Schrödinger Evolution Equations

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1 Introduction

The (linear) Schrödinger wave equation, first formulated by Erwin Schrödinger in 1925, provides a description of the time evolution of the wavefunction of a nonrelativistic quantum particle; for a free particle of mass m, it reads

$$i\hbar\frac{\partial}{\partial t}u + \frac{\hbar^2}{2m}\Delta u = 0.$$

From the outset, mathematicians and physicists have been concerned with the ways in which solutions to this equation can be associated to classical particle motion, and the ways in which they cannot: the classical dynamical behavior of particles is intermixed with *dispersive spreading* and interference phenomena in quantum theory.

More recently, *nonlinear* Schrödinger equations have come to play an essential role in the study of many physical problems. Nearly monochromatic waves with slowly varying amplitude occur frequently in science and technology. Second order expansions of physical models of wave phenomena around such waves lead naturally to the cubic nonlinear Schrödinger equation. Thus, the nonlinear Schrödinger equation is a *canonical wave model* since it emerges ubiquitously in the study of waves. Nonlinear Schrödinger (NLS) equations appear in such diverse fields as nonlinear optics, superconductivity, oceanography, and quantum field theory. The main themes of research discussed at this workshop concern the Cauchy or initial value problem for Schrödinger equations. Theoretical and applied aspects of nonlinear Schrödinger equations are nicely surveyed in the textbooks [4], [17].

Nonlinear Schrödinger evolutions involve a dynamical balance between *linear dispersive spreading* of the wave and *nonlinear self-interaction* of the wave. Generalizations of the physically relevant equations with nonlinear and dispersive parameters have been introduced to probe the interplay between these effects. For example, the semilinear initial value problem

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^{p-1}u \\ \quad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

may be viewed as a nonlinear generalization of the cubic problem on $\mathbb{R}^d$ corresponding to $p = 3$. The Laplacian term Δu generates the dispersion in this evolution equation. The cubic nonlinearity represents self-interaction of the wave. The choice of sign corresponds to (-) *focusing* and *defocusing* (+) nonlinearities. Some overlapping themes of the research discussed at the workshop may be outlined in the setting of (1):

- **Optimal Well-posedness**
  What are the minimal regularity assumptions on the initial data u_0 for which the initial value problem

(1) may be solved locally in time? What happens to rougher initial data? Are the local-in-time solutions in fact global-in-time? Spaces with norms which are invariant under the scaling symmetry of solutions play a crucial role in anwers to these questions.

- **Nonlinear dispersive systems**
Are the recent methods for solving (1) robust enough to also apply to systems of equations? Nonlinear dispersive systems, such as the Zakharov system and the Maxwell-Schrödinger system, more accurately model physical phemomena than the closely related cubic nonlinear Schrödinger equation. In certain regimes of physical parameters, some nonlinear dispersive systems are expected to be well-approximated by simpler problems. How do solutions of nonlinear dispersive systems behave as physical parameters are pushed toward extreme values?

- **Long-time behavior**
What happens? This is the main question to be addressed when considering an initial value problem. Fantastic progress over the past two decades has led to a nearly complete well-posedness theory for the equations in (1). Beyond existence and uniqueness, the issue is to provide qualitative descriptions of the evolution. For defocusing problems on $\mathbb{R}^d$, the expected behavior is similar to that expressed by the linear evolution. On compact domains, persisting nonlinear interactions are expected to generate oscillations on smaller and smaller scales. For focusing problems, the expected behavior includes the emergence of nonlinear coherent structures such as solitons and finite time explosions.

- **Linear equations**
Among the essential tools for recent progress into nonlinear Schrödinger evolutions are the Strichartz and dispersive smoothing estimates for linear Schrödinger equation. These and related estimates have thus been subjects of geat interest both on their own and for the light they shed on nonlinear phenomena. Under what conditions do the fundamental linear Schrödinger estimates extend to the setting of variable coefficients?

Talks at the workshop described significant progress in each of these four directions.

2 Optimal Well-posedness

When considering an initial value problem, such as (1), some basic questions arise: Does a solution exist? For which initial data does a solution exist? If a solution exists, is it unique? For how long does the solution last? How does the solution depend upon the initial data? Do smoothness properties of the intial data persist during the evolution? Answers to these and related questions are provided by the well-posedness theory for the initial value problem. For certain classes of initial value problems, such as (1), striking progress over the past two decades has culminated into a satisfactory local-in-time well-posedness theory. Over the past decade, methods for showing ill-posedness have emerged which have revealed the optimality of various known local-in-time well-posedness results and new ideas for establishing global-in-time well-posedness have been developed. Some of the talks at the workshop contributed in these directions.

Burq described a simplified proof, obtained in joint work with Gerard and Ibrahim, of ill-posedness results due to Lebeau [13] and Christ-Colliander-Tao [3]. An anisotropic scaling of a rather explicit solution provides a clear view into one low regularity mechanism causing havoc for the Cauchy problem. For cubic problems, Carles described an inspired geometrical optics based approach to proving similar results.

Gérard spoke about the cubic NLS in four dimensions, in various settings. See also the survey [9]. On S^4, together with Burq and Tzvetkov, he has obtained rather complete results on well-posedness: the equation is well-posed on H^s for $s > 1$, ill-posed for $s < 1$, and ill-posed in H^1, with the flow map failing to be continuous, even on small data. By contrast, in more recent work with Pierfelice, Gérard has shown that in the H^1 case, slightly relaxing the nonlinearity results in a qualitative change: if we replace $|u|^2u$ by certain homogeneous quadratic polynomials $q(u, \overline{u})$, then there is global well-posedness in H^1 provided that the *ODE* $i\partial_t u = q(u)$ does not blow up.

When the conserved quantities imply an apriori H^1 upper bound, standard local well-posedness results for H^1-subcritical initial value problems may be iterated to obtain global well-posedness for H^1 initial data.

Whether initial data of lower regularity for which local-in-time well-posedness holds evolves globally in time has been a topic of intense study over the past eight years.

In the case of the initial value problem for the L^2-critical equation

$$iu_t + \Delta u - |u|^{4/d} u = 0,$$

one can hope for global well-posedness and scattering all the way down to L^2. Staffilani announced joint work [16] with De-Silva, Pavlovic, and Tzirakis, in which the periodic initial value problem is studied in one dimension with data in H^s, and global well-posedness is shown for $s > 4/9$. This talk led to a very interactive question period, with discussion among Staffilani, Planchon, Burq, and Gérard on the relationship between bilinear estimates and local well-posedness.

For the defocusing L^2-critical NLS in four dimensions with radial data,

$$iu_t + \Delta u = |u|u,$$

Visan announced a breakthrough proof, obtained in joint with with Tao, that demonstrates global well-posedness and scattering. While the local theory dates back to the work of Cazenave-Weissler, the global theory is not yet well understood. Visan also described and compared this work with results she, her collaborators and others have recently obtained (see [21] and references therein) in the $\dot{H}^1$-critical case,

$$iu_t + \Delta u = |u|^{\frac{4}{d-2}} u$$

in dimension $d \geq 3$, for arbitrary data. The key ingredients here are an induction on energy strategy due to Bourgain [2] and a new interaction Morawetz inequality [5], [6]. The developments announced by Visan forecast profound improvements in our understanding of the L^2-critical nonlinear Schrödinger equations.

3 Nonlinear dispersive systems

The methods developed for scalar Cauchy problems like (1) have also been applied to more complicated, and more physically accurate, nonlinear dispersive systems. Dispersive systems lack certain simplifying features, such as scaling invariance, enjoyed by (1). Adaptations and innovations of the scalar techniques have recently been under investigation. Progress in this direction has demonstrated that the key insights are robust and extend to the setting of nonlinear dispersive systems.

Bejenaru recently proved [1] a global well-posedness result for the Schrödinger map problem posed on $\mathbb{R}^d, d \geq 3$, for small initial data in a scaling invariant Besov norm. A similar result has recently been obtained by Ionescu-Kenig. The proof relies upon structural properties of the nonlinearity and delicate bilinear estimates in (Besov variants of) $X_{s,b}$ spaces. This result is the Schrödinger analog of a celebrated result [19] of Tataru on wave maps.

Grillakis' talk discussed the evolution of a curve by binormal curvature flow and its relationship, under the Hasimoto transformation, to the cubic nonlinear Schrödinger equation on $\mathbb{R}$. He then derived a generalization to a curvature driven surface evolution. Establishing well-posedness for the surface evolution problem appears to be a difficult problem.

Ibrahim discussed recent work with Biryuk and Craig towards establishing that various approximation schemes converge to weak solutions of the Navier-Stokes system. A lively discussion following the talk hinted at the possibility that a decay property required for improvements to the convergence results is linked with spectral cluster estimates like those discussed by Smith.

In a rather different setting, Koch discussed joint work with Saut concerning the local smoothing and Strichartz estimates for a very wide class of third order dispersive equations in two dimensions (which include the linear parts of several equations describing surface gravity waves at various approximations). He obtains local smoothing and Strichartz estimates with a derivative gain, for the generic cases of these third order equations. This work demonstrates that the well-posedness theory based on dispersive estimates is robust and applies to a wide variety of model equations appearing the the applied mathematics and physics literature.

Nakamura described joint work [14] with Wada which established a low regularity local well-posedness result for the Maxwell-Schrödinger system in the Coulomb gauge. The result is based on ideas stemming from work of Koch and Tzvetkov. Estimates of energy type were also discussed which imply that the local solutions in fact extend globally in time.

Nakanishi's talk described joint work with Masmoudi which established that finite energy solutions of the Zakharov system converge to solutions of the focusing cubic nonlinear Schrödinger equation in the subsonic limit. Earlier work on this limit required regularity assumptions beyond finite energy. The Zakharov system models plasma in certain physical regimes. A sound speed parameter in the system moves toward infinity as the mass of the ions converges to the mass of the electrons. This result about the subsonic limit opens up the possibility that fine blowup properties of NLS and the Zakharov system can now be compared in the setting of finite energy solutons.

Tzirakis described a new method, developed with Colliander and Holmer [20], for globalizing certain nonlinear dispersive systems. The method exploits L^2 conservation on one of the system components and an almost conservation property for the other component. The method has been applied to the Zakharov system on $\mathbb{R}$ and to the Klein-Gordon-Schrödinger system on $\mathbb{R}^3$ to prove that the best known local-in-time solutions extend globally in time.

4 Long-time behavior

Aspects of the maximal-in-time behavior of solutions of nonlinear Schrödinger evolution equations were reported upon at the workshop. Besides the scattering and long-time existence results described previously, new results concerning the asymptotic behavior of soliton solutions and periodic-in-space solutions were discussed.

Holmer described work [12], with Marzuola and Zworski, which explains the behavior of fast solitons in the one dimensional cubic nonlinear Schrödinger equation interacting with a repulsive Dirac-mass singularity. Slow solitons will spend more time in the interaction region than fast solitons when passing through the Dirac singularity. Intuitively, slow solitons will have more back-reflected mass than fast solitons. Also, extremely fast solitons should have barely any back-reflected mass. The result described validates this intuition by showing that, in the high speed limit, the bulk of the soliton mass moves past the potential. However, the upper bound on the size of the back-reflected mass is larger than conjectured in the high speed limit.

In the other direction, Zhou's talk considered the behavior of a soliton trapped by a potential. Zhou discussed recent work [23] with Sigal which establishes asymptotic stability results for soliton solutions of NLS in the presence of an external potential. Under certain assumptions, their result shows that trapped solitons oscillate in a potential well and slowly shed excess energy to spatial infinity, eventually relaxing to an asymptotic equilibrium inside the well. The proof involves a clever application of normal forms reduction to rigorize intuition related to the Fermi golden rule.

Another new result giving insight into the qualitative behavior of global-in-time solutions was discussed by Tao. In joint work, Colliander-Keel-Staffilani-Takaoka-Tao have obtained results on cubic defocusing NLS on the two-torus which are a step toward the "weak turbulence conjecture," describing the movement of energy from low to high Fourier modes. In particular, this group has shown that for all $s > 0$, $\epsilon > 0$, $M \gg 1$ there is $u_0 \in \mathcal{C}^\infty(S^1 \times S^1)$ and $T > 0$ such that $\|u_0\|_s \leq \epsilon$ and $\|u(T)\|_s \geq M$, where u is the solution with initial data u_0. The idea is to convert to an immense system of ODE by expanding onto resonant Fourier modes, and then to make a rather delicate combinatorial construction.

5 Linear equations

A major recent thrust of work on linear Schrödinger equations has been to understand precisely the regularity of solutions and to obtain associated estimates, usually dispersive smoothing estimates and Strichartz estimates, that are of use in tackling nonlinear problems. For instance, it has been known since the work[1] of Constantin and Saut, Sjölin and Vega, that if $u(t)$ is a solution to the linear Schrödinger equation on $\mathbb{R}^n$, we have the "(local) dispersive smoothing estimate"

$$\int \int_{|x|<R} |\Delta_x^{1/4} u(t,x)|^2 \, dx \, dt \leq C_R \|u_0\|_{L^2}^2. \tag{2}$$

[1] The background results briefly described here are surveyed more completely in the textbook [4] of Cazenave.

In other words, we locally find that u is half a derivative smoother than its initial data, when averaged in time. Further refinements of this estimate are possible that are global in space (with a weight) and global in time. A microlocal version of it was first given, for variable coefficient equations, by Craig-Kappeler-Strauss[7].

Perhaps the key estimate on the linear equation for proving well-posedness is the *Strichartz* estimate. Again for the linear equation on $\mathbb{R}^n$, Strichartz deduced estimates of the form

$$\|u\|_{L^q_t L^q_x([0,1]\times M)} \leq C\|u(0)\|_{L^2(M)},$$

for an appropriate exponent q. Later, these estimates were extended to include norms of mixed Lebesgue exponent of the form

$$\|u\|_{L^q_t L^r_x([0,1]\times M)} \leq C\|u(0)\|_{L^2(M)},$$

for appropriate exponents q, r.

Much recent work has gone into generalizing the dispersive smoothing and Strichartz estimates to apply to variable coefficient operators, involving (possibly singular) potentials, and non-Euclidean metrics. One major tool for obtaining such results is *commutator estimates*, perhaps best considered as microlocal energy estimates, i.e., energy estimates localized in phase space. Another tack is to try to find a parametrix for the Schrödinger propagator e^{-itH} directly, and then obtain the estimates from its explicit form.

Tataru [18] presented new results on Strichartz estimates mostly following the latter approach. These estimates, generalizing the local in time variable coefficient estimates on asymptotically conic spaces obtained by Robbiano-Zuily [15] and Hassell-Tao-Wunsch [11], are global in time, and require only extremely weak estimates on the metric, much weaker than the usual "short-range" assumption. They also require very little differentiability. It should be noted that all of these constructions (and indeed, the validity of the usual Strichartz and dispersive smoothing estimates) rest on a crucial geometric assumption: the metric must be *non-trapping*, i.e. geodesics must approach spatial infinity as time goes to infinity.

One upshot of the parametrix construction of Hassell-Wunsch [10], described in its latest refinement in Hassell's talk, is a propagation theorem, describing the formation of singularities of $e^{-itH}u_0$ in terms of oscillatory behavior of u_0. Nakamura presented a different approach to such propagation results, giving a characterization of WFu_0 in terms of u_0 that rests on scattering-theoretic methods, furthermore yielding a generalization to long-range metrics. Doi also discussed new propagation theorems, yielding a very precise description of WF$e^{-itH}u_0$ in the case $H = (1/2)\Delta + V + W$ with V a harmonic oscillator potential and W a perturbation term. There are three different regimes, depending on the size of W: if W is of order $|x|^\rho$, and $\rho < 1$, then the perturbation is irrelevant to the formation of wavefront set. If $\rho = 1$ then there is a finite-speed correction to the propagation of singularities for the harmonic oscillator (see [8]). If $1 < \rho < 2$ then there is an infinite-speed correction. This generalizes a classic result of Zelditch [22], who took $\rho = 0$.

Robbiano reported on results generalizing the weighted, global in space dispersive smoothing estimates to a very broad class of operators, which most notably allows for potentials of any order (provided that the resulting operator has a self-adjoint extension).

6 New research directions

This workshop revealed new insights into the dynamical balance between nonlinear self-interaction and dispersive spreading of waves. It is of course impossible to predict precisely future research developments. However, recent significant developments and transparent gaps in the theory suggest two emergent research themes.

1. **Scattering at critical regularity.** Bourgain's induction on energy strategy [2] and subsequent developments [6], [21] in the energy critical semilinear Schrödinger problem forecast profound developments for the global-in-time theory of the defocusing problem (1) in the energy subcritical case. An optimal (at least in L^2-based Sobolev spaces) local well-posedness theory for (1) is now in place. The extant global well-posedness theory has relied upon a priori estimates inferred from energy conservation. Global results based on energy conservation (or almost conservation) have required regularity assumptions beyond what is necessary for local well-posedness. The talk by Visan hints that the induction strategy and virial or Morawetz-type estimates may perhaps be adapted to prove scattering and global

well-posedness results for the defocusing L^2-critical version of (1) (with $p-1=\frac{4}{d}$). Speculating further leads to the prospect that further development of these ideas may establish that optimal $H^{s_c}(\mathbb{R}^d)$ initial data for the defocusing version of (1), with $0 \leq s_c = \frac{d}{2} - \frac{2}{p-1} \leq 1$, evolves globally in time and scatters. Another outstanding open problem is to establish global well-posedness and scattering in the energy supercritical setting where $s_c > 1$.

2. **Variable coefficient nonlinear problems.** Strichartz and other dispersive linear estimates were first established in the constant coefficient setting. Bilinear and multilinear versions of the linear constant coefficient estimates underpin much of the recent progress on well-posedness and qualitative behavior. A thrust of recent work described at the workshop demonstrates that linear dispersive estimates are robust and extend to the variable coefficient setting. Further development of analogous bilinear and multilinear estimates will open up the study of many new problems in wave phenomena, and strengthen links with geometry, science and engineering.

If the recent past is a reasonable guide into the near future, there will be a continued rapid development of this field of research. The workshop at BIRS provided a splendid forum for the exchange of ideas and questions and contributed toward an improved understanding of Schrödinger evolution equations.

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