Geometrical Analysis in One and Several Complex Variables

Joseph A. Cima (University of North Carolina at Chapel Hill),
Ian Graham (University of Toronto),
Kang-Tae Kim (Pohang University of Science and Technology, Korea),
Steven G. Krantz (Washington University, St. Louis, Mo.)

May 22 - June 5, 2004

Of the topics we initially proposed for study, we spent most of our time considering holomorphic mappings in Banach and Hilbert spaces. Following some early developments dating back to about 1970, there has been a surge of recent activity in infinite-dimensional holomorphy. Of particular interest to us are problems which combine geometrical methods of one and several complex variables with operator theory and functional analysis. For example, the books [6] and [4] are written in this vein, and the papers [11] and [12] show that significant portions of the theory of normal families can be extended to separable Banach or Hilbert spaces. It should also be mentioned that some aspects of univalent function theory (the study of classes of conformal mappings of the unit disc) have been generalized to infinite dimensions (see the recent book [7] and the references there).

The Schwarz lemma is one of the most basic mapping results in one complex variable. Perhaps the most important part of this result is the uniqueness statement: if $f : D \to D$ is a holomorphic mapping from the unit disc D to itself, $f(0) = 0$, and $|f'(0)| = 1$, then f is a rotation. Generalizations to bounded domains in $\mathbb{C}^n$ have been considered by various authors, beginning with H. Cartan around 1930 (see [13], [18]). The result closest in spirit to our work in infinite dimensions is a theorem which states that a holomorphic self-map of a bounded domain in $\mathbb{C}^n$ with a fixed point p at which the eigenvalues of df_p have modulus 1 must be an automorphism of the domain. Using iteration techniques and the Cauchy estimates, one proves that the Jordan canonical form of df_p must be diagonal. Separate arguments are then needed to deal with the case of rational and irrational eigenvalues. Convergence questions for a suitable subsequence of the iterates of the mapping f must then be considered.

During the course of our Research in Teams program we studied versions of the rigidity result in the Schwarz lemma for bounded domains in a separable Hilbert space H. Bounds for the differential of the mapping are known; see [6] or [4]. Also, Harris [9] showed that if the domain in question is the unit ball and df_0 is an invertible isometry then $f = df_0$. We obtained the following theorem:

Theorem. *Let $\Omega \subseteq \mathcal{H}$ be a bounded convex domain. Fix a point $p \in \Omega$. Let $f : \Omega \to \Omega$ be a holomorphic mapping such that*
(a) $f(p) = p$;
(b) the differential df_p is triangularizable;
(c) $\sigma(df_p) \subseteq S^1$.
Then f is a biholomorphic mapping.

Convergence questions in infinite dimensions both for the powers of the elements of the point spectrum and for subsequences of the iterates of the mapping are quite subtle, and new techniques (whose genesis lies in recent work of Kim and Krantz [12]) must be introduced. We believe that these techniques will have further applications, and plan to explore them.

We also considered a basic open problem in infinite dimensional holomorphy: to show that the inverse of an injective holomorphic mapping from one bounded domain onto another in a separable Hilbert space H must be holomorphic. Examples dues to Suffridge [17] and Heath and Suffridge [8] show that some pathological behaviour is possible in non-separable Banach spaces. Some sufficient conditions are known for holomorphy of the inverse mapping [10]; but the general case remains open.

Members of the group also gave informal talks on topics of mutual interest: a generalization to infinite dimensions of a characterization of the unit ball by its automorphism group [11], the theory of Loewner chains in several variables [7], boundary behaviour of biholomorphic mappings between convex domains [15], and the approximation of vector fields in $\mathbb{C}^n$ by complete vector fields [2], [5]. We formulated a number of problems for further study in both finite and infinite dimensions, and believe that considering such results simultaneously may lead to a useful cross-fertilization of ideas:

1. If Ω_1 and Ω_2 are bounded domains in a separable Hilbert space H and $f : \Omega_1 \to \Omega_2$ is an injective holomorphic mapping of Ω_1 onto Ω_2, then is f^{-1} holomorphic ?

2. Is there a version of the Hopf lemma for plurisubharmonic functions on bounded domains in infinite dimensions ?

3. If B is the unit ball in a separable Hilbert space H, and if $f : B \to \Omega$ is a biholomorphic mapping of B onto a bounded convex domain, then does f^{-1} extend continuously to $\overline{\Omega}$?

4. With assumptions as in Problem 3, let U be a nonisotropic Koranyi ball in ∂B and suppose that $f(U) \subset \partial\Omega$. With respect to the appropriate Hausdorff measure μ, does there exist a constant $C > 0$ such that $\mu(U) \geq C\mu(f(U))$?

5. Suppose that $f : B \subset H \to H$ is a holomorphic mapping from the unit ball in a separable Hilbert space H into H with open image and that X is an analytic subvariety of B. When is $f(X)$ a variety ?

The group was very enthusiastic about the "Research in teams" format and felt that it provided an opportunity to carry out joint work which it would have been very difficult to accomplish without a period of intense concentration with all of us present. The setting, facilities, and staff were wonderful.

References

[1] E. Andersen, Volume-preserving automorphisms of $\mathbb{C}^n$, *Complex Variables Theory Appl.* **14**(1990), 223-235.

[2] E. Andersen and L. Lempert, On the group of holomorphic automorphisms of $\mathbb{C}^n$, *Invent. Math.* **110** (1992) 371-388.

[3] J. B. Conway, *A Course in Operator Theory*, American Mathematical Society, Providence, RI, 2000.

[4] S. Dineen, *The Schwarz lemma*, The Clarendon Press, Oxford University Press, Osford, 1989.

[5] F. Forstneric and J.-P. Rosay, Approximation of biholomorphic mappings by automorphisms of $\mathbb{C}^n$, *Invent. Math.* **112**(1993), 323-349.

[6] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.

[7] I. Graham and G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker Inc., New York, 2003.

[8] L. F. Heath and T. J. Suffridge, Starlike, convex, close-to-convex, spirallike, and Φ-like maps in a commutative Banach algebra with identity, *Trans. Amer. Math. Soc.* **250** (1979), 195-212.

[9] L. Harris, Schwarz's lemma in normed linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **62** (1969), 1014-1017.

[10] L. Harris and J.-P. Vigué, A metric condition for equivalence of domains, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8)**67** (1979), 402-403 (1980).

[11] K.-T. Kim and S. G. Krantz, Characterization of the Hilbert ball by its automorphism group,*Trans. Amer. Math. Soc.* **354** (2002), 2797-2818.

[12] K.-T. Kim and S. G. Krantz, Normal families of holomorphic functions and mappings on a Banach space, *Expo. Math.* **21**(2003), 193-218.

[13] S. G. Krantz, *Function Theory of Several Complex Variables*, American Mathematical Society-Chelsea, Providence, RI, 2001.

[14] L. Lempert, The Dolbeault complex in infinite dimensions, *J. Amer. Math. Soc.* **11**(1998), 485-520.

[15] P. Mercer, A general Hopf lemma and proper holomorphic mappings between convex domains in $\mathbb{C}^n$, *Proc. Amer. Math. Soc.***119**(1993), 573-578.

[16] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland, Amsterdam and New York, 1986.

[17] T. J. Suffridge, A holomorphic function having a discontinuous inverse, *Proc. Amer. Math. Soc.* **31** (1972), 629-630.

[18] H. Wu, Normal families of holomorphic mappings, *Acta Math.* **119** (1967), 193-223.