Threshold Dynamics, with Applications to Image Processing and Computer Vision

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1 Overview

Many important models of image processing and computer vision involve curvature dependent functionals. In segmentation, the Kass, Witkin, Terzopoulos method [8] of snakes originally calls for minimizing an active contour energy that involves integrating curvature squared along the curve. In segmentation with depth, the 2.1D Sketch model of Nitzberg, Mumford, and Shiota [10] involves the integral of a function of the curvature along the free discontinuity set. In the image inpainting application of Bertalmio, Sapiro, Caselles, and Ballester [2], Chan, Shen, and Kang [4] proposed generating the missing image information in the inpainting domain by minimizing Eulers elastica energy along image isophotes. Finally, in computer graphics, curvature dependent functionals have been proposed for surface denoising and smoothing.

One of the most successful techniques for minimizing variational models in image processing that involve unknown contours has been the level set method of Osher and Sethian [11]. Once the energies in question are written in terms of a level set representation for the unknown curves, optimality conditions (Euler-Lagrange equations) can be obtained in terms of the level set function, and a gradient descent procedure can be carried out. This involves solving non-linear, degenerate, fourth order parabolic PDE and can be computationally very expensive.

The idea behind threshold dynamics is to alternate the solution of a linear parabolic PDE, such as the heat equation, and thresholding to generate geometric motion of interfaces. The original idea is due to Merriman, Bence, and Osher [9], who proposed a technique for approximating the motion by mean curvature of an interface by alternating the solution of the heat equation (i.e. convolution by the Gaussian kernel) and thresholding. Convergence was proved by Evans [6], and by Barles and Georgelin [1]. There have been various generalizations of their method to other curvature dependent velocities, and a highly accurate version was developed by Ruuth in [12]; these offer an alternative to level set based techniques that require the solution of nonlinear second order equations.

On the novel side, this workshop proposes further investigation of non-standard thresholding that involves the geometry of the solution instead of only the pointwise values of the solution. To our knowledge, there has not been any existing working algorithms or analysis if the thresholding procedure is replaced, for example, by redistancing (reshaping the solution into the distance function of the shape it represents). This may be an important strategy to improve upon the existing algorithms and to solve more general problems.

Finally, there is a close link between variational approaches for image processing and more general inverse problems. The threshold dynamics approach described above can potentially be a useful tool for building up efficient solvers for inverse problems that require certain types geometric regularizations. Furthermore, in problems involving optimal shapes, there are ongoing discussions on suitable representations and the efficiency of their respective numerical algorithms. Typically these criticisms consist of algorithms being too costly or unable to obtain globally optimal shapes. Threshold dynamics offers a different way of building and analyzing the problems that may potentially lead to robust and efficient inverse problem solvers.

Motivated by the Merriman, Bence, Osher scheme, recently Esedoglu and Tsai proposed a technique for minimizing the piecewise constant versions of the Mumford-Shah segmentation functional that were introduced by Chan and Vese [3]. This new algorithm involves alternating the solution of a linear parabolic PDE and simple thresholding. It leads to a very efficient minimization of Chan and Veses Mumford-Shah energies. Based on this successful application of threshold dynamics ideas to an important image processing problem, we believe it is prudent to ask what other image processing and computer vision problems might benefit from this approach.

An important class of models in image processing and computer vision involve curvature dependent functionals. The minimization of these functionals involve the solution of fourth order geometric pdes. The numerical solution of such pdes with standard level set methods can be very costly.

2 Overview of the Research in Teams Event

Recently, Grzibovskis and Heintz [7] have found a threshold dynamics that approximates gradient flow for an important curvature dependent functional known as the Willmore energy. This energy consists of the integral of the square of a surface's mean curvature over that surface. Furthermore, it constitutes an essential part certain variational image models for segmentation with depth, disocclusion, and image inpainting. As a first step in bringing threshold dynamics to bear upon higher order models of image processing and computer vision, we recently generalized the work of Grzibovskis and Heintz to the reconstruction of an occluded binary image [5].

In this program we continued our earlier work by considering replacements to the characteristic function representations used in traditional threshold dynamics. We anticipate that by introducing new techniques to threshold dynamics we can broaden the appeal of the methods while simultaneously achieving improved accuracy to high order flows.

3 Developments and Progress

The first issue addressed by the BIRS event was how to evolve a curve with a normal velocity equal to the local curvature of the curve, according to a combination of convolution and redistancing[1]. “Motion by curvature” is precisely the motion that arises from a surface tension driven interface motion, and is central to many models in image processing as a method to regularize or smooth reconstructed images. For the prototype case of a circle, a consistent combination of convolution and redistancing was obtained. This result lead to a number of questions:

- Should the convolution and redistancing steps be combined into one? Will this give faster iterations and smoother solutions, thereby giving better results?
- Can we carry out extrapolation in the timestep size to obtain higher order methods, thereby obtaining more accurate approximations of mean curvature motion?
- Can combinations of different convolutions be considered which, taken together, give higher order convergence?
- Can combinations of different convolutions achieve more general high order motions, specifically the Willmore flow?

The answers we obtained for these questions follow.

3.1 Combining Convolution and Reinitialization Steps

A central question is whether to carry out convolution and reinitialization together as separate terms in the same PDE, or to carry out each separately in a traditional threshold dynamics fashion. It was found that by combining both we were able to obtain small errors for basic curvature flow, but that the combined approach was inferior when higher order accurate methods are sought. This further suggests that the combined approach would not be well-suited for obtaining motions for high order PDEs. A focus was therefore made on separate reinitialization and convolution steps.

3.2 Richardson Extrapolation and Methods for Canceling Error Terms

While consistent results were obtained with the basic approach, the results were low order accurate, specifically first order in the timestep size. Richardson extrapolation gave much better accuracy, as did combining different convolutions to cancel the dominant error terms. The former approach was extremely simple, however, the latter had the advantage of allowing clear extensions to the evolution by Willmore flow and flow by surface diffusion.

3.3 Willmore and Surface Diffusion Flows

Carrying out convolutions with multiple kernels of different widths, allows the construction of combinations which give Willmore flow and surface diffusion flow. Methods for both of these motion laws were obtained and found to be consistent for the prototype case of a circle.

[1]Redistancing is the process of determining a signed distance function to a curve from an earlier approximation.

4 Conclusions and Outstanding Issues

We have developed new algorithms based on combinations of convolution and redistancing. While both of these components are standard numerical techniques, the combinations considered are new. We obtained a variety of interesting motion laws including curvature flow (accurate to second order), Willmore flow and surface diffusion flow. The algorithms have the advantages that

- They use combinations of standard algorithms to obtain complicated flows.
- Stability of the methods is very good; indeed often better than existing algorithms. Thus, the approach holds strong promise for computing steady state flows in applications such as image processing.

Outstanding issues are primarily related to the speed of the algorithms. Traditional redistancing is slow to obtain accurate solutions, and the number of reinitialization steps was comparable to the number of grid points in a coordinate direction. We are interested in improving on this component in future work.

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