based on the belief that impacts do not change much from one search tree node to another but supporting evidence is not well documented in the scientific literature.

Nevertheless IBS generally performs very well and forms the basis of the default search heuristic in commercial CP solvers (e.g. IBM ILOG CP Optimizer).

Counting-Based Search Constraints have played a central role in CP because they capture key substructures of a problem and efficiently exploit them to boost inference. Counting-Based Search (CBS) proposes to do the same thing for search [22]. Up to now, the only visible effect of the consistency algorithms had been on the domains, projecting the set of tuples on each of the variables. Additional information about the number of solutions of a constraint (its *solution count*) can help a search heuristic to focus on critical parts of a problem or on promising solution fragments. At a more fine-grained level, the proportion of solutions to a given constraint in which variable x is assigned value d, termed the *solution density* of that variable-value pair, turns out to be a powerful indicator of the likelyhood that this assignment appears in a global solution.

For each family of constraints, providing a counting algorithm is sometimes trivial (e.g. `element` constraint), sometimes a simple extension of the filtering algorithm (e.g. `regular` constraint), and sometimes much harder to do exactly (e.g. `alldifferent` constraint). In the latter case, counting is only estimated through sampling, bounds, or relaxation.

Given the counting information, many different search heuristics can be considered. Despite trying several sophisticated ways to combine that information from each constraint, one of the conceptually simplest heuristics — branching on the highest solution density among every constraint, variable, and value (termed *maxSD*) — works just as well and often better.

Such a heuristic compares very well with the state of the art. Based on a varied set of benchmark problems, it has proved more consistent and often more efficient to solve satisfiable instances. More surprising, there is some empirical evidence that it can solve unsatisfiable instances very fast as well (i.e. provide a very small failed search tree).

This fairly recent line of research raises some questions. Among them:

- Why aren't more sophisticated ways of aggregating the counting information much better than the simpler *maxSD*?
- Why is a heuristic guiding the search toward satisfiable parts of the search tree also good to prove infeasibility?
- How can this approach be adapted to solve optimization problems?

Probabilistic Inference Techniques—Summary of Talk (E. Hsu)

In the last few years, the constraint satisfaction research community has begun to adapt a number of techniques that are normally associated with probabilistic reasoning over graphical models [25, 23, 24, 26, 28]. Interestingly, though, many aspects of these techniques can be be viewed in terms of soft analogues of existing constraint satisfaction concepts, for instance conditioning and inference. This is unsurprising given a long history of constraint satisfaction research dealing with graph structure–probabilistic distributions modeled as Bayes Networks [29] and Markov Random Fields [30] are continuous cousins to the discrete sets of solutions represented as constraint problems.

In particular, probabilistic models and constraint problems each express an otherwise intractable function over configurations of variables as a product of simpler functions. Using the factor graph formalism, and the algebraic notion of a semi-ring, then, we can make this comparison explicit. A *factor graph* [27] is a bipartite graph containing two types of nodes: *variables* and *factors* (i.e., *functions*). Associated with each such node is a variable or a function over variables, respectively; usually we will not need to distinguish between variables/factors and their associated nodes in the graph. The edges in a graph are defined unambiguously by the scopes of the functions. Thus, we can typically denote a factor graph $G = (X, F)$ solely in terms of its variables X and factors F. A *configuration* is an assignment to all the variables in X; the exponentially-sized set of all such configurations is denoted $\textsc{configs}(X)$. The semantics of a factor graph are such as to map each configuration of its variables to a real value; the graph's structure shows how this value factorizes into a product of individual instantiations. In particular, a factor graph G defined as above realizes a function W_G over configurations $\vec{x} \in \textsc{configs}(X)$:

$$W_G(\vec{x}) = \prod_{f_a \in F} f_a(\vec{x}_{|\sigma_a}) \tag{23.1}$$

Here "$|\sigma_a$" represents the projection of a configuration onto the domain of function a. In other words, to calculate the value of a configuration, we can just perform several function evaluations and multiply the results together. In its role as a probabilistic graphical model, a factor graph represents the joint probability distribution over X, whose contents are considered random variables. Thus W_G should be interpreted as issuing weights over configurations; such weights must then be normalized to form a proper probability distribution:

$$p(\vec{x}) = \frac{1}{\mathcal{N}} W_G(\vec{x}), \quad \text{where} \quad \mathcal{N} = \sum_{\vec{x} \in \text{CONFIGS}(X)} W_G(\vec{x}) \tag{23.2}$$

$\mathcal{N}$ is known as the normalizing constant or partition function (of G); its calculation is equivalent in structure and complexity to the marginal computation problem stated in the next section. Alternatively, the most basic insight of the student's research is that a factor graph can also be used to represent a constraint satisfaction problem:

A constraint satisfaction problem can be represented as a factor graph whose factors are all $0/1$ functions. That is, all factors evaluate to either 0, expressing the violation of a constraint, or to 1, expressing the satisfaction of a constraint. The weight function W_G for such a factor graph thus equals one if and only if a configuration meets all of the constraints, indicating the satisfaction of the problem as a whole. The normalizing constant $\mathcal{N}$ represents the number of solutions for such problems–accordingly the completed Equation 23.2 represents a uniform distribution over the set of solutions. (For unsatisfiable problems both the equation and the concept of a distribution over solutions are undefined.)

The term "algebraic semi-ring" is used to indicate the essential difference between probabilistic and combinatorial reasoning: in the former, sum and product are realized by standard arithmetic addition and multiplication, while in the latter they are realized by disjunction and conjunction. But by the simple definition given directly above, any constraint problem can be directly encoded in terms of addition/multiplication, as a uniform distribution over its set of solutions. Thus, we can ask standard queries defined over such distributions, and for a starting point we can consider standard probabilistic algorithms defined over graphical representations of such distributions.

Computer-Aided Algorithm Design: Principled Procedures for Building Better Solvers —Summary of Talk (H. Hoos)

High-performance algorithms can be found at the heart of many software systems; they often provide the key to effectively solving the computationally difficult problems encountered in the application areas in which these systems are deployed. Examples of such problems include scheduling, time-tabling, resource allocation, production planning and optimisation, computer-aided design and software verification.

Many of these problems are $\mathcal{NP}$-hard and considered computationally intractable. Nevertheless, these 'intractable' problems arise in practice, and finding good solutions to them in many cases tends to become more difficult as economic constraints tighten.

In most (if not all) cases, the key to solving computationally challenging problems effectively lies in the use of heuristic algorithms, that is, algorithms that make use of heuristic mechanisms, whose effectiveness can be demonstrated empirically, yet remains inaccessible to the analytical techniques used for proving theoretical complexity results. (We note that our use of the term 'heuristic algorithm' includes methods without provable performance guarantees as well as methods that have provable performance guarantees, but nevertheless make use of heuristic mechanisms; in the latter case, the use of heuristic mechanisms often results in empirical performance far better than the bounds guaranteed by rigorous theoretical analysis.) The design of such effective heuristic algorithms is a difficult task that requires considerable expertise and experience.

High-performance heuristic algorithms are typically constructed in an iterative, manual process in which the designer gradually introduces or modifies components or mechanisms whose performance is then tested by empirical evaluation on one or more sets of benchmark problems. During this iterative design process, the algorithm designer has to make many decisions. These concern choices of the heuristic mechanisms being integrated and the details of these mechanisms, as well as implementation details, such as data structures. Some of these choices take the form of parameters, whose values are guessed or determined based on limited experimentation.

This traditional approach for designing high-performance algorithms has various shortcomings. While it can and often does lead to satisfactory results, it tends to be tedious and labour-intensive; furthermore, the resulting algorithms are often unnecessarily complicated while failing to realise the full performance potential present in the space of designs that can be built using the same underlying set of components and mechanisms.

As an alternative to the traditional, manual algorithm design process, we advocate an approach that uses fully formalised procedures, implemented in software, to permit a human designer to explore large design spaces more effectively, with the aim of realising algorithms with desirable performance characteristics. This approach automates both, the construction of target algorithms as well as the assessment of their performance. Computer-aided algorithm design allows human designers to focus on the creative task of specifying a design space in terms of potentially useful components. This design space is then explored using powerful heuristic search and optimisation procedures in combination with significant amounts of computing power, with the aim of finding algorithms that perform well on given sets or distributions of input instances.

Our approach shares much of its motivation with existing work on automated parameter tuning, algorithm configuration, algorithm portfolios and algorithm selection, all of which can be seen as special cases of computer-aided algorithm design. Using this approach, we have achieved substantial improvements in the state of the art in solving a broad range of challenging combinatorial problems, ranging from SAT and mixed integer programming to course timetabling and protein structure prediction problems.

As a design approach, computer aided algorithm design is also more principled than the *ad-hoc* methods currently used, which makes it easier to disseminate and support, in the form of software systems for computer-aided algorithm design. Because of their more formalised nature, computer-aided algorithm design methods are also easier to evaluate and to improve. Their development and application therefore constitutes a key step in transforming the design of high-performance algorithm from a craft that is based primarily on experience and intuition to an engineering effort involving formalised procedures and best practices.

Drawing from methology and insights from a number of areas, including artificial intelligence, empirical algorithmics, algorithm engineering, operations research, numerical optimisation, machine learning, statistics, databases, parallel computing and software engineering, we believe that computer-aided algorithm design will fundamentally change the way in which we design high-performance algorithms. As a result, human experts will be able to more easily design effective solvers for computational problems encountered in application domains ranging from bioinformatics to industrial scheduling, from compiler optimisation to robotics, from databases to production planning.

Scientific Progress Made

The following two days were spent in subgroups exploring some topics identified at the end of the second day.

Learning during Search

Topic During the working session we concentrated our efforts on describing what would be the contributions of machine learning techniques, such as reinforcement learning, to the field of search in constraint programming. The focus of the discussion was put towards what could be achieved using learning *during* search, that is not using any offline training. We believe this is important to make such technique readily available as part of black box CP packages without requiring lengthy model training phases.

State Learning techniques are based on the principle that given that one can recognize that a process is in a given state, it is possible to learn the best suited action that should be performed. We consider the search state to be represented by the position in the search tree. Such position can be described by:

- the set of all branching decisions
- the set of all variable-value assignments
- the domains of all variables

Features Since learning mechanisms are subject to the curse of dimensionality and furthermore since the above state description has non constant dimensionality, it is desirable to succinctly describe the states as a short vector. To do so we defined features that could possibly abstract states by mapping them to real numbers. It is obviously crucial for performance that the features allow us to discriminate well between states. We have identified the following features:

- number of fixed variables
- size of the search space defined by a metric on domain sizes (like the Cartesian product)
- structure of the constraint graph
- solution space defined by information based on solution counting
- lower bound on the objective value
- backdoor information (probing, information based on the path from root to failure)

Actions Given a state identified by a set of features, the possible actions that can be taken during search are typically:

- branching on a variable/value assignment
- branching on a constraint (like precedence or domain splitting)
- restarting
- changing inference levels (from bound to arc consistency, or *vice versa*)
- changing the amount of propagation
- changing the search heuristics (variable and value selection heurisitics)
- changing the tree traversal strategy (DFS, LDS, etc.)

Reward function We discussed the possible reward functions that could be used during learning. When a reward is given after following an action, this reward would be back propagated up in the tree to previously taken decisions and visited states. If a failure has clearly a negative reward, it is not obvious how to define positive rewards. Given that we are solving a CSP, finding a feasible solution is not only a success but also the goal of the search... It was suggested to use as positive rewards something similar to pseudo costs in MIP. In CP this could be the difference in feature value for two successive states like, for instance, what is done during impact based search.

Notwithstanding the focus of the discussion, we agreed that training the reward function offline, perhaps using sophisticated parameter tuning packages such as paramILS[14] would probably be a good idea.

Algorithm We agreed that temporal difference learning will not be sufficient as propagation is expensive and it would be impractical to evaluate all possible branching decisions. Therefore it was suggested that a variant of Q learning would thus be more appropriate in such a case. There are a few papers on learning for Job Shop Scheduling by Zhang and Dietterich which seems apply to such a framework.

Empirical Models for Local Search

Topic The group addressed the issue of developing measurements of local search behaviour that could make a contribution to the principled understanding of local search performance. For example, there are a considerable number of papers that make appeals to two ill-defined notions (intensification and diversification) as a basis for motivating new local search algorithms and variations or as a basis for an intuitive explanation of search performance. There appears to be a notion that it is useful to balance intensification and diversification. To some extent such work is problematic as there is no formal definition (or even agreement) on what these measurements really are or how to measure them. Therefore, controlled experiments that seek to test if the superior performance of a given algorithm is related to its ability to, say, intensify, better than another algorithm cannot be done and the “empirical science” of heuristic search is not well developed.

Group The working group consisted of: Chris Beck, University of Toronto; Holger Hoos, University of British Columbia; Eric Hsu, University of Toronto; Serdar Kadioglu, Brown University; and Steve Smith, Carnegie Mellon University.

Summary of Important Points A detailed discussion over approximately three hours, developed the following main points:

- **Search Trajectory Features** We are primarily interested in characteristics of the states visited by a search method and the order in which the states were visited. Focusing on such measures, it was felt, would abstract away from algorithm details as well as from landscape features (e.g,. “the big valley”). While both algorithm details and landscape features are important it was generally agreed that they only impact search performance via the search trajectory and so focusing on the trajectory may help in developing and testing more precise ideas.

- **Quality and Distance Measures** It is important to distinguish quality and distance measures. A distance measure has something to do with the number of steps (or an estimate or bound on the number of steps) that it takes to get from one state to another. For example, one measure of stagnation might be that the maximum Hamming distance between the starting solution and any solution visited is not growing very quickly. In contrast, we have measures that include the cost function such as statistics about the quality of local optima (e.g., mean or variance or coefficient of variation of the costs of the local optima encountered). We also have potential measures that include them both such as statistics about the distance between consecutive local optima. Quality and distance metrics may reflect the same underlying search behavior in different ways. For example, if a local search is lost of a large plateau, a distance-based measure would show progress (e.g., getting increasingly further from the starting solution) while a cost-based measure would show stagnation (e.g., narrow variance in the cost of local optima). The group discussed intensification and diversification from the perspective of both quality and distance. It seems reasonable that for distance-based measures, intensification and diversification should be inverses: the more search visit states close (in e.g. Hamming distance) to a starting point (intensification), the less it explores different solutions (diversification). Quality-based intensification would seem to have the flavour of aggressively following a cost gradient whereas quality-based diversification might be measured by the variance of the costs of states (or local optima) that have been visited. It was felt that these are not directly inverses. No clear formal definition of these notions was developed–when such a definition is, different names should be used.

- **Measures of Local Optima Trajectory** Search trajectory features should be broad enough to include sub-trajectories. It was suggested that rather than looking at the state-by-state trajectory, it might be valuable to abstract to a local optimal-by-local optima trajectory: look at the characteristics of consecutive local optima. The local search features used in SATzilla [13] were given as examples. Over one local search run, measures include: change in cost from starting solution to minimum local optima found, number of search steps to first local optima, number of steps to minimum local optima, ratio of the difference in cost from starting solution to first local optima to the difference in cost from the starting solution to the minimum local optima, coefficient of variation of the cost of local optima. Distance and quality measures that can be defined on the overall search trajectory could be defined for the local optima trajectory and may be more meaningful/indicative.

- **Desired Characteristics of Measures** There are a variety of desirable qualities for any measures that we suggest. In particular, a measure over time ism ore useful than a single summary number. To take one example, it would seem that some sort of moving coefficient of variation of local optima qualities, visualized in a graph, is more meaningful than a single number for the entire trajectory. Measures should also “make sense” for particular, simple local search algorithms. For example, random walk intuitively should have some medium-level of diversity (if measured as a distance metric) while random sampling should have a relatively high measure (though perhaps not as high as a more sophisticated sampling method specifically designed for coverage of a space). Any suggested measure should be tested to confirm such “intuitive” behaviour.

- **Specific Measure Suggestions** One starting point for specific measures is the work of Schuurmans & Southey [12] who suggest: mobility (average Hamming distance between states that are k-steps apart in the trajectory), coverage (a measure of the maximal distance between visited and unvisited states), and depth (average objective value over the trajectory). One suggestion, based on mobility, is the expected number of steps to encounter a state with a Hamming distance of more than k from the starting point.

Further Information It is intended that a more in-depth description of the above points will be posted to the website of the Constraint Programming Society of North America.

Algorithm (CP solver) configuration

Topic The group discussed the application of automatic algorithm design to Constraint Programming. Automatic Algorithm design opens a new paradigm of thinking about and writing solvers. Usually the few parameters in solvers that are left open to the user for configuration have some “semantics”. When writing a highly configurable solver to be tuned by an automatic algorithm design, the designer can expand the exposed tunable parameters to aspects that do not have clear “semantics”. In complex configurable algorithms, one needs a way to express conditional dependencies between parameters. This can be handled currently by methods such as ParamILS [14]. In a complex design space, it might help to be able to express preferences over the design space such as “I expect this parameter with this value to have negative interactions with this other parameter-value pair.” or “If you change this parameter, you might need to change this other parameter.” One can also consider specifying a prior over the parameter space.

The challenge in applying automatic algorithm design to CP, as opposed to SAT, is that in CP there is a tight connection between the formulation of the model and the solving method applied (e.g. constraints (model) and propagators (algorithmic)).

Design Space of CP Solvers The group discussed the possible design space for a CP solver and identified relevant paper references:

- encoding/formulation/modeling alternatives (ref. [15] for automatic reformulation)
- ordering of propagation of constraints (ref. [18] for configurable priority queue of constraint groups in Gecode)
- restarts
- branching heuristics (e.g. automatic selection between the different variants of impact-based heuristics)
- filtering levels of constraints (ref. [20] for automatically choosing propagators)
- overall search approach (i.e. local search versus tree search)

Relevant Work We identified some existing work and systems that include partial automated algorithm design: AEON [19], Essence [16], Minion [17] (highly parametrized, reformulation SAT/LP, different propagations for alldiff), and Tailor [15] (direct compilation to Gecode and Minion from model language).

Challenge Problem To make this study more concrete, it was suggested that we pick a particular problem and have different people create different reconfigurable solution methods with exposed parameters. Then once these models are submitted, we could put everything as the components of a global design space that can be then explored through the automatic algorithm design framework. It was suggested that the challenge problem could be quasi-group with holes (QWH).

Instance-based Algorithm Configuration We also had a brief discussion on instance-based automatic algorithm selection and configuration and in particular the instance features of CP problems important for algorithm selection. We identified relevant work by Pascal van Hentenryck on the use of syntactic features of scheduling problems in COMET. One could also consider algorithmic features that are more informative but maybe more expensive to compute than the ones used in COMET, such as some of the features used in Satzilla for SAT, e.g. running a local search on the instance and recording runtime until first feasible solution found.

Outcome of the Meeting

We expect that some of the discussions started during the workshop will be continued and eventually lead to scientific publications. Many informal discussions in smaller groups also took place and the general feeling was that this had been a great opportunity to network within the north-american constraint programming community and to spark new collaborations. Some of these could also lead to publications in the next few years. We agreed to repeat the experience.

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List of Participants

Beck, Chris (University of Toronto)
Chapados, Nicolas (Universite de Montreal)
Dilkina, Bistra (Cornell University)
Hooker, John (Carnegie Mellon University)
Hoos, Holger (UBC)
Hsu, Eric (University of Toronto)
Hutter, Frank (University of British Columbia)
Kadioglu, Serdar (Brown University)
Leyton-Brown, Kevin (University of British Columbia)
Malitsky, Yuri (Brown University)
Pesant, Gilles (École Polytechnique de Montréal)
Rousseau, Louis-Martin (École Polytechnique de Montréal)

Sabharwal, Ashish (Cornell University)
Sellmann, Meinolf (Brown University)
Smith, Stephen (Carnegie-Mellon University)
van Hoeve, Willem-Jan (Carnegie Mellon University)

Two-day Workshop Reports

Chapter 24

Ted Lewis Workshop on SNAP Math Fairs 2009 (09w2150)

Apr 24 - Apr 26, 2009

Organizer(s): Tiina Hohn (Grant MacEwan University) Ted Lewis (University of Alberta) Andy Liu (University of Alberta)

This was the seventh year that math fair workshops have been held at BIRS. This year two workshops were held with seventeen participants in the first workshop and sixteen in the second. The first workshop was directed mainly towards math fairs for elementary, junior, and senior high schools, the other was concerned with math fairs for pre-service teachers attending colleges and universities.

The participants came from elementary schools, junior-high and high schools, from independent organizations, and from universities and colleges. The purpose of the workshop was to bring together educators who are interested in using our particular type of math fair, called a SNAP math fair, to enhance the mathematics curriculum. (The name SNAP is an acronym for the guiding principles of this unconventional type of math fair: It is student-centered, non-competitive, all-inclusive, and problem-based.) The projects at a SNAP math fair are problems that the students present to the visitors. In preparation, the students will have solved chosen problems, rewritten them in their own words, and created hands-on models for the visitors. At a SNAP math fair, all the students participate, and the students are the facilitators who help the visitors solve the problems. This process of involving students in fun, rich mathematics is the underlying vision that makes the SNAP program so unique and effective.

At the BIRS workshop, the participants learn about and try math-based puzzles and games that they can use in the classroom. More importantly, they have a chance to see how other teachers have organized math fairs at their schools, how the SNAP math fair fits the curriculum, and what some schools have done for follow-ups.

At the first workshop, Garnik Tonoyan from Yerevan State University in Armenia was a special guest speaker. Dr. Tonoyan has been involved with the International Math Olympiad for many years and he described some of his work Armenia.

The concept of the SNAP math fair originated in Edmonton with Andy Liu and Mike Dumanski, and it has proved so successful that it led to the formation of a non-profit organization, the SNAP mathematics foundation, which has helped promote mathematics in schools around the world. As well as the SNAP foundation, the Calgary-based Galileo Education Network Association (GENA) helps schools organize math fairs, and provides valuable lesson-study follow-ups. The founder of GENA, Sharon Friesen, described the work of GENA in Alberta and BC.

Altogether, the BIRS math fair workshops are having a noticeable impact on mathematics education. The BIRS math fair workshops have contributed greatly to the proliferation and popularization of the SNAP math fair. SNAP math fairs have been held at schools in Alberta, Ontario, British Columbia, as well as in several countries other than Canada. In some places, the use of a SNAP math fair to change children's attitudes about mathematics has almost become a ”grass-roots” movement. Although there is only a small amount of research concerning SNAP math fairs, several participants have strong anecdotal evidence that student achievement in mathematics improves after they have participated in a SNAP math fair, and that the problem solving that they do in preparation for the math fair transfers to other areas of the curriculum.

List of Participants

Ditlof, Gail (Edmonton Public Schools)
Francis-Poscente, Krista (St. Mary’s University College)
Graves, Sean (University of Alberta)
Hamilton, Gordon (Masters Academy and College)
Hoffman, Janice (Edmonton Public Schools)
Hohn, Tiina (Grant MacEwan University)
Kirsch, Julie (Calgary Board of Education)
Lewis, Ted (University of Alberta)
Martin, Judy (University of Calgary)
Motoska, Kimberly (Edmonton Catholic Schools)
Nichols, Ryan (Edmonton Schools)
Pasanen, Trevor (University of Alberta)
Shaw, Dolph (Edmonton Public Schools)
Thomas, Teena (Edmonton Public Schools)
Tonoyan, Garnik (Yerevan State University)
Tonoyan, Lusine (University of Alberta)
Wheeler, Dallas (Calgary Board of Education)

Chapter 25

Workshop on Stochasticity in Biochemical Reaction Networks (09w2142)

Sep 25 - Sep 27, 2009

Organizer(s): David Thorsley (Biotechnology HPC Software Applications Institute), Brian Munsky (Los Alamos National Laboratory)

Overview of the Field

Cellular processes are subject to vast amounts of random variation, which can cause isogenic cells to respond differently, despite identical environmental conditions. Recent experimental techniques make it possible to measure this variation in gene expression, protein abundance, and cellular behavior. Combined with computational modeling, these techniques enable us to uncover the causes and effects of stochastic cellular dynamics. Depending on cellular function, biochemical processes may act to minimize stochastic variations or exploit them to the cell's advantage; in both cases, cellular processes have evolved to be remarkably robust to both intrinsic and extrinsic noise. By exploring this robustness in naturally occurring biological systems, we hope not only to improve our understanding of cellular biology, but also to formulate the "design principles" necessary to build similarly robust biochemical circuits and nanoscale devices. The second workshop on **Stochasticity in Biochemical Reaction Networks** held at Banff International Research Station, 25-27, September 2009 served as an excellent venue to discuss the multifaceted progress in this field.

The exciting research topic of stochasticity in biochemical networks combines many different aspects from multiple disciplines. First, experimental molecular biologists have begun to develop and perfect new quantitative techniques to observe single cell and single molecule dynamics. Tools such as flow cytometry and fluorescence activated cell sorting (FACS) enable researchers to measure the protein levels for millions of individual living cells in the time span of a single minute–thus conducting millions of simultaneous experiments. Time-lapse fluorescence microscopy and microfluidics have made it possible for researchers to measure, track and manipulate the behavior of single cells in carefully controlled micro-environments. Single molecule fluorescence *in situ* hybridization (FISH) techniques enable researchers to explore the spatial distributions of specific RNA molecule within a cell.

Next, theorists and mathematicians have derived new quantitative methods to analyze and explain the vast amounts of statistical data gathered from such experiments. It is known that stochasticity in cells is caused in part by what is referred to as "intrinsic noise" - the variability caused by the statistical dynamics of a chemical reaction with a small number of reactants - and in part by "extrinsic noise" - the variability caused by random fluctuations in a cell's environment. The participants in this workshop have developed many methods to understand and differentiate between these types of variability in experimental data.

Finally, collaborations between theorists and experimentalists can enable the multidisciplinary community to understand how, why and when different cellular mechanisms transmit variability in different ways, i.e. some suppress it while others amplify or exploit it. For example, control theory can help us understand feedback and feedforward regulatory motifs in cellular architectures, while an information theoretic perspective can help us to understand how cells in a developing multicellular organism can determine their exact spatial location. These analyses suggest new methods and appropriate models for mathematically demonstrating how certain motifs are useful for dealing with cellular uncertainties. Such analyses are then directly applicable to the work of more applied researchers, who can use these theories to better constructing synthetic biological circuits and devices at the nanoscale level, including biomolecular motors and DNA molecular machines.

This workshop highlighted many of these recent improvements to measure, analyze, understanding and/or implement stochasticity in biological systems and has served as a starting point to devise the next crucial steps in this progress. A brief summary of some of the specific topics in these categories are discussed in more detail in the following section.

Presentation Highlights

The participants of this workshop form an intellectually diverse group of researchers united by their interest in the subject of stochasticity in biochemical reaction networks; they represent the fields of biology, biophysics, engineering, chemistry, mathematics, and computer science. Each has contributed to the field of biochemical networks in either the theoretical or experimental sphere and many have contributed in both areas.

Kyung-Hyuk Kim, University of Washington derived sensitivity analyses to better understand the effects of cellular variability as it passes through biochemical networks. In theory, these sensitivities could later be used in synthetic biological design to control cellular fluctuations while minimizing changes in mean concentration levels.

Mary Dunlop showed how temporal measurements of gene expression fluctuations could help scientists to determine the existence and form of regulatory links. She showed that natural stochastic noise aids in this process by exciting the systems dynamics. Then as these stochastic dynamics pass through the network it is possible to follow the signature of that noise and determine the underlying sequence of protein (in)activation. Using single cell microscopy and a well characterized three color synthetic gene regulatory construct, she validated the usefulness of these approaches with *in vivo* experiments. She then applied these approaches to discover the regulatory mechanisms in a natural galactose metabolism network.[**?**]

Aleksandra Walczak used analytical tools from the theory of information processing to understand how cell regulatory networks transmit information in order to process external stimuli and initiate cellular responses. Certain biological tasks require more precision and thus more information than others. However, small concentrations of regulatory components at the cellular level (and the resulting intrinsic noise) place strong limitations on cells' abilities to conduct this information transmission. Thus, by understanding how much information is necessary to complete a given biological function, one may be able to predict the qualitative and even quantitative form of the network necessary to complete that function [**?**]. By generalizing the theoretical considerations of information flow, Walczak also formulated spectral method to compute the joint stationary probability distribution of gene regulatory cascades [**?**].

James Faeder illustrated the vast complexity that can arise in signaling networks involving myriad protein-protein interactions. Through combinatorial complexity, the number of distinct chemical species in a given biochemical reaction may exceed any reasonable number, while the mechanics of these reactions and species can be understood with a handful of reaction "rules." In turn these rule-based models can be efficiently simulated with on-the-fly generation of the chemical species as they become important [**?**]. Faeder has successfully applied these rules to develop models to cell-surface receptor aggregation under typical experimental conditions [**?**]. Although biochemical networks like the ones discussed by Faeder can be incredibly complicated, they can often be reduced to much simpler systems as Ilya Nemenman illustrated in his presentation. The key component of this work was to eliminate the many fast chemical species without losing information regarding the slow, non-Poissonian fluctuations of the slow chemical species [**?**]. In related work, Nemenman and collaborators have shown the even very complicated multi-step processes can reduce down to much simpler dynamical systems [**?**, **?**].

Arjun Raj presented a novel approach that he has developed to detect and count individual mRNA molecules inside a single living cell. This process known as single molecule fluorescence *in situ* hybridization promises to revolutionize the study of stochasticity gene regulatory networks [**?**]. Raj then uses this approach to study the gene regulatory network responsible for the robust gut formation during early embryonic development in *C. elegans*.

Also using the FISH approach to detect single mRNA molecules, Gregor Neuert has studied the high-osmolarity glycerol (HOG) pathway, which is one of the mitogen-activated protein kinase (MAPK) pathways in Saccharomyces cerevisiae yeast cells. While the components of this regulatory network are known from many years of previous work, the dynamics of stochastic gene expression in single cells were previously unknown. With the precise single-cell experimental procedures offered by FISH, and careful modeling, a simple intuitive model has been formulated to capture and predict the all observable aspect of the single cell dynamics.

Inspired by the new wealth of quantitative single cell experiment data offered by flow cytometry and single cell microscopy experiments, Brian Munsky showed how one could use this data to better identify the parameters of gene regulatory systems. With theoretical studies, Munsky showed how the distributions of single cell population responses at a few transient time points could provide a lot of information about the underlying system's dynamics, much more information than is obtainable from just the mean behavior or even distributions taken at a stationary time point. These theoretical studies help to establish experimental guidelines that have been used help to identify and test predictive models for (*i*) *lac* regulation in E. coli using flow cytometry experiments [**?**] and (*ii*) the HOG pathway in yeast using Neuert's single molecule mRNA measurements.

Narendra Maheshri investigated the role that stochasticity plays in the positive feedback loop motif, that is prevalent in genetic regulatory networks. Maheshri demonstrated experimentally and in simulation that network with positive feedback can exhibit a bimodal distribution when noise is present in the feedback loop, even if the corresponding deterministic system does not exhibit bistability. Theoretical studies indicate that in order for the bimodal behavior to occur, the promoter in the feedback loop should be expressed in infrequent, large bursts and decay rapidly.

On the topic of molecular computation, David Soloveichik reported on the computational properties of stochastic chemical reaction networks and highlighted the connections between standard models of stochastic chemical kinetics and well-known computation models such as Boolean Logic Circuits, Vector Addition Systems, Petri nets, Primitive Recursive Functions, Register Machines, and Turing Machines [**?**]. Marc Riedel elaborated on this issue of molecular computation from the point of view of circuit design, proposing methods for automated synthesis of stochastic biochemical networks that perform mathematical computations with a high degree of accuracy.

The next pair of talks considered the role of stochasticity in molecular engineering and, specifically, the design of nanostructures and nanotransporters. William Shih presented novel results in the self-assembly of DNA structures. Building on previous results on programmable self-assembly of two-dimensional structures, Shih demonstrated how, by using stacks of flat layers of DNA, custom-designed three-dimensional structures can be made to self-assemble and explained how to control the curvature of the DNA strands in order to design complex shapes [**?**]. Henry Hess discussed the construction and control of molecular shuttles, consisting of cargo-binding microtubules that are propelled by surface-immobilized kinesin motor proteins. Ideally such nanoscale system can be selectively activated at programmable locations and times [**?**]. By controlling the sequestration of the activator compound using an enzymatic network, Hess develops a scheme for sharpening the concentration profile of the diffusing activator at the cost of decreasing activator utilization.

Michael Samoilov discussed the connections between classical, deterministic modeling of "large molecular systems," i.e., chemical reactions in which all of the reacting species are abundant, and stochastic modeling of these systems. Samoilov demonstrated that stochastic effects in large molecular systems are not, as commonly assumed, the results of low molecular counts of some species or of transient effects, but can occur in stationarity even for large systems.

David Thorsley investigated the problem of determining the state of a stochastic chemical kinetic system using time-lapse microscopy data. Because most chemical species in a single-cell experiment cannot be directly observed, Thorsley developed the concept of an observer for a stochastic chemical kinetic system and demonstrated how it could be used for state estimation, parameter estimation, and hypothesis testing.

The last two talks of the workshop focussed on approaches for simulating stochastic chemical reactions. In the basic stochastic simulation algorithms, the chamber in which the reactions occur is assumed to be well-mixed. Sotiria Lampoudi presented a spatio-temporal variant of the stochastic simulation algorithm [**?**]. Michael Chevalier discussed the issue of time-scale separation in stochastic biological systems. The existence of reactions on different time scales results adversely affects the computation time needed for basic stochastic simulations, and Chevalier proposed a new decomposition technique that allows for approximate solutions that trade off between computation time and guarantees of accuracy.

Outcome of the Meeting

The workshop emphasized recent improvements in the theoretical, computational, and experimental investigation of stochasticity at the cellular and nanoscale levels. Each of the participants at the meeting contributed to this progress in at least one, and in many cases two or three, of these advances. The workshop promoted cross-disciplinary communication and collaboration between researchers in mathematical fields such as stochastic processes, Markov models, stochastic simulation and information theory, engineering fields such as control theory, computer science, and circuit design, and scientific fields such as computational biology, nucleic acid research, biophysics, biochemistry, and nanotechnology. The event was highly successful in encouraging the development of a research community uniquely qualified to investigate the phenomenon of stochasticity in biochemical reaction networks.

In addition to presenting significant progress on the topics of stochasticity in biochemical reactions, the workshop also highlighted the persisting need for continued improvements in the analysis of such reactions. For example, combining new techniques for measuring spatial variability in cellular components with spatially non-homogenous analyses may yield new insights into cell regulatory behaviors. Similarly, the expanding usage of experimental techniques such as flow cytometry, time-lapse fluorescence microscopy, and other techniques involving the use of fluorescent proteins leads to a demand of a much more quantitative characterization of these important proteins. Finally, with researchers from many diverse disciplines exploring stochasticity in the fields of synthetic and computational biology, a real need is arising for an improved and standardized toolkit for researchers to describe and computationally analyze cellular variability. These and other discussion topics that arose during the meeting will be revisited in the next workshop on stochasticity in biochemical reaction networks.

List of Participants

Chevalier, Michael (University of California, San Francisco)
Dunlop, Mary (University of Vermont)
Faeder, James (University of Pittsburgh)
Hess, Henry (University of Florida, Columbia University)
Jang, S. Shelly (University of Washington)
Kim, Kyung Hyuk (University of Washington)
Klavins, Eric (University of Washington)
Lampoudi, Sotiria (University of California Santa Barbara)
Maheshri, Narendra (Massachusetts Institute of Technology)
Munsky, Brian (Los Alamos National Laboratory)
Murray, Richard (California Institute of Technology)

Nemenman, Ilya (Emory University)
Neuert, Gregor (Massachusetts Institute of Technology)
Popovic, Lea (Concordia University)
Raj, Arjun (University of Pennsylvania)
Rao, Christopher (University of Illinois Urbana-Champaign)
Riedel, Marc (University of Minnesota)
Samoilov, Michael (QB3, UC Berkeley)
Scott, Matthew (University of Waterloo)
Sheppard, Patrick (University of California)
Shih, William (Harvard Medical School)
Soloveichik, David (California Institute of Technology)
Thorsley, David (Biotechnology HPC Software Applications Institute)
Walczak, Aleksandra (Ecole Normale Superieure)

Chapter 26

Northwest Functional Analysis Seminar (09w2156)

Oct 16 - Oct 18, 2009

Organizer(s): Douglas Farenick (University of Regina), Berndt Brenken (University of Calgary), Heath Emerson (University of Victoria), Vladimir Troitsky (University of Alberta)

Overview of the Field

The field of functional analysis is a central and thriving branch of modern mathematics. Western Canada is particularly strong in the field, as there are researchers in the region who are internationally recognised for their contributions to Banach space geometry, noncommutative harmonic analysis, operator algebras, noncommutative geometry, and operator theory.

The region is home to a number of Tier I Canada Research Chairs in the field of functional analysis, and has a significant number of additional distinguished faculty. A PIMS Collaborative Research Group in Geometric and Harmonic Analysis (led by researchers in Calgary and Edmonton) wrapped up its activities in 2008, and a PIMS Collaborative Research Group in Operator Algebras and Noncommutative Geometry (led by researchers in Victoria, Edmonton, and Regina) commenced its activities in May 2009.

Scientific Developments

The Northwest Functional Analysis Seminar (NWFAS) is a bi-annual regional scientific meeting of researchers (faculty and postdoctoral) and graduate students in functional analysis. Participants are drawn mainly from universities in Western Canada. At this meeting the scientific program addressed: Banach space geometry, ring theoretical notions in Banach algebras, classification of C^*-algebras, complex manifolds, dynamical systems, invariant subspace theory, noncommutative geometry, noncommutative harmonic analysis, and applications of functional analysis to seismic imaging and models for rotating fluids.

A novelty of NWFAS is that young researchers form the bulk of the featured speakers, providing them with a venue to communicate their research results and to form contacts with other functional analysts within the region. The program also regularly features lectures by two or three senior researchers who address topics that are currently attracting high levels of interest.

Of the fifteen lectures at this meeting, 5 were presented by graduate students, 3 by postdoctoral researchers, 5 by recent tenure-track appointments, and 2 by senior researchers.

Presentation Highlights

Banach algebras and ring theory

Alex Brudnyi, in joint work with, A. Sasane, discussed topological conditions on the maximal ideal space of a unital semi-simple commutative complex Banach algebra A that imply that A is a projective, free ring. Several examples were presented, most notably the Hardy algebra of bounded holomorphic functions on coverings of a Riemann surface of finite type.

Bogdan Nica followed the notion general stable rank, which encodes the passage from stably free to free finitely generated modules. He explained that the right perspective on the general stable rank is to view it as a member of a quartet of stable ranks—the other three being the Bass, the topological, and the connected stable ranks. Nica's lecture presented some properties of the general stable rank, as well as some exact computations.

Banach spaces and operators

Vlad Yaskin investigated which Banach spaces embed into L_p with some $p < 0$. Using methods of Harmonic Analysis, he proved that for any two integers k and m, $0 < k < m < n-3$, there is a norm on $\mathbf{R^n}$ such that the resulting normed space embeds into L_{-m} but not in L_{-k}.

Alexey Popov discussed algebras $\mathcal{A}$ of operators on a Banach space X having an almost invariant halfspace. That is, a subspace Y of X of infinite dimension and co-dimension, such that for every $T \in \mathcal{A}$ there exists a finite-dimensional subspace F of X such that $TY \subseteq Y + F$. Popov proved that if $\mathcal{A}$ is generated by a finite number of commuting operators then it also has a common invariant subspace.

C*-algebras

Cristian Ivanescu spoke on the Cuntz semigroup, which has recently received intensive study in the Elliott classification program. In his lecture, Ivanescu presented an existence theorem and the uniqueness theorem in connection with the classification up to isomorphism of simple separable projectionless C*-algebras.

Brady Killough lectured on the class of hyperbolic dynamical systems known as Smale spaces. There is also a well-known construction that takes a Smale space and produces three C*-algebras, each associated with a natural equivalence relation on the space. Integration against the Bowen measure (or its expanding/contracting part) yields a trace on each of these three algebras. Killough presented a result relating these integration traces to an asymptotic of the canonical trace of a bounded operator on a Hilbert space.

Function theory

Damir Kinzebulatov spoke on his joint work with Alex Brudnyi devoted to the study if holomorphic almost periodic functions on coverings of complex manifolds, their function-theoretic properties, and the 'sprouts' of the theory of analytic sheaves on the corresponding Bohr compactifications of the coverings. This work provides a natural link between (i) holomorphic almost periodic functions on tube domains and (ii) almost periodic functions on topological groups.

Noncommutative geometry

Robert Yuncken spoke on an application of noncommutative harmonic analysis to index theory that aims to provide a convenient construction of the infamous gamma element in KK-theory for semisimple Lie groups. After introducing the relevant C*-algebraic structures for the special case of the groups SL$(n, \mathbf{C})$, Yuncken described an explicit construction of the gamma element for SL$(3, \mathbf{C})$.

Robin Deeley, inspired by work of Baum and Douglas, introduced a geometric model for K-homology with coefficients in $\mathbf{Z_k}$. Fundamental to this model is the replacement of spin^c manifold theory with spin^c $\mathbf{Z_k}$ manifold theory. Deeley then used inductive limits to obtain models for any countable abelian coefficient group.

Noncommutative harmonic analysis

Ebrahim Samei, in joint work with Michael Brannan, proved that certain co-representations of the von Neumann algebra generated by the left regular representation of a locally compact group are in fact unitary representations for large classes of groups that includes SIN-groups, maximally almost periodic groups, and totally disconnected groups.

Yin Hei Chen lectured on a geometric property, called the H-separation property, of closed subgroups H in a locally compact group G. Chen explained how this property is related to the duality of subgroups in an operator algebraic setting.

Noncommutative probability theory

Serban Belinschi opened the meeting with a lecture on operator-valued free probability, concentrating on the ideas behind, and some of the applications of, freeness with amalgamation. He described some of the (very many) open problems in the field, and concluded with an interpretation of classical normal distribution from this perspective.

Omar Rivasplata, in joint work with Alexander Litvak, found estimates on the least singular value of a random matrix. He considers large matrices of size $N \times n$, with $N \geq n$, with entries being independent random variables. Unlike in previous studies, he allows some of the entries to be zeros.

Applications

Illia Karabash described his spectral analysis of a partial differential operator L. arising in the theory of rotating liquid films. The periodic conditions cannot be formulated in weighted L^2-spaces in a manner that makes L self-adjoint. However, Karabash explained how periodic conditions can be partially saved in the space $L^2(0, 2\pi)$, even though the operator L becomes "highly non-self-adjoint," and he identifies the natural domain of L and proves that the set of eigenfunctions is complete, but does not form an unconditional basis.

Michael Lamoureux presented techniques developed for numerical modeling of wave propagation and source-signature removal in seismic imaging, based on a class of linear operators known as Gabor multipliers. He discussed boundedness and stability properties for these operators, approximations to PDEs and pseudodifferential operators, and an approximate functional calculus.

Funding

In addition to the support provided by BIRS, the PIMS Collaborative Research Group in Operator Algebras and Noncommutative Geometry provided financial support for travel expenses to graduate students, to postdoctoral researchers, and to new and as yet unfunded faculty members.

Outcome of the Meeting

This meeting of the Northwest Functional Analysis Seminar was the fourth, the first three having taken place at BIRS in 2003, 2005, and 2007. Like the first three meetings, the 2009 meeting was very successful in its aim to play a formative role in developing the profiles of young researchers and graduate students. In addition to a strong scientific program, the seminar provided the only venue in 2008 and 2009 at which the region's researchers in functional analysis came into contact collectively at one meeting.

List of Participants

Belinschi, Serban (Department of Mathematics and Statistics, University of Saskatchewan)
Binding, Paul (University of Calgary)
Brenken, Berndt (University of Calgary)
Brudnyi, Alex (University of Calgary)
Cheng, Michael Yin Hei (University of Alberta)

Deeley, Robin (University of Victoria)
Emerson, Heath (University of Victoria)
Erlijman, Juliana (University of Regina)
Farenick, Douglas (University of Regina)
Floricel, Remus (University of Regina)
Haagerup, Uffe (University of Southern Denmark)
Ivanescu, Cristian (Grant MacEwan University)
Karabash, Illia (University of Calgary)
Killough, Brady (Mount Royal University)
Kinzebulatov, Damir (University of Toronto)
Lamoureux, Michael (University of Calgary)
McCann, Shawn (University of Calgary)
McLean, Doug (University of Calgary)
Nica, Bogdan (University of Victoria)
Phillips, John (University of Victoria)
Popov, Alexey (University of Alberta)
Rivasplata, Omar (University of Alberta)
Runde, Volker (University of Alberta)
Samei, Ebrahim (University of Saskatchewan)
Sourour, Ahmed Ramzi (University of Victoria)
Spektor, Susanna (University of Alberta)
Troitsky, Vladimir (University of Alberta)
Yaskin, Vladyslav (University of Alberta)
Yuncken, Robert (University of Victoria)
Zhang, Dali (University of Calgary)

Focused Research Group Reports

Chapter 27

Residually finite groups, graph limits and dynamics (09frg147)

Apr 12 - Apr 19, 2009

Organizer(s): Miklos Abert (University of Chicago) Balazs Szegedy (University of Toronto)

Overview of the Field

The Focussed Research Group brought together seven active researchers in the following fields: asymptotic group theory (Abert, Jaikin-Zapirain and Nikolov), ergodic theory (Bowen), discrete mathematics (Szegedy) and probability theory (Lyons and Virag).

The common object of interest is residually finite groups, that each field investigates from a different angle. An infinite group Γ is called residually finite, if the intersection of its subgroups of finite index is trivial. This means that finite images approximate the group structure of Γ. Important examples are lattices in linear Lie groups. More generally, finitely generated linear groups, and specifically, arithmetic groups.

From the abstract group theoretical point of view, residual finiteness is a natural condition that allows one to analyze such groups using finite group theory and discrete mathematics. A natural generalization of residual finiteness is soficity. The definition comes from Gromov and means that the group can be approximated by finite structures in a strong sense. The notion is wide enough to put amenable groups in the net – in fact, no finitely generated non-sofic groups are known. On the other hand, it is strong enough to prove general results. For instance, every sofic group satisfies the Kaplansky direct finiteness conjecture.

The topics addressed in the meeting included covering towers, graph limits, weak containment, entropy, groups acting on rooted trees, spectral measure, free spanning forests, percolation, L2 Betti numbers, unimodular random networks, cost, rank gradient, property (τ) and expander graphs.

Recent Developments and Open Problems

The field can be described as something lying at the crossroads of graph theory, group theory, ergodic theory and percolation theory. The field is only half-existing in the sense that while there are already many exciting results and even more questions, some of the researchers active in the area have not assimilated each other's point of view and major directions of research are waiting to be explored. The Focussed Research Group aimed to address these problems.

We now quote some important recent results and problems in the field; most connect to the work of one of the participants.

By the work of Abert and Nikolov, the growth of rank (called rank gradient) equals the cost of the corresponding boundary action of Γ. It is not known whether the rank gradient depends on the chain; both possible answers would solve a distinguished problem, one in 3-manifold theory and the other in topological dynamics. One way to solve this is to decide whether the cost-1 is multiplicative for arbitrary free actions.

Dense graphs have been investigated successfully with analytic methods. For graphs of bounded degree, there are strong hints of the existence of such analysis, but it has not yet been born. A crucial challenge is to understand the shape of random d-regular graphs in some sense. A major test problem here is to show that the independence ratio converges on a random graph sequence.

The residually finite group Γ acts by automorphisms on the corresponding coset tree (a locally finite rooted tree). The action extends to a measurable action on the boundary of the tree. One can connect the dynamics of this boundary action to asymptotic properties of the chain. In particular, the boundary action gives us a graphing (a measurable graph) that is the limit of the graphs coming from the actions on the levels of the coset tree. These special kind of graphings (profinite graphs) need to be investigated in depth. When taking a random point of the boundary, the rooted graph starting there gives us a unimodular random network.

Under mild conditions, the spectral measure of the Markov operator on finite quotients converges to the spectral measure of the Markov operator on Γ. In some cases, this allows one to compute the spectral measure. The core of the Lück approximation theorem is that the spectral measures converge even in a stronger sense. There is a lot of math waiting to be explored here. By Lück approximation, the growth of the first $\mathbb{Q}$-homology equals the first L^2-Betti number of Γ. In particular, the homology growth does not depend on the chain. When taking mod p homology, it is not even known whether the limit always exists.

Lattices in $SL_2(C)$ deserve a special attention among residually finite groups. For instance, the growth of the Heegaard genus on a covering tower of 3-manifolds can be analyzed using spectral properties of the corresponding chain. These topological investigations have already lead to new, exciting pure group theoretical results and more is expected in this direction.

Free spanning forests of $\mathbb{Z}^d$ are widely investigated in probability theory, because of their connection to random walks and percolation. Maybe the most direct way to introduce the first L^2 Betti number of a group is from the expected degree of a free spanning forest on a Cayley graph of it. There are also higher dimensional analogues. A good direction is to exploit this connection and prove new results on L^2 Betti numbers using percolation theory. The cost is also involved in this game, as it gives strong general estimates between the critical values of percolation. There are various beautiful natural problems in this area: for instance, show that $p_c(G) = 1$ implies that G has two ends.

Presentation Highlights

We had numerous three hour long presentations, typically in the mornings. These presentations were very enjoyable, with a lot of questions and dialogue. Speakers usually provided a general picture on the subject and then went into proofs, detailed as the audience requested. The list of three hour presentations included:

- Lyons on percolation and factors of i.i.d.;
- Szegedy on graph limits;
- Abert on rank gradient, weak containment and cost;
- Bowen on entropy in the non-amenable setting;
- Jaikin-Zapirain on Luck approximation and Lackenby's results;
- Lyons on L2 Betti numbers;

The afternoons were typically discussion sessions. People digested each others questions – sometimes answered them, sometimes found new ones.

Scientific Progress Made

Most importantly, people in the group learned each other's angles on problems that were interesting to everyone involved. The group found an array of new questions – in fact, the final collection of questions counts around forty. Some were solved on site, but many remained unanswered. Naturally, it is hard to judge now how hard these will prove to be. We quote some of the new questions, either found at this workshop and some that already existed but have not been publicized widely.

1. Can every d-regular graphing be properly edge colored by $d + 1$ colors (measurable Vizing theorem).
2. For a nonamenable Cayley graph G let $U(G)$ denote the set of factors of i.i.d. on G with a unique infinite cluster. Is the density bounded below on $U(G)$?
3. Let G be a non-amenable Cayley graph with one end. Does there exist a $d' < d$ such that if ω is a factor of i.i.d. with expected degree at least d', then ω has a unique infinite cluster?
4. Are factors of i.i.d. on the 3-regular tree closed in the weak topology? In particular, look at the weak limit of majority functions on n-balls. Is that a factor of i.i.d.?
5. Is it true that $\mathrm{cost}(\Gamma, X) = \mathrm{cost}(\Gamma, X^2)$ for free actions?
6. Suppose G and H are Cayley expanders on n vertices and you can almost match them. Is it true that they are isomorphic?
7. Can you show $\beta_1^2(\Gamma) \leq \mathrm{cost}(\Gamma) - 1$ by combinatorial means (say, free spanning forest)?
8. Let Γ be amenable, acting ergodically and essentially faithfully on X (every nontrivial element moves a set of positive measure). Is the cost of the action 1? If Γ is finitely presented, is it true for any infinite ergodic action?
9. Let G be a Cayley graph and G_n be a sofic approximation of G. Can you label G_n so that it soficly approximates the Cayley diagram?
10. Let Γ be a Property (T) group and let G_n be a sofic approximation to G, that is, a sequence of finite graphs that converges to a Cayley graph of G. Surely, G_n does not have to be an expander family. But can we modify the sequence by an asymptotically vanishing amount (in the edit distance) to a sequence G'_n such that any subsequence of connected components of G'_n is an expander family? To put it another way, can one 'pullback' the ergodic decomposition of the invariant measure on the ultraproduct space?
11. Let Γ be a nonamenable group. Does $\{0, 1\}^\Gamma$ factor onto $\{0, 1, 2\}^\Gamma$?
12. Let S be a finite set and let $\Gamma = (F_S, R)$ be a presentation. For $r > 0, \varepsilon > 0$ and n let $\mathrm{Sof}(r, \varepsilon, n)$ denote the set of sofic approximations of Γ with degree n up to radius r with error at most ε. That is, a function $f : S \to \mathrm{Sym}(n)$ belongs to $\mathrm{Sof}(r, \varepsilon, n)$, if for all $w \in F_S$ with $|w| \leq r$ we have $\mathrm{fix}(w(f)) < \varepsilon$ if $w \notin R$ and $\mathrm{fix}(w(f)) > 1 - \varepsilon$ if $w \in R$. Here $\mathrm{fix}(\sigma)$ denotes the fixed point ratio of σ. Now define the *sofic dimension* of Γ as

$$\sigma(\Gamma) = \inf_r \inf_\varepsilon \lim_{n\to\infty} \frac{\log |\mathrm{Sof}(r, \varepsilon, n)|}{\log |\mathrm{Sym}(n)|}$$

This is a finitary version of the free entropy dimension. Relate it to the first L^2 Betti number of Γ. Can one define the sofic dimension of a m.p. action accordingly?
13. If an invariant percolation on a Cayley graph dominates the FSF and is finitely dependent, must it be connected a.s.?
14. Take an action of a free group with positive f-invariant (in terms of Bowen). Does it factor on an i.i.d.?
15. Let G be a strongly ergodic, bounded degree graphing that weakly contains a finite graph H. Does G factor on H?
16. Are periodic measures dense among all invariant measures for one relator groups?

Outcome of the Meeting

The meeting was a clear success in every aspect. The participants learned a lot of exciting new math. It was nice to observe as people in somewhat distant subjects, like Bowen, Szegedy, Lyons and Abert, had very similar questions and were speaking each other's mathematical language naturally. A lot of new questions (and some answers) have been found. As a result of the meeting, we expect further interaction and maybe collaboration among the participants.

List of Participants

Abert, Miklos (University of Chicago)
Bowen, Lewis (University of Hawaii)
Jaikin-Zapirain, Andrei (Universidad Autonoma de Madrid)
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Nikolov, Nikolay (Imperial College)
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Virag, Balint (University of Toronto)

Chapter 28

Indecomposable binary structures (09frg149)

Jun 14 - Jun 21, 2009

Organizer(s): Pierre Ille (Pacific Institute for the Mathematical Sciences) Gena Hahn (University of Montreal)

Overview of the Field

The notion of *interval* is well-known for linear orders. The analogue for (undirected) graphs is called *module* [25] or *homogeneous* set [6]. One uses also *autonomous* set [16, 21, 22] for partially ordered sets. It is still called interval for relations and multirelations [14, 15], and for directed graphs [18, 24]. For 2-structures [11, 13], it is called *clan*. In our framework, it is easier and more efficient to consider labelled 2-structures [13], simply called binary structures [19].

Given a binary structure, a quotient is naturally associated with a partition in clans of its vertex set. The notions above were mainly introduced to obtain a simple notion of quotient. A binary structure admitting a non-trivial quotient is *decomposable*, otherwise it is *indecomposable* (or *prime* or *primitive*).

Discussion Highlights

Weakly Partitive Families

Ille recalled the basic decomposition theorem of Gallai [16, 22] and its generalisation to binary structures [19]. To obtain decomposition results, it is sufficient to consider weakly partitive families (i.e. families of subsets with the same set properties as the families of the clans of binary structures) without an underlying binary structure. From a weakly partitive family on a finite set, we can apply several times the clan decomposition to obtain its decomposition tree. Then the problem is to construct a binary structure whose family of clans coincides with the initial weakly partitive family. In the finite case, this result is classic and easy. In the infinite one, Ille and Woodrow [20] showed that such a binary structure of rank 3 (that is a 3-labelled 2-structure) exists. Villemaire presented the main points of [20] and gave a nice and short proof of this theorem when an infinite rank is allowed.

Rao presented a generalisation of weakly partitive families, the weakly bipartitive families. They arise from the bipartitions obtained for instance from the splits [9, 10] or from the bijoins [23].

Critical binary structures

Ille recalled the first important results on the indecomposable substructures of an indecomposable binary structure (for instance, see [12]). It results that an indecomposable binary structure contains an indecomposable substructure obtained by deleting one or two vertices. Whence the following definition: an indecomposable binary structure is *critical* if all of its substructures obtained by removing one vertex is decomposable. Schmerl and Trotter [24] characterised the critical binary relational structures. Bonizonni [1] extended their characterisation to 2-structures. Boudabbous and Ille [2] use the *indecomposability graph*, introduced by Ille [17], to obtain a much simpler characterization of critical binary sructures. Ille presented their approach which is based on the characterisation of the connected components of the indecomposability graph.

For tournaments, Culus and Jouve [8] consider *linear clans*, that is, clans inducing subtournaments which are linear orders. They obtain a weaker indecomposability for which they characterised the critical tournaments. Jouve presented their arduous proof.

Duality theorems

Duality theorems are the analogues for specific classes of directed graphs of the classic result of Gallai [16, 22]: given two partially ordered sets with the same comparability graph, if one of them is indecomposable, then they are equal or dual. Boussaïri, Ille, Lopez and Thomassé [4] obtained a similar result for tournaments by considering the family of the 3-cycles instead of the comparability graph. A. Boussaïri and Ille [3] found a very succinct proof of this result by using the minimal indecomposable tournaments [7] and established other duality theorems. Boussaïri presented their work.

Scientific Progress Made

Brignall [5] proposed a nice and natural conjecture on the minimal prime extensions of a graph. During the week, Boussaïri and Ille answered positively to the conjecture and extended their answer to 2-structures.

Outcome of the Meeting

After the presentation of Rao (see Subsection 2.1), the group discussed the possible relationships between the indecomposability (for the clans) and that for the splits or for the bijoins. This constitutes a new area of research.

By considering constant or linear clans, Ille and Jouve will try to extend to binary structures the characterisation obtained in [8] (see Subsection 2.2).

After the presentation of Boussaïri (see Subsection 2.3), Boussaïri and Ille tried to extend duality theorems to binary structures. It is difficult and they will probably have to begin with the extension to binary structures of the characterisation of minimal indecomposable graphs [7].

The French participants will apply for support provided by the French Research National Agency to pursue their joint work in this area.

List of Participants

Boussairi, Abderrahim (University of Casablanca)
Hahn, Gena (University of Montreal)
Ille, Pierre (Pacific Institute for the Mathematical Sciences)
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Leblet, Jimmy (Institut Telecom)
Rao, Michael (LaBRI - CNRS - Université Bordeaux 1)
Villemaire, Roger (University of Quebec in Montreal)
Woodrow, Robert (University of Calgary)

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Research in Teams Reports

Chapter 29

Multiscale statistical analysis for inverse problems in correlated noise (09rit136)

Feb 08 - Feb 15, 2009

Organizer(s): Rafal Kulik (University of Ottawa) Marc Raimondo (University of Sydney) Justin Wishart (University of Sydney)

Nonparametric curve estimation is nowadays a classical topic, which nevertheless still brings a lot of challenges, both in theoretical and applied statistics. Particular problems arise, when one considers

- correlated (in particular, long memory) errors, and/or
- adaptive estimation, and/or
- inverse problems, and/or
- estimation of higher order derivatives.

Overview of the Field

Existing methods of nonparametric estimation include (among others) classical kernel methods, orthogonal series approach and wavelet thresholding algorithms. In particular, in case of long memory errors, the kernel method was studied in [4] (fixed-design regression) and [5] (random-design regression). The wavelet thresholding was studied in [14] (fixed-design case). One has to mention at this point, that in case of long memory errors, fixed-design and random-design regression has to be treated in a completely different way, unlike in case independent, identically distributed (i.i.d.) errors.

The fixed-design nonparametric regression is often referred as *a direct problem*, since we observe a curve (signal) directly, with a noise added. On the other hand, in case of inverse problems, a curve is subjected to a linear operator, which makes an estimation problem much more difficult. Inverse problems may be studied using e.g. kernel methods, but since the work done by Donoho and Johnstone, wavelet adaptive estimation became very popular, see e.g. [7], [8].

Furthermore, recently there has been also an increasing interest in nonparametric change point estimation in a curve and its derivatives, both in a direct setting, as well as in inverse problem set-up, see e.g. [6].

Recent Developments and Open Problems

In case of nonparametric regression in random-design and direct fixed-design setting, there has been a growing interest in variance estimation in heteroscedastic models, see [1]. A general message is, that an estimation of a conditional mean, does not have too much influence on variance estimation. However, to best of our knowledge, very little has been done in case of long memory errors and/or predictors. One should expect that in this case variance estimation may not have an oracle properties, i.e. estimation of conditional mean *does have* influence on variance estimation.

Furthermore, an adaptive wavelet estimation is still not very well understood in case of random-design, even when errors and predictors are i.i.d., see [9].

In case of inverse problem, although the theory of adaptive wavelet estimation is quite well-understood, there has been still some work on numerical performance of suggested algorithms. Especially, the problem of adaptiveness to an unknown Degree of Ill Posedness creates a lot of challenges, even in case of i.i.d. errors. The reader is referred to the recent work in [2]. Needless to say, the case of long memory errors is almost untouched. There, one has to construct an estimator which is adaptive to an unknown Degree of Ill Posedness and unknown long memory parameter.

As for change point estimation, a procedure from [6] does not seem to be easily applicable in practice. To account for that, in [3] the authors proposed and studied, both theoretical and numerical properties, of a kernel-based estimator in case of fixed-design and i.i.d. errors. However, the case of dependent errors (and predictors in random-design case) is almost untouched, except of the recent work [15].

Scientific Progress Made

During the meeting we had focused on two topics:

- Estimating jump points in derivatives in nonparametric regression with dependent noise and predictors, and

- Adaptive estimation in inverse problems with correlated errors.

In case of the first topic, we note that a fixed-design case with long memory errors had been considered in [15]. There, the rates of convergence are influenced by the long memory parameter. In the random-design case, if the errors have long memory and predictors are i.i.d., we were able to prove that the rates of convergence of the appropriately constructed kernel estimator are the same as in the case of i.i.d. errors. In other words, long memory in errors does not affect the rates. In particular, the rates of convergence match the optimal ones in [6]. On the contrary, if the predictors have long memory, then this influences the rates of convergence.

In due course we had also noticed that the estimator from [3], suitable for a fixed-design setting, does not work very well in case of random-design regression. To accommodate that, we modified the estimator, by combining it with quantile estimation.

As for the second topic, as has been mentioned above, while adaptive estimation has been derived in certain inverse problems or in direct regression models with correlated noise, the combined effect of dependence and Degree of Ill Posedness on adaptive estimation remains largely unstudied. We were able to provide the final version of a theorem, which describes rates of convergence in such inverse problems with long memory errors, see [11]. To do that, we utilized a wavelet representation of a Fractional Brownian motion, and we showed that inverse problem with long memory errors can be written equivalently, in a sequence space, as another inverse problem with independent errors. This allowed us to use optimal results from [7]. Furthermore, this lead one of the participants to study a multichannel inverse problem and illustrate very nice theoretical phenomena related to a number of channels, long memory parameters and Degree of Ill Posedness. We have also constructed a modified version of `WaveD` algorithm, which allows us to get better numerical performance in case of long memory errors.

Outcome of the Meeting

Based on the scientific progress described in the previous section, we were able to prepare a preliminary version of the paper on a jump point estimation in random-design regression with correlated noise and predictors, see [13]. It includes development of theory and numerical procedures based on the kernel method. This paper has been in fact finalised during Justin Wishart's stay at the University of Ottawa.

Furthermore, during our meeting we were able to revise two papers on adaptive wavelet estimation with long memory errors: [10] in random-design case, and [11] in fixed-design case. Both papers have been already accepted for publication. The latter paper has in fact an immediate extension to multichannel deconvolution, see [12].

Note

This meeting had been originally scheduled as Research in Teams, with Marc Raimondo and Rafał Kulik as participants. Unfortunately, Marc Raimondo passed away few weeks after our proposal had been accepted. Because of that unfortunate event, the original focus of this meeting, i.e. adaptive estimation in inverse problems with correlated errors (based on a joint work of Marc Raimondo and Rafał Kulik), had to be shifted somehow, to accommodate a joint work of Marc Raimondo and his Ph.D. student, Justin Wishart.

Last but not least, the participants would like to thank BIRS for hospitality. It was for both of us a great opportunity to focus on research for the entire week.

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Kulik, Rafal (University of Ottawa)
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Chapter 30

Modular invariants and twisted equivariant K-theory (09rit146)

Apr 26 - May 3, 2009

Organizer(s): Terry Gannon (University of Alberta), David Evans (Cardiff University)

Overview of CFT and twisted equivariant K-theory

Conformally invariant quantum field theory in 2 dimensions (CFT for short) is by now a well-established area of mathematical physics, with profound relations to several areas of pure mathematics. The two easiest classes of examples are associated to finite groups G (*holomorphic orbifolds*) and to the loop group $LG = \{f : S^1 \to G\}$ of compact Lie groups G (*Wess-Zumino-Witten models*), at some level $k \in \mathbb{Z}$. New examples can be constructed from old ones through the orbifold and GKO coset constructions.

A CFT consists of two halves, called *vertex operator algebras* (VOA), which are linked together by a *modular invariant*. Typically these two VOAs are isomorphic. For the nicest VOAs (called *rational*), e.g. those associated to finite groups or loop groups, the modules form a modular tensor category, and so among other things come with representations of braid groups and other mapping class groups such as the modular group SL(2,$\mathbb{Z}$). The Grothendieck ring of this category is called the *Verlinde ring*. In these rational theories — the only ones we consider — each Verlinde ring is finite-dimensional, associative, commutative and is perhaps the simplest algebraic structure associated to the CFT.

The modular invariant should be thought of as the glue linking together the two VOAs (or more specifically their modules) into the full CFT. The possible modular invariants for a finite group G are parametrized by pairs (H, ψ) for a subgroup H of $G \times G$ and $\psi \in H^2_H(\text{pt}; S^1)$. No such parametrisation is known for loop groups: the modular invariants at all levels k are known only for $LSU(2)$ (which have an A-D-E classification) and $LSU(3)$. More generally, we know that the generic loop group modular invariants are associated to affine Dynkin diagram symmetries. Those symmetries of the unextended Dynkin diagram are associated to outer automorphisms of G; the remaining symmetries are associated to subgroups Z of the centre of G, and yield the so-called *simple current modular invariants*. The remaining modular invariants — the *exceptional* ones — are primarily due to *conformal embeddings* (certain subgroups H of G) and *rank-level duality*. For example, in the A-D-E list of $LSU(2)$, the outer automorphisms of $SU(2)$ are trivial and give rise to the A-series of modular invariants, the only nontrivial subgroup of the centre $\mathbb{Z}_2$ of $SU(2)$ gives rise to the D-series, conformal embeddings give rise to the E_6 and E_8 modular invariants, while E_7 is due to rank-level duality.

The Verlinde ring is associated to each half. The analogous structure, associated to the full conformal field theory (or if you prefer, the modular invariant), is called the *full system* or *algebra of defect lines*. The *nimrep* or *boundary data* is a module for both the Verlinde ring and the full system. In some sense, every modular invariant of a pair of VOAs comes from a restriction of a larger VOA, twisted by an automorphism of the larger VOA. The problem in general is then to find such extensions. In practise (and in theory) the reverse procedure (inducing rather than restricting) is more valuable and is called *alpha induction*.

Much of this data is beautifully captured by *subfactors* (a subfactor is a containment $N \subset M$ of *factors*, which are simple von Neumann algebras), and *sectors* (equivalence classes) of endomorphisms. Here the Verlinde algebra is represented by sectors ${}_N\mathcal{X}_N$ on a III_1 factor N which are nondegenerately braided; multiplication is composition. In this picture, every modular invariant arises from a subfactor $N \subset M$ and an alpha-induction up to a closed (but unbraided) system ${}_M\mathcal{X}_M$ of sectors on M, and the nimrep to a system ${}_N\mathcal{X}_M$ of maps $N \to M$ closed under left compositions by ${}_N\mathcal{X}_N$ and right compositions by ${}_M\mathcal{X}_M$.

The other ingredient in our story is K*-theory*, which on a compact Hausdorff space X looks at the vector bundles over X, or equivalently the finitely generated projective modules over the C^*-algebra $C(X)$ of complex valued continuous functions on X. This gives the abelian group $K^0(X)$, as the Grothendieck group of vector bundles or modules. If a group G acts on our space X, we can define *equivariant* K-theory $K^0_G(X)$ for equivariant bundles, e.g. as the K-theory corresponding to the crossed product $C(X) \rtimes G$. For locally compact spaces, we need to be a bit careful, e.g. by inserting and removing one-point compactifications, but once we've done that we can define the group $K^1_G(X)$ as $K^0_G(\mathbb{R} \times X)$. These C^*-algebras (thought of as spaces of sections of the trivial bundle over X with fibres the compacts $\mathcal{K}$) can be twisted, by taking a non trivial bundle $\mathcal{K}_\tau$ over X. This results in *twisted* (equivariant) K-theory ${}^\tau K^*(X)$ (or ${}^\tau K^*_G(X)$ in the equivariant case). The possible twists τ are classified by a $\check{\mathrm{C}}\mathrm{ech}$ cohomology class of X, the Dixmier–Douady invariant $H^3(X;\mathbb{Z})$ (or $H^3_G(X;\mathbb{Z})$).

In a similar way, twisted equivariant K*-homology* ${}^\tau K^G_*(X)$ can be defined; these are related by Poincaré duality. The most important property of K-theory (or K-homology) is *Bott periodicity*, which says ${}^\tau K^{i+2}_G(X) \cong {}^\tau K^i_G(X)$ and ${}^\tau K^G_{i+2}(X) \cong {}^\tau K^G_i(X)$.

For example, let G be a compact connected simply connected Lie group. The equivalence classes of its finite-dimensional representations under direct sum and tensor product form the *representation ring* R_G. This ring can be realized as the equivariant K-group $K^0_G(\mathrm{pt})$ of G acting trivially on a point pt; the other K-group is $K^1_G(\mathrm{pt}) = 0$.

Recent Developments and Open Problems

The recent work of Freed-Hopkins-Teleman (see e.g. [5]) gives a K-theoretic interpretation for the Verlinde ring $\mathrm{Ver}_k(G)$ of a loop group LG at level k: $\mathrm{Ver}_k(G)$ is the twisted equivariant K-group ${}^{k+h^\vee}K^{\dim(G)}_G(G)$ where G here acts adjointly on itself, $h^\vee$ is the dual Coxeter number of G, and the twist $k + h^\vee$ lies in $H^3_G(G;\mathbb{Z}) \cong \mathbb{Z}$. The multiplication in $\mathrm{Ver}_k(G)$ is recovered from the push-forward of group multiplication. The other K-group, namely ${}^{k+h^\vee}K^{1+\dim(G)}_G(G)$, is 0.

A natural extension of Freed-Hopkins-Teleman would be to realise in a similar spirit (e.g. K-theoretically) the other data, such as the full system, nimreps, and alpha induction, for the modular invariants associated to loop groups. Freed-Hopkins-Teleman were helped to their loop group theory, through considering a toy model: the finite group G case, where it is much easier to see that the Verlinde ring is isomorphic to $K^0_G(G)$. But in [1], the finite group story is developed much more completely, guided by the braided subfactor approach. Consider a modular invariant associated to subgroup $H < G \times G$ and, for simplicity, trivial cocycle ψ in $H^2_H(\mathrm{pt}; S^1)$. Then the full system can be identified with $K^0_{H\times H}(G \times G)$, where $H \times H$ acts on $G \times G$ diagonally on the left and right, and $K^1_{H\times H}(G \times G) = 0$. The nimrep is $K^0_H(G)$, and again $K^1_H(G) = 0$.

We would expect something similar for loop groups. But one of the many ways in which finite groups G are easier than loop groups is that uniform parametrisation of modular invariants. For loop groups, we would expect a different description of the full system etc, for each class of modular invariants (namely those coming from outer automorphisms of G, from subgroups of the centre of G, from conformal embeddings, from rank-level duality,...).

Our recent paper [2] confronted these questions for the loop groups. It's long and technically complicated, and took us over 3 years to write, but will provide the foundation for all of our future work. In it we focussed primarily on what we thought would be the class closest to Freed-Hopkins-Teleman, namely conformal embeddings $H \subset G$; we expected the full system to be related to some twist of $K_H(G)$. This turned out to be far from straightforward, for reasons we only now understand, and we could only obtain partial matches.

[2] also constructed the relevant Dixmier-Douady bundles realising the twists, and studied orbifold examples (again with only partial results).

Scientific Progress Made

We arrived in BIRS with several questions and some ideas. Our intention was to begin a sequel to our paper [2], which we had recently completed. A week later we left with 50+ pages of notes and the core of the sequel [3] worked out. Considering how hard [2] had been to write, we were both completely amazed at how much progress we made in so little time.

We have a new and much more promising approach to conformal embeddings, namely ${}^{\tau}K^0_{H\times H}(G\times G)$ where the action is diagonal: $(h_1,h_2).(g_1,g_2)=(h_1g_1h_2^{-1},h_1g_2h_2^{-1})$ (implicit here is the map $H\to G$). But we now realise that exact K-theoretic descriptions of conformal embeddings will require addressing the Clifford algebras implicit in [5].

But much more important, we obtained a complete understanding of the full system (including nimreps, alpha-induction,...) corresponding to the generic modular invariants, i.e. those coming from outer automorphisms and subgroups of the centre. For example the full system corresponding to a subgroup Z of the centre will be ${}^{\tau}K^0_{G\times G}(G/Z_0\times G/Z_0)$, again using the diagonal action, where Z_0 is a certain subgroup of Z, and τ some twist. The nimrep is ${}^{\tau}K^{\dim G}_{G/Z}(G)$. We accomplished this by first working out the complete picture for the special case of tori, which have a geometric description in terms of lattices. Furthermore, we obtained the K-theoretic description for the Verlinde ring of an infinite class of (non-holomorphic) orbifolds. We failed to do this for even one example in [2].

Outcome of the Meeting

Once we left BIRS we began fleshing out the details. We applied our K-theoretic descriptions to dramatically simplify nimrep formulas appearing in the CFT literature, and recovered K-theoretically formulas for D-brane charges which appeared in the CFT literature. The resulting paper [3] has been submitted it to Commun. Math. Phys. (We also began an unrelated paper, [4], which slowed somewhat our completion of [3].)

There are still some open questions left in [3] (e.g. we only have a partial understanding of rank-level duality and hence of the E_7 modular invariant of $LSU(2)$), but we both feel that the K-theoretic story is now close to complete, and the next step is to obtain direct KK-theoretic descriptions of the various maps here, namely the modular invariant, alpha inductions, the modular group representation, etc. These should be analysed via spectral triples, Fredholm modules and Dirac operators. Given the success of [3], developing this picture is the natural next step.

[2] took over 3 years to write. Partly this is because of its length (88 pages) and complexity, but partly it was because we work on opposite sides of the Atlantic and our visits together are diluted somewhat by other obligations (teaching, grad students, family etc). By contrast our week at BIRS was intense and distraction-free. It was a fabulous and invaluable experience, which pushed our desired extension of Freed-Hopkins-Teleman to new levels. [3] is a fine paper; it could not have been written in anything like this timeline without this Research-in-Teams at BIRS.

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Chapter 31

Prime number races and zeros of Dirichlet L-functions (09rit148)

May 03 - May 10, 2009

Organizer(s): Nathan Ng (University of Lethbridge) Greg Martin (University of British Columbia)

Overview of the Field

This Research in Teams meeting focused on the finer behaviour of the function $\pi(x;q,a)$, which denotes the number of prime numbers of the form $qn+a$ that are less than or equal to x. Dirichlet's famous theorem on primes in arithmetic progressions asserts that that there are infinitely many primes of the form $qn+a$ when a is a reduced residue modulo q (that is, when a and q are relatively prime), and so $\pi(x;q,a)$ is unbounded.

If a and b are reduced residues modulo q, then we may ask whether the inequality

$$\pi(x;q,a) > \pi(x;q,b) \tag{31.1}$$

is satisfied for arbitrarily large x. Chebyshev observed that for the triple $(q;a,b) = (4;3,1)$, the inequality (31.1) holds for all small x. In fact, he asked whether this inequality would continue to hold for all x. However, in 1914 Littlewood proved that for each of the triples $(4;1,3)$ and $(4;3,1)$, there are arbitrarily large values of x such that the inequality (31.1) holds. These inequalities can be thought of as a "prime number race" between two contestants, Team 1 and Team 3. In these terms, Chebyshev observed that Team 3 usually leads Team 1; Littlewood's theorem asserts that each team takes turns leading the prime number race infinitely often.

Over the years, researchers have attempted to prove that there are triples $(q;a,b)$ such that (31.1) holds for arbitrarily large x. However, only a few such results have been established. For many triples $(q;a,b)$ with values of q ranging up to 100, it is known that the inequality (31.1) holds for arbitrarily large x; these results, however, depend on lengthy computer calculations of zeros of Dirichlet L-functions.

One can further generalize to prime number races with more than two contestants. Let $a_1,\dots,a_r$ be distinct reduced residues modulo q. A natural question is whether the system of inequalities

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r) \tag{31.2}$$

holds for arbitrarily large x; this question can be interpreted as a prime number race among r teams. This generalized problem has also received considerable attention without many proven results. The primary goal of *comparative prime number theory* is to establish that any set of inequalities of the form (31.2) has arbitrarily large solutions x. In particular, if we focus upon a particular r-way prime number race (that is, a particular r-tuple $\{a_1,\dots,a_r\}$ modulo q), we can hope to prove that the inequalities (31.2) will be satisfied for arbitrarily large values of x no matter what permutation of the a_i we choose: we call such a prime number race *inclusive*. If for some permutation of the a_i, the inequalities (31.2) are never satisfied when x is sufficiently large, we call the prime number race *exclusive*.

Recent Developments and Open Problems

Over the years, more and more prime number races were proved to be inclusive, but this progress came at the cost of assuming stronger and stronger hypotheses on the locations of the zeros of Dirichlet L-functions. Rubinstein and Sarnak established that all prime number races are inclusive, but only by requiring two strong hypotheses: the Generalized Riemann Hypothesis (GRH), the assertion that all nontrivial zeros of Dirichlet L-functions have real part equal to $1/2$; and a Linear Independence hypothesis (LI), the assertion that the imaginary parts of these nontrivial zeros are linearly independent over the rational numbers.

It is natural to wonder: are such strong hypotheses really necessary to prove these results in comparative prime number theory? Two recent papers of Ford and Konyagin [1, 2] illustrate the difficulty of these problems in the absence of such hypotheses. They show that certain hypothetical GRH-violating configurations of zeros would in fact result in exclusive prime number races. In fact these malicious zeros can be arbitrarily close to the center of the critical strip and arbitrarily far from the real axis; therefore no prohibition on the zeros that was limited to a strip of width less than $1/2$, or of finite height, can suffice to prove that a race game is inclusive. In this sense, their work shows that a hypothesis almost as strong as GRH is necessary to establish this type of result.

One can similarly ask if the LI hypothesis is necessary to prove that prime number races are inclusive, even if we assume GRH. Indeed, Rubinstein and Sarnak deemed LI to be a “working hypothesis”. We believe that it should be possible to construct an infinite set of hypothetical violations of LI that forces a prime number race to be exclusive; doing so would show that a hypothesis almost as strong as LI is also necessary to establish exclusivity results. The main goal of our Research in Teams week was exactly this extension of the Ford–Konyagin work: we planned to search for constructions of hypothetical configurations of zeros, satisfying GRH but violating LI, that force prime number races to be exclusive.

Scientific Progress Made at the Meeting

Our Research in Teams meeting was extremely productive and successful—although none of our successes, as it turns out, involved progress towards the main goal of the week! As it happened, we did prove one result directly related to the main goal, although it actually demonstrates that the goal is harder to achieve than we first imagined. We also established several other results related more fundamentally to the LI hypothesis.

Since our main goal was to show that “enough” linear dependences among the imaginary parts of zeros of Dirichlet L-functions could hypothetically cause prime number races to become exclusive, it made sense for us to first ask: how much is “enough”? In other words, we wanted to show that the presence of only a small set of linear dependences could not force a prime number race to be exclusive, so that we would have a better idea of how complicated our construction would have to be. One of our stated objectives for the meeting was to show that a finite set of linear dependences would not suffice; we quickly realized that we could show that even a “density zero” set of linear dependences would not impact whether a race was inclusive or exclusive. Very surprisingly, however, we were able during the week to improve this result even further, to show that even certain “density one” sets of linear dependences would not impact a race.

To be more precise, assume the generalized Riemann hypothesis, and let γ be an imaginary part of a zero of a Dirichlet L-function $L(s,\chi)$, where χ is a character modulo q. We say that $\frac{1}{2}+i\gamma$ is *self-sufficient* if there is no linear combination of other imaginary parts of zeros of Dirichlet L-functions modulo q that equals γ (so that the LI hypothesis is the assumption that every such γ is self-sufficient). We proved that if every L-function modulo q has a dense enough set of self-sufficient zeros—for example, at least $\varepsilon T/\log T$ zeros with imaginary part bounded by T in absolute value when T is sufficiently large—then every prime number race modulo q, including the full $\phi(q)$-way race, is inclusive. (Note that the total number of zeros of $L(s,\chi)$ with imaginary part bounded by T has order of magnitude $T\log T$, and so quite a thin set of self-sufficient zeros suffices to make a race inclusive.) In fact, for particular races modulo q, we don’t even require this property for all L-functions modulo q, but only for certain subsets of the L-functions depending on the r-tuple $\{a_1, \ldots, a_r\}$ of residue classes. This counterintuitively strong theorem suggests that it could be very difficult to find configurations of zeros of Dirichlet L-functions, satisfying GRH but violating LI, that force prime number races to be exclusive: any such construction would have to involve almost all of the zeros.

Our other successes involved modest (yet still groundbreaking) progress towards actually establishing the LI hypothesis. Two consequences of LI would be that all zeros of Dirichlet L-functions are simple and that $L(\frac{1}{2}, \chi)$ never equals zero; both consequences have been proved to hold a positive proportion of the time in a suitable sense. However, almost no theoretical progress had been made towards showing that more complicated linear dependences seldom occur. Our work during the Research in Teams week advanced this knowledge in multiple ways.

For example, if we consider a fixed arithmetic progression $\{s_k\} = \{\frac{1}{2} + i(\alpha + k\beta)\}$ and a particular function $L(s, \chi)$, it is not possible for every single s_k to be a zero of $L(s, \chi)$. Lapidus and van Frankenhauser have shown [3, chapter 9] that there are at least $T^{5/6}$ values of k between 1 and T for which $\zeta(s_k) \neq 0$ (although their method requires that $\alpha = 0$) and a similar result for Dirichlet L-functions. During the meeting, we were able to show that $L(s_k, \chi) \neq 0$ for at least $cT/\log T$ values of k between 1 and T for some positive constant c, with no restriction on α.

One can also consider a fixed linear form and investigate how often the corresponding linear combination of zeros could be another zero. For concreteness, consider a fixed k-tuple $(\alpha_1, \ldots, \alpha_k)$ of real numbers strictly between 0 and 1 (the case where the α_j are rational has a direct bearing on the LI hypothesis, but our result holds for any real numbers). We proved, assuming the Riemann hypothesis, that among all k-tuples $(\gamma_1, \ldots, \gamma_k)$ in the box $[T, 2T]^k$ such that $\zeta(\frac{1}{2} + i\gamma_j) = 0$ for each $1 \leq j \leq k$, at least $T^{k-\varepsilon}$ of them have the property that $\zeta(\frac{1}{2} + i(\alpha_1\gamma_1 + \cdots + \alpha_k\gamma_k))$ is nonzero.

Our method for establishing this last result can accommodate real numbers α_j that exceed 1, as long as they satisfy a particular bound depending on k; however, there are still natural cases to consider that fall outside the scope we could treat. For example, if $\frac{1}{2} + i\gamma$ is a zero of $\zeta(s)$, then can $\frac{1}{2} + 2i\gamma$ also be a zero of $\zeta(s)$? The LI hypothesis predicts that the answer is always “no”. We were able to reduce the problem of showing that the answer is often “no” to bounding a mysterious exponential sum over primes that was considered by Vinogradov, namely $\sum_{T<n<2T} \Lambda(n) n^{-1/4} e^{2\pi i m\sqrt{n}}$. The conjectured upper bound for that exponential sum would provide a lower bound for the number of expressions of the form $\frac{1}{2} + 2i\gamma$ that are nonzero.

Future Directions

The accomplishments of our Research in Teams meeting have left a collection of accessible further questions to address. Above we described how we showed that at least $cT/\log T$ of the first T elements in an arithmetic progression on the critical line are not zeros of $L(s, \chi)$; we believe that by overcoming certain technicalities, we can actually improve this result to show that a positive proportion of points in any such arithmetic progression are not zeros of $L(s, \chi)$. A similar remark applies to our result on the number of values of a fixed linear form that are not zeros of $\zeta(s)$: by striving for some technical improvements, we hope to increase the lower bound for the number of such values, ideally to $(T \log T)^k$ which is the order of magnitude of the sample space itself. We also plan to see if we can extend this latter result from the Riemann zeta function to all Dirichlet L-functions.

Regarding our last result, which depends upon the exponential sum considered by Vinogradov, one obvious avenue of research is to try to establish the conjectured upper bound for that sum. On the other hand, for our application we need such an upper bound only on average over m, which makes a successful analysis seem more likely. We also plan to generalize our approach to consider $\frac{1}{2} + \alpha i\gamma$ rather than simply $\frac{1}{2} + 2i\gamma$. Finally, our original goal still remains open: can we construct a configuration of hypothetical zeros of Dirichlet L-functions whose linear dependences would force a corresponding prime number race to be exclusive?

With this combination of established results and fruitful additional lines of inquiry, all created in one productive (and luxurious) week at BIRS, we certainly deem our Research in Teams meeting an unqualified success.

List of Participants

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Chapter 32

Measure algebras and their second duals (09rit139)

May 17 - May 24, 2009

Organizer(s): Harold Garth Dales (University of Leeds) Anthony To-Ming Lau (University of Alberta)

Overview of the Field

Let A be a Banach algebra. Then there are two natural products on the second dual A'' of A; they are called the *Arens products*; we here denote the products by $\Box$ and $\Diamond$, respectively. For definitions and discussions of these products, see [2, 4, 5], for example. We briefly recall the definitions. As usual, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\Phi \in A''$, define $\lambda \cdot \Phi \in A$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle , \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle \quad (a \in A) .$$

For $\Phi, \Psi \in A''$, define

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle \ (\lambda \in A') ,$$

and similarly for $\Diamond$. The *left topological centre* of A'' is defined by

$$Z^{(\ell)}(A'') = \{ \Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in A'') \} ,$$

and similarly for the *right topological centre* $Z^{(r)}(A'')$. The algebra A is said to be *Arens regular* if $Z^{(\ell)}(A'') = Z^{(r)}(A'') = A''$ and *strongly Arens irregular* if $Z^{(\ell)}(A'') = Z^{(r)}(A'') = A$. For example, every C^*-algebra is Arens regular [2].

There has been a great deal of study of these two algebras, especially in the case where A is the group algebra $L^1(G)$ for a locally compact group G. Results on the second dual algebras of $L^1(G)$ are given in [2, 16, 17], for example.

More recently, the three participants have studied [5] the second dual of a semigroup algebra; here S is a semigroup, and our Banach algebra is $A = (\ell^{\,1}(S), \star)$. We see that the second dual A'' can be identified with the space $M(\beta S)$ of complex-valued, regular Borel measures on βS, the Stone–Cech compactification of S. In fact, $(\beta S, \Box)$ is itself a subsemigroup of $(M(\beta S), \Box)$. (See [15] for background on $(\beta S, \Box)$.)

Let A be a Banach algebra which is strongly Arens irregular, and let V be a subset of A''. Then V is *determining for the topological centre* if $\Phi \in A$ for each $\Phi \in A''$ such that $\Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in V)$. Recently it has become clear that various 'small' subsets of A'' are determining for the topological centre in the case of some of the above algebras. For example, in [5], we showed that, for a wide class of semigroups including all cancellative semigroups, there are just two points in the space βS that are determining for the topological centre of $\ell^1(S)''$. For an extension of these results to the case of various weighted convolution algebras, see [4] and [3].

Let G be a locally compact group. The measure algebra $M(G)$ of G has also been much studied. This algebra is the multiplier algebra of the group algebra $L^1(G)$. Even in the case where G is the circle group $\mathbb{T}$, the Banach algebra $M(G)$ is very complicated; its character space is 'much larger' than the dual group $\mathbb{Z}$ of $\mathbb{T}$ [14].

Starting at a BIRS 'Research in Teams' in September, 2006, the three participants have been studying the algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$. The report on that week discussed our progress in 2006. Work by the participants continued, and in 2007 and 2008 we established a number of other results that are now all contained in a memoir [6].

The first part of our memoir studies the second dual space of $C_0(\Omega)$, where Ω is a locally compact space. This second dual is identified with $C(\widetilde{\Omega})$ for a certain compact hyper-Stonean space $\widetilde{\Omega}$. The seminal paper on this space is the classic [10] of Dixmier, but we were able to establish some results in this setting that go beyond [10]. We then turn to the algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$ when G is a locally compact group. For example, [6] contains many results on the semigroup structure of $\widetilde{G}$, which is the natural analogue of βS in the non-discrete case. Indeed it is shown in [6, Chapter 8] that $(\widetilde{G}, \Box)$ is semigroup if and only if G is discrete, and in [6, Chapter 7] that the space $\widetilde{G}$ determines the locally compact group G, a result that was already known in the case where G is compact [13].

The plan for the present workshop had two aspects: (1) to complete the memoir [6]; (2) to study the spaces $AP(M(G))$ and $WAP(M(G))$ of almost periodic and weakly almost periodic functionals, respectively, on $M(G)$, where G is a locally compact group.

Recent Developments and Open Problems

There have recently been three very striking advances in our area. These all occurred after our proposal to BIRS was written, and so that that document does not take account of them.

1) Let G be a locally compact group. M. Daws of Leeds has made a dramatic advance [8] on the study of $AP(M(G))$ and $WAP(M(G))$: by using results on Hopf–von Neumann algebras and his earlier work [7], he showed that both of these spaces are C^*-subalgebras of the space $M(G)'$, so resolving a central problem that had been raised in our proposal.

Nevertheless, many problems about $AP(M(G))$ and $WAP(M(G))$ remain open. For example, we know that

$$X_G \subset AP(G) \subset AP(M(G)) \subset WAP(M(G)) \subset M(G)' = C(\widetilde{G}),$$

where X_G is the closed linear span of the character space of $M(G)$. Is it true that $AP(G) = AP(M(G))$ only if G is discrete? If H is a subgroup of G, what is the relation of $AP(M(H))$ to $AP(M(G))$? Several related questions are stated in [6], and may form the basis of a future proposal to BIRS.

2) As stated above, it is known that the group algebra $L^1(G)$ is strongly Arens irregular for each locally compact group G. It is a next obvious question to determine some 'small sets' that determine the topological centre of $L^1(G)''$. Let Φ be the character space, or spectrum, of the commutative C^*-algebra $L^\infty(G)$.

First suppose that G is compact. Then a proof in [16] shows that the family of right identities in $(M(\Phi), \Box)$, a subset of Φ, is determining for the topological centre of $L^1(G)''$.

Second, suppose that G is a locally compact, non-compact group. Then a set which is determining for the topological centre of $L^1(G)''$ is specified by Neufang in [18, Theorem 1.1] (with a certain set-theoretic condition). A further paper of Filali and Salmi [11] establishes in an attractive way that $L^1(G)$ is strongly Arens irregular, and unifies this result with several related results.

Third, our memoir [6] proves that the space Φ (together with two further points in the non-compact case) is determining for the topological centre.

Shortly before the meeting in Banff, we received the very impressive paper of Budak, Işik, and Pym [1] that proves a much stronger result in the case where G is not compact, namely that there are just two points $\varphi_a, \varphi_b \in \Phi$ such that $\{\varphi_a, \varphi_b\}$ is determining for the topological centre of $L^1(G)''$.

The above all leave open the question in the case where the group is compact. Let G be a compact group (such as $\mathbb{T}$). Could it be that a smaller set than Φ is sufficient to determine the topological centre? In fact, at least in the case where G has a at most $\mathfrak{c}$ Borel subsets, it is shown in [6] that we only need $\mathfrak{c}$ points, whereas the fibre has cardinality at least $2^{\mathfrak{c}}$. (Here $\mathfrak{c}$ is the cardinal of the continuum.) However the main open **question** is: *Is there always a finite or countable set S of points in Φ such that S is determining for the topological centre of $L^1(G)$?*

3) The question whether or not the Banach algebra $M(G)$ is strongly Arens irregular for each locally compact group was raised in [12]. This question was resolved positively for non-compact groups G by Neufang in [19], leaving open the question for compact groups. Our proposal stated that we planned to study the Banach algebra $M(G)$, and in particular to seek to show that $M(G)$ is strongly Arens regular for each compact group G. We were not able to resolve this question, although some partial results are given in [6, Chapter 10].

Very shortly before the meeting in Banff, we received from Matthias Neufang an announcement of the following result [20]. Let G be a compact, infinite group of non-measurable cardinality at most $\mathfrak{c}$. Then $M(G)$ is strongly Arens irregular. As yet, we have not had an opportunity to study the proof of this exciting result. Again it suggests the quest of finding small subsets of $\widetilde{G}$ that are determining for the topological centre of $M(G)$.

There is a variety of open questions at the end of [6]. One which we find attractive is the following. Let X be a compact space such that $C(X)$ is isometrically isomorphic to the second dual space of a Banach space. Is it necessarily true that there is a locally compact space Ω such that $X = \widetilde{\Omega}$? Some partial results are given in [6].

Presentation Highlights

Since this was a workshop for three people assembled for 'Research in teams', there were no formal presentations.

Scientific Progress Made

We made progress in two related areas.

First, we studied the draft of the memoir [6] carefully, and made a number of minor corrections, clarifications, and extensions; we included also references to recent results. This memoir has now been submitted to the *Memoirs of the American Mathematical Society*, and is available at the website:

http://www.amsta.leeds.ac.uk/ pmt6hgd/dales.html.

Second, we made further study of the space $\widetilde{\Omega}$ mentioned above. There is an equivalence relation $\sim$ on the set $\widetilde{\Omega}$, defined by saying that $\varphi \sim \psi$ if $\varphi, \psi \in \widetilde{\Omega}$ are not separated by the images of the bounded Borel functions on Ω. The subset U_Ω of $\widetilde{\Omega}$ is the union of the sets Φ_μ, where Φ_μ is the character space of the C^*-algebra $L^1(\mu)'$ for μ a positive measure, and $[U_\Omega]$ is the collection of points of $\widetilde{\Omega}$ that are equivalent to a point in U_Ω. We established the following theorem, and several similar results; it seems that results of this type were not considered before, perhaps rather surprisingly.

Theorem Let Ω be an uncountable, compact, metrizable space. Then

$$|\beta\Omega_d \setminus [U_\Omega]| = |[U_\Omega]| = \left|[U_\Omega] \cap \widetilde{\Omega}_c\right| = \left|\widetilde{\Omega}_c \setminus [U_\Omega]\right| = 2^{2^{\mathfrak{c}}} .$$

Further, suppose that $\varphi \in \widetilde{\Omega}_c$. Then

$$\left|[\varphi] \cap \widetilde{\Omega}_c\right| = 2^{2^{\mathfrak{c}}} .$$

Here $\widetilde{\Omega}_c$ and $\beta\Omega_d$ are the subset of $\widetilde{\Omega}$ corresponding to the space of continuous and discrete measures on Ω, resepctively.

Outcome of the Meeting

The three participants have submitted for publication the memoir [6] that they produced.

All three participants will attend a meeting in Cambridge in July, 2009, in honour of the 75th anniversary of Dr. Dona Strauss; we shall meet several experts in area, including Pym and Neufang, and we expect to have useful discussions. Dales will speak on some recent work in [6].

Dales and Daws will attend the 19th International Conference on Banach algebras in Bedlewo, Poland, in July, 2009, and will have discussions on their work there.

Lau will visit the other two authors in England in November 2009; we expect to discuss further related topics during his visit. We expect that at that time we shall prepare a proposal for a future visit to BIRS for 'Research in Teams'. Further, Lau will speak on work in [6] at a conference in Taiwan in December, 2009.

List of Participants

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Chapter 33

Global Dynamics of Stochastic Differential Delay Equations (09rit141)

Jun 21 - Jun 28, 2009

Organizer(s): Anatoli Ivanov (Pennsylvania State University)

Overview of the Field and Some Open Problems

Stochastic functional differential equations represent a relatively new field of the qualitative theory of differential equations. Their significance has become more evident in recent years due to a great variety of their applications in modeling real life phenomena. Delays are intrinsic features in a multitude of processes in applied sciences and engineering. Uncertainty of the available data and/or the randomness of aspects of the processes themselves lead to the presence of random elements in the models, thus resulting in stochastic differential delay equations. Though the theory of both deterministic functional differential equations and the stochastic ordinary differential equations are rather well developed areas of research, the qualitative theory of stochastic differential delay equations is largely in its infancy stage. Partial explanation of such state of things is the sometimes enormous difficulties facing the researchers, in their approaches and attempts to solve even simply formulated problems. For example, conditions for the stability of the following simple linear stochastic differential delay equation with constant coefficients

$$dx = (ax(t) + bx(t-\tau))\,dt + (\sigma_0 x(t) + \sigma_1 x(t-\tau))\,dW(t) \tag{33.1}$$

are not derived yet. Many aspects of the basic theory of stochastic functional differential equations still need to be developed. Note that the book [5] contains most of the basic theory presently available for stochastic functional differential equations. See also works [1, 3, 7] for additional related details and open problems.

Our intention as a group is to approach some of the problems in the field from a unified point of view, as small stochastic perturbations of some well known deterministic processes. From this prospective, and as a part of the program of our RiT Workshop at the BIRS, we are trying to study the effects of stochastic elements on one-dimensional dynamical systems and continuous time difference equations [4, 6].

Scientific Progress Made

During the meeting we have discussed a number of models appearing in recent applications that are described by stochastic functional differential equations. Among others, the models include equations frequently used in finance applications, such as the geometric Brownian motion, the Ornstein-Uhlenbeck process, the Vasicek process, and the continuous GARCH process. We have developed a unified approach to tackle a range of problems related to the above processes and a detailed plan of their future studies. In particular, we are aiming at solving a range of control problems for the general stochastic equation with delay

$$dS(t) = a(S_t, S(t), u(t))dt + b(S_t, S(t), u(t))dW(t), \tag{33.2}$$

where one is looking to optimize a certain related functional (such as a cost functional, a consumption functional, etc.). Some of the ideas to be used and further developed are based on our recent paper [2].

We have also looked at several specific problems of global dynamics in the stochastic differential delay equation

$$dx(t) = [f(x(t-\tau)) - a\,x(t)]dt + g(x_\tau)dW(t). \tag{33.3}$$

The problems include the global stability of a unique steady state, instability and bifurcation of equilibria, existence of periodic solutions and their stability and shape, and dependence of solutions on parameters. Some of those problems have been stated and partial solutions derived for some of the respective deterministic equations. To the best of our knowledge, those questions are not addressed yet for the stochastic equation (33.3). We have achieved a good progress in solving several of those problems, in particular by treating equation (33.3) as a perturbation of the limiting difference equation $x(t) = 1/a\ f(x(t-\tau))$. This work should result in a joint publication (to be submitted soon). We have also developed a plan of further joint studies in this direction.

Outcome of the Meeting

The purpose of the one-week meeting at the BIRS has been two-fold. The first one was to develop a program of joint research in particular directions of stochastic differential delay equations that are in the intersection of mutual interests of the participants. We have achieved this goal by identifying a number of applied models with the related equations that we will approach to study various aspects of their dynamical behavior.

The second part of the main objective was to further advance and to complete several aspects of joint ongoing research that have been in the working lately between the participants. We have succeeded in this part too. In particular, Ivanov and Swishchuk have completed a typescript dealing with the problem of global stability in a stochastic differential delay equation which is a singular and random perturbation of a continuous time difference equation. Ivanov and Khusainov have completed their work on certain representations of solutions for partial differential equations with delay. Both works have been submitted for publication. Two more manuscripts are near completion.

List of Participants

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Chapter 34

Connections between Minimum Rank and Minimum Semidefinite Rank (09rit155)

Sep 20 - Sep 27, 2009

Organizer(s): Shaun Fallat (University of Regina), Francesco Barioli (University of Tennessee at Chattanooga), Lon Mitchell (Virginia Commonwealth University), Sivaram Narayan (Central Michigan University)

Let G be a (simple, undirected, finite) graph, denote the order of G by $|G|$, and let S_n denote the set of real symmetric $n \times n$ matrices. We use the notation, $G(A)$, to describe *the graph of* A, and by this we mean the graph on vertices $\{1, 2, \ldots, n\}$ and with ij an edge of $G(A)$ if and only if $i \neq j$ and $a_{ij} \neq 0$. The *minimum rank* of G is

$$mr(G) = \min\{rank(A) : \ A \in S_n \text{ and } G(A) = G\}.$$

The *maximum nullity* of a graph G (over $\mathbf{R}$) is defined to be

$$M(G) = \max\{dim\,(ker(A)) : \ A \in S_n \text{ and } G(A) = G\}.$$

Clearly,

$$mr(G) + M(G) = |G|.$$

Two other families of matrices associated with a graph are subsets of the real $n \times n$ positive semidefinite matrices, which we denote by PSD_n, and the complex $n \times n$ positive semidefinite matrices, which we denote by $HPSD_n$. The *set of symmetric positive semidefinite matrices of graph* G is

$$SD(G) = \{A \in PSD_{|G|} : \ G(A) = G\},$$

and the *set of Hermitian positive semidefinite matrices of graph* G is

$$HSD(G) = \{A \in HPSD_{|G|} : \ G(A) = G\}.$$

Then we define the *minimum semidefinite rank of a graph* G, denoted by $msr(G)$, the smallest rank over all matrices in SD(G). It is clear that $mr(G) \leq msr(G)$. Along these lines, we define $M_+(G)$ to be the maximum nullity over all matrices in SD(G). It is evident that $M(G) \geq M_+(G)$, for all graphs G.

During the week at BIRS our team considered a number of important open problems regarding minimum rank and minimum semidefinite rank. One such issue, which is of current interest, was to consider the class of graphs known as outerplanar. A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices lie on the same face. It is worth noting that all trees and all unicyclic graphs are outerplanar.

Associated with any graph is an important graph parameter, known as the path cover number. The *path cover number* of a simple graph G, $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all of the vertices of G. It is known that $M(T) = P(T)$ for every (simple) tree T [2]. Sinkovic has recently demonstrated that for a (simple) outerplanar graph G, $M(G) \leq P(G)$ and has given a family of outerplanar graphs for which equality holds [5].

One of our main objectives for the week was to gain a better understanding on the possible connections between the minimum rank and minimum semidefinite rank of a graph, and this is exactly what we accomplished in the case of outerplanar graphs. We began by discovering a new graph parameter, known as the tree cover number and used it in connection with $M_+(G)$ when G is outerplanar.

The *tree cover number* of a graph G, possibly with multiple edges but no loops, denoted $T(G)$, is the minimum number of vertex disjoint simple trees occurring as induced subgraphs of G that cover all of the vertices of G.

Our main result is a complete characterization of the maximum nullity over all positive semidefinite matrices whose graph is outerplanar. Namely, we proved that $M_+(G) = T(G)$, for all outerplanar graphs G. This is a significant result, as like the case of trees, it establishes, a direct link between the algebraic quantity, nullity, to a combinatorial quantity, namely the tree cover number. Moreover, this result verifies an equation between msr and a graph parameter. The main tool used in the proof of this theorem is the notion of orthogonal removal of a vertex, which was developed in the context of finding the minimum semidefinite rank of chordal multigraphs [1].

We also studied the tree cover number in general, and compared with other known graph parameters and to M as a completeness exercise. We were also faced with a number of interesting open questions, such as studying the graph complement conjecture for outerplanar graphs, along with many other issues.

Given a set $\mathbf{X}$ of n nonzero column vectors in $\mathbf{C}^d$, $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, let X be the matrix $\begin{bmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_n \end{bmatrix}$. Then X^*X is a psd matrix called the *Gram matrix* of $\mathbf{X}$ with regard to the Euclidean inner product. Its associated graph G has n vertices $\{v_1, \ldots, v_n\}$ corresponding to the vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, and edges corresponding to nonzero inner products among those vectors. By the *rank* of $\mathbf{X}$, we mean the dimension of the span of the vectors in $\mathbf{X}$, which is equal to the rank of X^*X. Consequently, $\mathbf{X}$ is called a *vector representation* of G. Vector representations have been a key tool in recent advances in minimum semidefinite rank problems (see, for example, [2, 3]).

At BIRS, it was shown that vector representations can be used in conjunction with unitary matrices to solve or give new approaches to open problems:

Given a vector representation of a graph G, let X be the matrix mentioned above. Let

$$P = \begin{bmatrix} X \\ E \end{bmatrix},$$

where E is a matrix whose zero/nonzero pattern is that of the edge-vertex incidence matrix of G. We first notice that, since the columns of X give a vector representation of G, and since the rows of E have only two nonzero entries each corresponding to a nonzero inner product of columns of X, the nonzero entries of E can be specified so that the columns of P are pairwise orthogonal. After normalizing the columns of P, P may be completed to a unitary matrix

$$V = \begin{bmatrix} X & ? \\ E & L \end{bmatrix},$$

where the rows of L must then be a vector representation of the line graph of G, $L(G)$. Inspecting the sizes of the various blocks of V, this gives the following result, obtained at BIRS this year: For any graph G, $|G| - \text{msr}(G) \leq |L(G)| - \text{msr}(L(G))$.

Written slightly differently as

$$\text{msr}(L(G)) \leq |L(G)| - (|G| - \text{msr}(G)),$$

this is reminiscent of the δ-conjecture, one of the two most well-known open conjectures in minimum rank: For any graph G, $\operatorname{msr}(G) \leq |G| - \delta(G)$, where $\delta(G)$ is the smallest degree among the vertices of G. Significant progress was made in special cases of the δ-conjecture at BIRS: including a proof of the conjecture when $\delta \leq 3$. Further, it was conjectured that a stronger result is true, namely, that $M_+(G) \geq \tilde{\delta}$, where $\tilde{\delta}$ is the maximum degree of a vertex v that has minimum degree among its neighbors, and $D(v)$ (i.e., the graph obtained from G by deleting v and all of the neighbors of v) is connected.

Another well-known open conjecture is the *Graph Complement Conjecture*: For any graph G, if $\overline{G}$ is the complement of G, then $\operatorname{msr}(G) + \operatorname{msr}(\overline{G}) \leq |G| + 2$.

As noted at BIRS, both the graph complement conjecture (GCC) and the δ-conjecture can be transformed into associated unitary matrix completion problems. This idea is similar to one previously explored in the context of finding the msr of bipartite graphs [4]. Here, we will demonstrate how to approach GCC:

Let X be a matrix whose columns form a minimal vector representation of G. Construct a matrix E whose rows have exactly two nonzero entries, and where each row of E corresponds to either an edge of G or an edge of $\overline{G}$. Thus E will be a $\begin{pmatrix} |G| \\ 2 \end{pmatrix} \times |G|$ matrix. Let

$$M = \begin{bmatrix} X \\ E \end{bmatrix}.$$

Choose the nonzero entries of E, row by row, so that if the columns of X corresponding to the two nonzero entries of a row in E are not orthogonal, then the corresponding columns of M are, and vice-versa. At the end of this process, the columns of M will be a vector representation of $\overline{G}$, but most likely not a useful one, as it will no doubt have a high rank. Now, find a matrix N so that the rows of the matrix $\begin{bmatrix} M & N \end{bmatrix}$ are pairwise orthogonal and all have the same length (we discuss how to do this below). Having done so, normalize the rows of $\begin{bmatrix} M & N \end{bmatrix}$, and extend to a unitary matrix

$$U = \begin{bmatrix} M & N \\ V & ? \end{bmatrix}.$$

By construction, the columns of V give a vector representation of $\overline{G}$ with rank bounded by a function of the size of N. In particular, if such an N can be selected to have $\begin{pmatrix} |G| \\ 2 \end{pmatrix} + 2$ columns, then

$$\operatorname{msr}(\overline{G}) \leq |G| + 2 - \operatorname{msr}(G),$$

establishing GCC. If any such N must have more than $\begin{pmatrix} |G| \\ 2 \end{pmatrix} + 2$ columns for a particular graph, then that graph will give a counterexample for GCC.

We note that such an N may always be found, as the question of simultaneously normalizing and orthogonalizing a set of vectors can be phrased as the matrix equation

$$M^*M + N^*N = cI,$$

where M is known. Since M^*M is a positive semidefinite matrix, choosing any $c > \max \sigma(M^*M)$ (where σ is the set of eigenvalues) will make $cI - M^*M$ a positive semidefinite matrix, and guarantee the existence of an N with $N^*N = cI - M^*M$.

We can phrase this question, then, in a number of different but equivalent ways: Given a zero/nonzero pattern, what is the size of the smallest pattern containing the original that is the pattern of a unitary matrix? What is the largest multiplicity of the largest eigenvalue of a Hermitian matrix with given zero/nonzero pattern? What is the smallest rank matrix N that will solve the matrix equation $M^*M + N^*N = cI$ for given M and arbitrary c?

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Chapter 35

Exceptional Dehn filling (09rit158)

Oct 25 - Nov 1, 2009

Organizer(s): Cameron Gordon (University of Texas at Austin), Steve Boyer (Université du Québec à Montréal)

Overview of the Field

This Research in Team workshop focused on several problems in the theory of *exceptional Dehn fillings* in 3-dimensional topology. Dehn filling is the construction in which you take a 3-manifold M, with a distinguished torus boundary component T, and glue a solid torus V to M via some homeomorphism from ∂V to T. The resulting manifold depends only on the isotopy class (*slope*) α on T that is identified with the boundary of a meridian disk of V, so we denote it by $M(\alpha)$. The construction goes back to Dehn in 1910, who introduced it in the special case where M is the exterior of a knot in S^3. The Lickorish-Wallace theorem of the early 1960's showed that any closed, connected, orientable 3-manifold can be obtained by Dehn filling the boundary tori of the exterior of some link in S^3. Consequently, many of the basic problems in 3-manifold topology can been analysed in terms of the operation.

Renewed interest in the construction arose with the ground-breaking work of Thurston in the 1970's, who used it to study hyperbolic geometric structures on 3-manifolds. In particular, Thurston showed that if M is hyperbolic then $M(\alpha)$ is also hyperbolic for all but finitely many slopes α on T. When M is hyperbolic but $M(\alpha)$ is not, one says that $(M;\alpha)$ is *exceptional*. Although it is clear that one cannot hope to classify all exceptional $(M;\alpha)$'s, it turns out that it is relatively rare for a hyperbolic 3-manifold to have two distinct exceptional slopes α and β on T, and it is not too unreasonable to try to classify all such $(M;\alpha,\beta)$'s. This has usually been approached by considering the various different ways in which $M(\alpha)$ and $M(\beta)$ can fail to be hyperbolic, and there has been a lot of progress along these lines. The cases about which least is known is when the boundary of M is a torus and one of the fillings, say $M(\beta)$, is a small Seifert fiber space. This was the focus of the workshop

Recent Developments and Open Problems

As stated in the Overview, the focus of the workshop was the classification of triples $(M;\alpha,\beta)$ where M is a hyperbolic 3-manifold with torus boundary and $M(\alpha)$ and $M(\beta)$ are distinct non-hyperbolic Dehn fillings on M. There has been considerable progress on this classification over the last 30 years or so, but some problems remain. Let $\mathcal{E}(M) = \{\alpha \mid M(\alpha) \text{ is not hyperbolic}\}$ be the set of exceptional slopes on the boundary of M. Two problems which have been the focus of intense research are:

(A) Understand the structure of $\mathcal{E}(M)$.

(B) Describe the topology of M when $|\mathcal{E}(M)| \geq 2$.

See the survey [6] for instance.

An essential quantity used to describe results on these problems is the notion of the *distance* $\Delta(\alpha, \beta)$ (minimal geometric intersection number) between two exceptional slopes α and β. Two results which exemplify what can occur when the situation described in problem (B) arises are Gordon's theorem: *if two toroidal filling slopes are of mutual distance at least* 6*, then* M *is one of four specific manifolds* M_1, M_2, M_3, M_4 [7], and Ni's recent theorem: *if* M *is the exterior of a knot in the* 3*-sphere which has a non-meridional slope whose associated filling yields a lens space, then* M *fibres over the circle* [11]. One of the key conjectures concerning problem (A) is the following:

Conjecture [C.McA. Gordon] $\#\mathcal{E}(M) \leq 10$ and $\Delta(\mathcal{E}(M)) \leq 8$. Moreover, if $M \neq M_1, M_2, M_3, M_4$, then $\#\mathcal{E}(M) \leq 7$ and $\Delta(\mathcal{E}(M)) \leq 5$.

It is shown in [4] that the conjecture holds if the first Betti number of M is at least 2. (Note that it is at least 1.) Lackenby and Meyerhoff have recently announced a proof that the first statement of the conjecture holds in general [8]. Agol has shown that there are only finitely many hyperbolic knot manifolds M with $\Delta(\mathcal{E}(M)) > 5$ [1], though there is no practical fashion to determine this finite set.

It is a consequence of the Geometrization Conjecture, recently proved by Perelman (c.f. [9, 10]), that if a 3-manifold is not hyperbolic, then it is either

(1) reducible (contains an essential sphere), or

(2) toroidal (contains an essential torus), or

(3) a Seifert fiber space over S^2 with at most 3 exceptional fibers.

Moreover, a manifold of type (3) is either

(a) S^3, or

(b) a lens space, or

(c) a *small* Seifert fiber space (SSFS), i.e. one with base S^2 and exactly 3 exceptional fibers.

Finally, case 3(c) splits into two subcases (i) finite fundamental group, and (ii) infinite fundamental group.

The problems have usually been approached by considering the various possibilities (1), (2), (3)(a), (3)(b), (3)(c)(i) or 3(c)(ii), for the pair of non-hyperbolic manifolds $M(\alpha)$ and $M(\beta)$. Perhaps surprisingly, the least tractable cases are when $M(\alpha)$ or $M(\beta)$ (or both) is a SSFS. When neither is a SSFS, the best possible upper bounds for $\Delta(\alpha, \beta)$ are essentially known, as a result of the work of several people (there are only two cases left). Further, the conjecture above has been verified in all cases. When $M(\beta)$ is a SSFS and $M(\alpha)$ belongs to one of the classes of non-hyperbolic manifolds listed, an important problem is to obtain the optimal upper bound on $\Delta(\alpha, \beta)$.

The case where $M(\alpha)$ is reducible was considered in [2] and [3]. Building on [3], but introducing new tangle-theoretic techniques, the optimal bound $\Delta(\alpha, \beta) = 1$ in the case where $M(\beta)$ has finite fundamental group was established in [5]. Suppose now that $M(\alpha)$ is toroidal. In this case there is an m-punctured essential genus 1 surface F of slope α properly embedded in M. Assume that F is chosen to minimize m among all such surfaces. It is expected that if $\Delta(\alpha, \beta) > 5$, then M is the exterior of the figure eight knot and F is separating with $m = 2$.

Scientific Progress Made

The precise focus of this Research in Team workshop was the case where $M(\alpha)$ is toroidal and $M(\beta)$ is a SSFS. During our week at BIRS, we essentially completed the proof that $\Delta(\alpha, \beta) \leq 5$ if F does not split M into I-bundles and $m \geq 3$. This was achieved by enhancing the method used in [2, 3] based on the JSJ decomposition. A paper *Characteristic Submanifold Theory and Toroidal Dehn Filling I* by S. Boyer, C. McA. Gordon, and X. Zhang is currently being prepared which details this advance. We also made significant advances on the case $m \leq 2$ where a combination of the JSJ technique and tangle-theoretic technique introduced in [5] is used. This work will appear in a second paper *Characteristic Submanifold Theory and Toroidal Dehn Filling II* that we are currently working on.

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List of Participants

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Summer School Reports

Chapter 36

The Mathematics of Invasions in Ecology and Epidemiology (09ss128)

May 10 - May 17, 2009

Organizer(s): Troy Day (Queens University) James Watmough (University of New Brunswick) Jianhong Wu (York University) Fred Brauer (University of British Columbia) Rachel Bennett (Queen's University)

Overview of the Field

This summer school was a continuation of the MITACS summer school series on disease modelling. Previous schools have been held at BIRS (2004), York University (2006), Xian Jiaotong University (2006), the Atlanta Center for Disease Control (2007), the University of Edmonton (2008) and the University of Ottawa (2009). In contrast to these schools, which focused on the mathematics of epidemiology and public health, this school focused on the dynamics of invasions and evolution. The school was organized jointly by two MITACS research groups: a group of researchers working on mathematical models of infectious diseases (www.liam.yorku.ca/research/MADI/) and a group of researchers working on mathematical models of biological invasions and dispersal (www.unb.ca/bid).

Historically, the fields of mathematical ecology and the dynamics of evolution have developed separately and it is only recently that work has been done to begin to bridge these two fields. Models for the evolution of populations assumed slowly changing or constant populations, and models for ecological populations assumed evolution took a much longer time scale than population dynamics. Recent theoretical work has begun to bridge these two approaches allowing population traits to change on the same timescale as population size. This advance is necessary for a theoretical framework for pathogen evolution in many systems. The influenza virus provides a pressing example. The timescale of viral evolution is similar to the rate of spread of the virus though the host population. Any control measures, such as vaccines or antiviral medications, must take into account the rapid appearance of drug resistant strains. Other examples presented in lectures include weedy species [7], HIV and vector-borne parasites such as malaria.

The lectures were divided roughly along two lines: the evolution of pathogens; and the spread of an invading pathogen. The mathematical foundations of pathogen evolution were outlined in the lectures of Day and Gomulkiewicz, and applied in the lectures of Gilchrist, Reisberg and Reid. The lectures of Allen and Brauer introduced the basic stochastic and deterministic models for pathogen invasion and spread, while the lectures of Arino and Gourley covered spatial approaches to modelling, first in the context of metapopulations, and the latter in the continuous space. Nelson applied these methods to the spread of a forest insect and introduced additional modelling techniques for structured host populations. The lecture of Zou extended the models introduced by Gourley to include delays.

Recent Developments and Open Problems

Several key open problems were introduced in the lectures.

1. A single theoretical framework for pathogen evolution and spread has yet to be developed.

2. New modelling paradigms are necessary for pathogen spread and evolution that incorporate both the scales of within-host and between-host into a single theoretical framework [3].

3. Key public concerns surrounding treatment and vaccination require models for the evolution of resistance, such as a resistance to treatment or vaccine in a human infectious disease.

4. Analytical tools are needed to address the spatial component of resistance evolution. For example, Chloroquine resistance in mosquitoes seems to arise in areas of low transmission [1], which suggests that control measures must take spatial dynamics and evolution of the pathogen and vector into account.

5. Much more analytical work is needed to understand the path of spread of a pathogen through a structured population. Two examples given were the global spread of novel human pathogens [4] and the spread of the mountain pine beetle [5]. In the first case, the host population is spatially structured, and in the second, the population is structured by host defenses.

6. Most diseases involve a delay between infection and onset. This leads to many open mathematical problems in a spatial setting [6]

Presentation Highlights

As with previous schools, there were many more applications than could be accepted. In attendance were 35 students from mathematics and ecology of which 8 were from the US and 4 from outside north america.

The format for the school consisted of a series of short courses, case studies and student group projects. Short courses were an important component, providing students with the basic theory on a topic. The participants were able to apply the theory in group projects. Each short course consisted of one 90 minute lecture and one 60 minute tutorial. They covered topics on basic modelling, both deterministic and probabilistic, and specialized modelling topics such as metapopulations, evolution and dispersal. Case studies consisted of 90 minute lectures by leading international researchers. This summer school covered relevant mathematical background and recent progress in the fields of biological invasions in ecology and epidemiology

The highlight of the meeting was the presentations by the students. These students varied from upper level undergraduates to Postdoctoral and young researchers. The projects were assigned on the second day of the workshop, giving a mere five days for development and analysis of an appropriate model. As in the past, the projects were well done, with some exceptional presentations.

The student projects identified six open problems:

- the role of disease in the decline of amphibian species, specifically the role of *Chytridiomycosis* as a cause of species extinction;

- the role of a pathogen in the spread of an invasive forest or agricultural insect;

- the evolution of virulence in the spread of syphilis;

- the reasons for spectacular failures of biological control using exotic species;

- the evolution of a species in a changing habitat;

- the spread of a novel strain of influenza.

Scientific Progress Made

The lectures and case studies presented techniques, ideas and problems at the leading edge of mathematics and its applications to ecology and evolution. Most of the case studies covered applications of mathematics from an ecological perspective, encouraging all participants to think about new mathematical approaches to problems. Some of the project reports covered new ground, and we encourage the students to continue their collaboration wih the goal of publishing their results in a peer reviewed journal.

Outcome of the Meeting

In summary, the school brought together students and researchers in evolution, ecology and epidemiology using a variety of modelling and analytical techniques, including game theoretic, statistical, pde, ode and dynamical systems. We hope that many students will continue to work in this emerging research area of theoretical epidemiology at the interface of mathematics, ecology and evolution.

List of Participants

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Anvari, Vahid (York University)
Arino, Julien (University of Manitoba)
Ashander, Jaime (University of Alberta)
Biernaskie, Jay (University of Toronto)
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Crespo Perez, Veronica (University Paris VI/Pontifical Catholic University of Ecuador)
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Gourley, Stephen (University of Surrey)
Hammond, Rebecca (Acadia University)
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"Chytridiomycosis as a cause of species extinction?" used a simple differential equation model to study the role of disease in the decline of amphibian species. Many amphibian pathogens can live freely in the host environment as well as within the host. This increases the chance of species extinction.

War of the Worlds: Modelling the spread of an invasive species and a pathogen in a host population.

Syphilitic Strategies

Biological Control of Invasive Species: Why doesn't it work?

Mathematical Models Of Predator-Prey Systems