The Many Aspects of Mahler's Measure

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1 Introduction

The idea behind the workshop was to bring together experts specializing in many different fields: dynamical systems, K-theory, number theory, topology, analysis, to explore some of the many apparently different ways that Mahler's measure appears in different areas of Mathematics. The hope was to encourage cross-fertilization between these disciplines and increase our understanding of Mahler's measure.

Our plans did not anticipate that on the day that we were to gather for our workshop, nature would decide to dump 60 cm of snow on Calgary, shutting down Calgary International Airport to most incoming flights. As a result, the workshop started a day late with a somewhat diminished attendance. Fortunately, Dale Rolfsen had arrived in Banff a day early and was able to take over until some of the organizers were able to arrive. In spite of the late start, everyone who had planned to speak was able to give a lecture and the outcome exceeded all expectations, as we will see below.

It should be mentioned that many of the participants and many others, including students and junior faculty, took the opportunity to continue their study of Mahler's measure at a PIMS conference in June at Simon Fraser University, organized by Peter Borwein and Stephen Choi. A highlight of this conference was the series of four 90 minute lectures by Jeff Vaaler, each treating a different aspect of Mahler's measure.

Given a polynomial $P(z_1, \ldots, z_n)$ with complex (or for us usually integer) coefficients, the *logarithmic Mahler measure* is defined to be the average of $\log|P|$ over the real n-torus, i.e.

$$m(P) := \int_0^1 \cdots \int_0^1 \log|P(e(t_1), \ldots, e(t_n))| \, dt_1 \ldots dt_n,$$

where as usual $e(t) := \exp(2\pi i t)$. The quantity actually defined by Mahler was $M(P) = \exp(m(P))$, i.e. the geometric mean of $|P|$ over the n-torus, but it seems that $m(P)$ is really the more fundamental quantity. We will simply refer to $m(P)$ as the Mahler measure of P. Below we will always assume that P has integer coefficients unless otherwise stated. There is no harm in allowing P to be a Laurent polynomial, i.e. a polynomial in $z_1, 1/z_1, \ldots, z_n, 1/z_n$ since these can be converted to ordinary polynomials by multiplication by a monomial in $(z_1, \ldots, z_n)$ and a monomial has logarithmic Mahler measure 0.

2 Polynomials in one variable

For polynomials in one variable, Jensen's formula shows that if $P(z) = a_0(z - \alpha_1)\dots(z - \alpha_d)$, one has $M(P(z)) = |a_0| \prod_{j=1}^{d} \max(|\alpha_j|, 1)$.. If P has integer coefficients, this shows that $M(P)$ is an algebraic integer so $m(P)$ is the logarithm of an algebraic integer. It also shows that $m(P) \geq 0$ and that if $m(P) = 0$ then P is a monic polynomial all of whose roots are roots of unity (briefly, a *cyclotomic* polynomial).

The quantity $m(P)$ for polynomials in one variable occurs naturally in many problems of number theory or dynamical systems, as a growth rate or an entropy. Lehmer [19] encountered $m(P)$ in his study of the integer sequences $\Delta_n = \text{Res}(P(z), z^n - 1) = \prod_{j=1}^{d}(\alpha_j^n - 1)$ for monic P. It is clear that if P does not vanish on the circle then $\lim |\Delta_n|^{1/n} = M(P)$ and this is true even if P does vanish on the circle, but the proof is certainly not obvious. Lehmer was led by this to ask whether there is a constant $c > 0$ such that $m(P) > c$ provided P is not cyclotomic. This question of Lehmer is still unanswered and provided early motivation for the study of $m(P)$. The positive answer is the one expected and is known as *Lehmer's conjecture* (although Lehmer did not conjecture this in print). Lehmer provided an example of the polynomial $L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ for which $m(L) = 0.162357\dots$ is the smallest value known for non-cyclotomic polynomials. Note that this polynomial is reciprocal (palindromic). Smyth [29] showed in his Ph.D. thesis that if P is non-reciprocal then the smallest value attainable is $m(z^3 - z - 1) = 0.281199\dots$, almost twice as large.

Lehmer's conjecture is not yet proved and is still a question of much interest. An important and useful result of Edward Dobrowoski [13] gives the estimate $m(P(z)) \geq c(\log\log n / \log n)^3$ for a non-cyclotomic polynomial in one variable with degree n, where $c > 0$ is an explicit constant. This result does not extend to polynomials in many variables, but there is another estimate also due to Dobrowolski [14] that gives an explicit lower bound for $m(P(z))$ depending only on the number of terms in the polynomial. Using the limit theorem described in the next section, this does extend to polynomials in many variables giving exactly the same estimate.

At the workshop, Peter Borwein described work with Kevin Hare and Michael Mossinghoff [2] giving the explicit (and best possible) lower bound of $(1 + \sqrt{5})/2$ for the Mahler measure of non-reciprocal polynomials with odd integer coefficients, applying in particular to Littlewood polynomials (those with all coefficients in $\{-1, 1\}$. The question of how to extend this to reciprocal polynomials was partially settled during the workshop in a collaboration between Borwein, Mossinghoff and Dobrowolski [3]. They succeeded in proving the result for reciprocal polynomials which have no cyclotomic factor, giving an explicit lower bound which they do not believe to be best possible. The proof uses a clever choice of some auxiliary polynomials and work continues to find an even more clever choice which will improve the estimate.

Jeff Vaaler and his student Shey-Jey Chern [9] have recently proved some remarkable results about the distribution function of the values of the Mahler measure of a polynomial in one variable considered as a function of its coefficients. The methods bear a family relationship to those from the geometry of numbers, but here the fundamental shape is not a convex body. He described these results in one of his lectures at the conference at SFU in June. His student Christopher Sinclair [28] has proved analogous results restricted to the set of reciprocal polynomials of a given degree and spoke on this at the BIRS workshop.

3 Many Variable Polynomials

Mahler [22] introduced his measure for polynomials in many variables as a device to provide a simple proof of Gelfond's inequality for the product of polynomials in many variables. This uses the obvious $M(PQ) = M(P)M(Q)$ together with the fact that the coefficients of a polynomial can be bounded from above in terms of $M(P)$. This is still an important tool in transcendence theory.

However, it became apparent in the late 1970's that $m(P)$ has has a much more fundamental significance. In studying the *spectrum* (range) of $m(P(z))$ for polynomials in one variable, Boyd noticed that if $P(z_1, \dots, z_n)$ is a polynomial in many variables, then there are sequences of one variable polynomials $P_{\mathbf{a}} = P(z^{a_1}, \ldots, z^{a_n})$ for which $m(P_{\mathbf{a}}(z))$ converges to $m(P(z_1, \ldots, z_n))$ provided $\mathbf{a} \to \infty$ in a suitable way. Some special cases were proved in [4] and the most general case was proved by Lawton [18]. This shows that in order to study the set of values of $m(P(z))$, it is natural to look at the larger set $\mathbb{L}$ of values of $m(P(z_1, \ldots, z_n))$ as P varies over all polynomials with integer coefficients in an arbitrary number of variables. It was conjectured in [4] that the set $\mathbb{L}$ is closed. A trivial consequence of this would be a proof of Lehmer's conjecture but without any explicit lower bound.

Lind, Schmidt and Ward [20] used Lawton's limit theorem in their proof that $m(P(z_1, \ldots, z_n))$ is the topological entropy of a $\mathbb{Z}^n$ action defined by P. At the workshop, Ward explained how topological entropy is defined, for the benefit of those working in other areas. So the set $\mathbb{L}$ is the set of entropies of these actions and hence occurs in nature. In the final lecture of the workshop, Lind explained what Lehmer's problem should be in the context of compact abelian groups, where the integration over the torus is replaced by integration over the group. He gave an interesting computation showing that the *Lehmer constant* for the group $\mathbb{T} \times (\mathbb{Z}/2\mathbb{Z})$ is strictly less than $m(L(z))$, the presumed minimum for the group $\mathbb{T}$.

4 Explicit Formulas – 2 variables

At the time Boyd was formulating these ideas it was fortunate that Chris Smyth was visiting on a sabbatical and became intrigued by the question of finding explicit formulas for $m(P)$ for polynomials in more than one variable. During that time, he provided a proof of the remarkable formulas

$$m(1 + z_1 + z_2) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}),$$

where $L(s, \chi_{-3})$ is the L-function for the odd quadratic character of conductor 3, and

$$m(1 + z_1 + z_2 + z_3) = \frac{7}{2\pi^2} \zeta(3),$$

where ζ is the Riemann zeta function [30]. Note that in these formulas the measure of a certain 2 variable polynomial is a *di*logarithm and the measure of a certain 3 variable polynomial is a *tri*logarithm, whereas the measures of one variable polynomials are all *uni*logarithms. Since such quantities are widely believed to be algebraically independent, this surely suggests that something deeper is going on.

The recent revival in the interest in $\mathbb{L}$ was due to Christopher Deninger [12], who showed that if $P(z_1, \ldots, z_n)$ does not vanish on the n-torus then $m(P)$ is a Deligne period of the motive associated to the variety defined by $P = 0$. Thus in this case, $m(P)$ is related to Beilinson's higher regulators. In that same paper, Deninger responded to a challenge from Boyd and produced the following remarkable conjecture:

$$m(1 + z_1 + 1/z_1 + z_2 + 1/z_2) = r \frac{15}{4\pi^2} L(E, 2),$$

where E is the elliptic curve of conductor 15 defined by $1 + z_1 + 1/z_1 + z_2 + 1/z_2 = 0$ and r is an (unspecified) rational number. Calculations verify this formula is correct with $r = 1$ to 100 decimal place accuracy. Note that once again we have a dilogarithm of a sort, but here an elliptic dilogarithm.

In a large scale numerical experiment, Boyd [5] computed the measures of polynomials of the form $P_k(z_1, z_2) = k + Q(z_1, z_2)$ where Q is a Laurent polynomial and k is an integer parameter and found families of (conjectural) formulas of this general type relating $m(P)$ to $L(E, 2)$, where $P = 0$ is of genus 1 or 2 and E is a factor of the Jacobian variety of this curve. The conjectured formulas are true to many decimal place accuracy but for the most part have not yet been rigorously proved. Such formulas fall into the general framework of *periods* introduced by Knotsevich and Zagier [16]. If their meta-conjecture is correct, all these formulas should be provable by *elementary calculus*.

Rodriguez-Villegas [25] carried this study a step further and produced formulas based on the theory of modular forms for $m(P_k)$ for many of the families considered in [5]. He thus obtained expressions for the $m(P_k)$ as Kronecker–Eisenstein series and hence very rapidly converging series for the $m(P_k)$ in terms of a modular parameter. In favourable circumstances, this has allowed him to give rigorous proofs for some of the formulas in [5]. He also made explicit the connection between m and the regulator map from the K-group $K_2(E)$ to $\mathbb{R}$.

5 Hyperbolic volume

In a different but related direction, again motivated by Smyth's formula for $m(1 + z_1 + z_2)$, one recognizes that the quantity $3\sqrt{3}L(2, \chi_3)/(4\pi)$ appearing there is, except for the factor π in the denominator, the volume of an equilateral ideal hyperbolic tetrahedron. The set of volumes of hyperbolic 3-manifolds has a well known structure studied by Thurston and Jorgensen. In particular it has a non-zero minimum. An intriguing idea is then that there may be a polynomial P in two variables related to each hyperbolic 3-manifold M in some way so that the relationship $\pi m(P) = \mathrm{vol}(M)$ holds. Perhaps one could go in the other direction from P to M with suitable restrictions on P. For example, what is the manifold related to $1 + z_1 + z_2$?

There are of course many polynomial invariants connected to manifolds. One promising candidate for the relation $\pi m(P) = \mathrm{vol}(M)$ is the so-called A-polynomial $A(x, y)$ defined in [11] for every one-cusped hyperbolic 3-manifold. This does indeed have an intimate connection with volume as explored in [6] and under special circumstances one really does have $\pi m(A) = \mathrm{vol}(M)$, but this is true only under some rather special circumstances. In general all that one can say is that $\pi m(A)$ can be written as the sum of values of the Bloch–Wigner dilogarithm function $\mathcal{D}(z)$ evaluated at certain algebraic numbers (the shapes in certain *pseudo-triangulations* of the manifold) [7]. This result is ultimately a consequence of Schläfli's famous formula for the differential of the volume of a deformed polyhedron.

This connection between $m(A)$ and volume has some bearing on a conjecture of Chinburg that, given an odd primitive quadratic character χ, there is a polynomial $P_\chi(z_1, z_2)$ and a rational number r_χ such that $m(P_\chi) = r_\chi L'(-1, \chi)$, which would generalize Smyth's formula for $P = 1 + z_1 + z_2$. (The relation between $L'(-1, \chi)$ and $L(2, \chi)$ is due to the functional equation for $L(s, \chi)$ and nicely takes care of the factor $d^{3/2}/(4\pi)$ that would otherwise occur. However, it does disguise the fact that the right hand side is a dilogarithm.) At the workshop, Rodriguez-Villegas discussed the proof of a special example of this sort, where one can deduce from K-theory (the theory of the Bloch group) that $m(P(z_1, z_2)) = rL'(-1, \chi)$ for a certain P and χ but with an unspecified rational number that is not computable from a theorem of Borel that is used in the proof. However, in this example one is able to prove that $r = 1/6$ by showing that P is in fact the A-polynomial of a certain hyperbolic manifold constructed by Nathan Dunfield [8].

At the workshop, a number of talks had a bearing on this. Adam Sikora lectured on the A-polynomial and the colored Jones polynomial J_n of a knot, a *quantum invariant* of a knot. There is a conjecture of Kashaev [15] that the volume of a hyperbolic knot complement can be recovered from the limit of the values of the colored Jones polynomial evaluated at roots of unity. This has been proved for a few simple knots by direct calculation. Murakami [23] has made the related conjecture that $2\pi m(J_n)/\log n$ converges to the volume of the knot complement as $n \to \infty$. This has not been verified for any knot. It would be very interesting to prove this result at least for those knots for which $\pi m(A)$ is the volume, by relating the colored Jones polynomial to the A-polynomial.

6 More Hyperbolic geometry

The Chern-Simons invariant can be considered as a complexification of the Volume. Walter Neumann lectured on the connection between the Chern-Simons invariant and a generalization of the Bloch group which requires a complexification of the Bloch–Wigner dilogarithm [24]. One would like to be able to calculate the Chern–Simons invariant by an integration of some polynomial computable from a triangulation of the manifold similar to that defining Mahler's measure. Since $m(P)$ is real-valued, it is apparent that a complex-valued generalization of $m(P)$ would be required for this purpose. Perhaps the Ronkin function from the new theory of *amoebas* provides a clue [32].

Another polynomial invariant of a 3-manifold is the Alexander polynomial. For example, Lehmer's polynomial turns out to be the Alexander polynomial of the pretzel knot $P(-2,3,7)$. At the workshop, Dan Silver and Susan Williams discussed the Alexander polynomial of the complement of a link with n components [26, 27]. This is an n-variable Laurent polynomial. Susan Williams gave a geometric interpretation of the Mahler measure of the Alexander polynomial in terms of the homology of cyclic covers, relating this to the dynamical systems results of Lind, Schmidt and Ward. The fact that Lehmer's polynomial arises in this way suggests that perhaps other small measure polynomials can be constructed in this way, perhaps opening the way to a proof of Lehmer's conjecture.

Remarkably, many of the polynomials identified in [4] as being small limit points of $\mathbb{L}$ turn out to arise as Alexander polynomials of simple links. It was conjectured in [4] that Smyth's numbers $m(1+z_1+z_2)$ and $m(1+z_1+z_2+z_3)$ are the minimal elements of the 2nd and 3rd derived sets of $\mathbb{L}$, (the n-th derived set is the set of limit points of the $(n-1)$-st derived set). It would be natural to speculate that $m(1+z_1+\ldots+z_n)$ is the smallest element of the n-th derived set for $n \geq 2$, but Silver and Williams cast some doubt on this with their example of the Alexander polynomial of a minimally twisted 4-link which is $1+z_1+z_2+z_3+z_4-z_1^{-1}z_2^{-1}z_3z_4$, which seems to have slightly smaller measure than that of $1+z_1+z_2+z_3+z_4$. The difficulty in numerically computing the Mahler measure of 4-variable polynomials makes this difficult to verify numerically.

7 Explicit Formulas – many variable polynomials

Apart from Smyth's formula for $m(1+z_1+z_2+z_3)$, there were for many years no explicit formulas for the Mahler measure of polynomials in more than 2 variables. The situation has changed drastically in recent years. The first formula of this type was again proved by Smyth, [31], another completely different 3 variable polynomial whose Mahler measure is a rational multiple of $\zeta(3)/\pi^2$. He spoke on this and other examples at the workshop and provided the example

$$(z_1+1/z_1)\cdots(z_{n-2}+1/z_{n-2})+2^{n-3}(z_{n-1}+z_n),$$

of a polynomial whose Mahler measure can be expressed in terms of an n-logarithm. The workshop provided a forum for further progress along these lines.

In his paper [25] devoted to modular formulas for families of elliptic curves, Rodriguez-Villegas mentioned that the same method would apply to certain modular K3 surfaces, for example that defined by the polynomial $(z_1+1/z_1)(z_2+1/z_2)(z_3+1/z_3)+k$. Marie José Bertin [1] has followed up this suggestion and worked out all the details of two examples different from this. Using results of of Verill, she was able to compute Kronecker–Eisenstein series for the Mahler measure for polynomials

$$P_k = z_1+1/z_1+z_2+1/z_2+z_3+1/z_3-k.$$

For certain values of k the explicit formula can be expressed in terms of the value at $s=3$ of certain Hecke L-series and modular forms. She spoke on this work at the BIRS workshop and also the SFU conference.

Matilde Lalín described a new approach to evaluating the Mahler measure of some families of polynomials in many variables in terms of polylogarithms [17]. A highlight of this is the formula

$$\pi^4 m((1+z_1)(1+z_2)(1+z_3)+(1-z_1)(1-z_2)(z_4+z_5)) = 93\zeta(5).$$

The connection between the Mahler measure of some 2-variable polynomials and the volume of hyperbolic 3-manifolds leads one to wonder whether this can be extended to higher dimensions. Ruth Kellerhals gave an instructive introduction to hyperbolic volume in higher dimensions explaining how the parity of the dimension plays an important role. The fact that the volume of polyhedra in hyperbolic 5-space can be expressed in terms of polylogarithms of order ≤ 3 suggests a possible connection with $m(P)$ for polynomials in 3 variables. She presented, for example, an orthoscheme with 5-dimensional volume $5\zeta(3)/4608$. Is there a polynomial $P(z_1,z_2,z_3)$ constructible from this orthoscheme for which $\pi^2 m(P(z_1,z_2,z_3)) = 5\zeta(3)/4608$?

8 Mahler measure and motivic cohomology

In an inspiring lecture, Vincent Maillot went beyond Deninger's [12] framework to provide an explanation of many of the formulas presented by other speakers in terms of the cohomology of the varieties defined by the polynomials. His approach is particularly successful in the case of non-reciprocal polynomials and explains the difference between formulas in which higher L-functions such as $L(E, s)$ appear and formulas in which only only polylogarithms appear. The point is that the Mahler measure only detects the intersection of the variety $P = 0$ with the real torus, and hence the quantities that appear in the right hand side of the formulas should be related to the variety that is the intersection of $P(z_1, \ldots, z_n) = 0$ and $P(1/z_1, \ldots, 1/z_n) = 0$, (an observation that Maillot attributed to Darboux (1875)).

After the workshop, Rodriguez-Villegas pondered what this would mean for the simple polynomials $1 + z_1 + \ldots + z_n$ and came up with the remarkable conjectures that

$$m(1 + z_1 + \ldots + z_4) = L'(f_4, -1),$$

where f_4 is a normalized cusp form of weight 3 and conductor 15, and

$$m(1 + z_1 + \ldots + z_5) = 4L'(f_5, -1),$$

where f_5 is a normalized cusp form of weight 4 and conductor 6. (Note that the functional equation relates these to the values $L(f_4, 4)$ and $L(f_5, 5)$, as one would expect). He verified these results numerically to 28 decimal places. It is a non-trivial problem to compute these Mahler measures numerically. Fortunately, Rodriguez-Villegas, Tornaria and Vaaler had just recently developed a series for $m(1 + z_1 + \ldots + z_n)$ which gave sufficient numerical accuracy for the purpose. (This series was the topic of one of Vaaler's lectures at the SFU conference). These formulas do not seem to extend to $n \geq 6$ because the corresponding space of cusp forms has dimension greater than 1.

In a different direction, Rodriguez-Villegas and Boyd made a search for families of non-reciprocal polynomials in 3 variables for which the intersection of $P(z_1, z_2, z_3) = 0$ and $P(1/z_1, 1/z_2, 1/z_3) = 0$ is an elliptic curve E, in which case it is conceivable that $m(P)$ could be expressible in terms of $L(E, 3)$, that is, as a rational multiple of $L'(E, -1) = (N^2/(8\pi^4))L(E, 3)$, where N is the conductor of E. We found 5 examples where to 40 decimal place accuracy, such formulas seem to be true. For example,

$$m((z_1 + 1)^2 + z_2 + z_3) = L'(E_{24}, -1),$$

where E_{24} denotes an elliptic curve of conductor 24. None of these have yet been rigorously proved.

In one degenerate case, the intersection variety was not an elliptic curve but rather a rational curve, suggesting perhaps a formula in terms of $\zeta(3)$, and leading to the conjecture

$$m((1 + z_1) + (1 - z_1)(z_2 + z_3)) = \frac{28}{5}\frac{\zeta(3)}{\pi^2}.$$

In spite of the intriguing resemblance to some of Lalín's formulas, it appears not to be accessible by her method. Boyd presented this at a lecture at the SFU conference attended by John Condon, a student of Rodriguez-Villegas. Remarkably, a few months later, Condon had found a classical but extremely ingenious (and *long*) proof of this identity which forms the basis for his recent Ph.D. thesis [10].

9 Conclusions

In the space of 4 days at BIRS, many ideas were exchanged and collaborations formed. The few examples presented above should be evidence of the value of this sort of workshop. Continuing in the spirit of international collaboration, Marie José Bertin and Vincent Maillot were inspired to organize a similar gathering at CIRM in Luminy in May of 2005. It is to be hoped that the research inspired by such workshops will continue to unravel the mysteries of Mahler's marvelous measure.

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