North American Workshop on Tropical Geometry

I. Itenberg (Université Louis Pasteur, Strasbourg),
G. Mikhalkin (University of Toronto),
Y. Soibelman (Kansas State Univeristy)

March 5 - March 9, 2007

1 Overview of the Field

Tropical Geometry is a branch of Geometry that has appeared just recently. Formally, it can be viewed as a sort of Algebraic Geometry with the underlying algebra based on the so-called tropical numbers. The tropical numbers (the term "tropical" comes from Computer Science and commemorates Brazil, in particular a contribution of the Brazilian school to the language recognition problem) are the real numbers enhanced with negative infinity and equipped with two arithmetic operations called tropical addition and tropical multiplication. The tropical addition is the operation of taking the maximum. The tropical multiplication is the conventional addition. These operations are commutative, associative and satisfy the distribution law.

It turns out that such tropical algebra describes some meaningful geometric objects, namely, the Tropical Varieties. From the topological point of view the tropical varieties are piecewise-linear polyhedral complexes equipped with a particular geometric structure coming from tropical algebra. From the point of view of Complex Geometry this geometric structure is the worst possible degeneration of complex structure on a manifold. From the point of view of Symplectic Geometry the tropical variety is the result of the Lagrangian collapse of a symplectic manifold (along a singular fibration by Lagrangian tori).

The easiest to describe are tropical varieties in dimension 1, i.e. tropical curves. These are the so-called "metric graphs", i.e. finite graphs equipped with an inner metric such that all "leaves", i.e. the edges adjacent to 1-valent vertices have infinite length. There is a finite-dimensional moduli space of tropical curves once we fix the number of cycles in the graph (this is the tropical counterpart of the genus) and the number of leaves (this is the tropical counterpart of the number of punctures). Such a moduli space is itself a tropical orbifold and there is a certain intersection theory on it.

From the point of view of Toric Geometry tropical varieties are limiting shapes of the amoebas of algebraic varieties under the deformation degenerating the argument torus. Such degeneration can be described by varying the base of the logarithm in the amoeba map to infinity. In toric geometry such construction is known as "the patchworking", it was introduced by O. Viro in 1979 for the needs of real algebraic geometry to give a way to construct real forms of complex algebraic varieties with controlled topology. Historically this was perhaps the first time of implicit appearance of tropical geometry. Since this appearance there were several discoveries and proposals that stirred the research related to the area, most notably the introduction of amoebas by I. M. Gelfand, M. Kapranov and A. Zelevinski (and, in particular, the introduction of non-Archimedean amoebas by Kapranov), the proposal to use tropical curves in the context of Mirror Symmetry (particularly, for computation of the Gromov-Witten invariants) by M. Kontsevich, the introduction of the Morse category by Fukaya and the introduction of tropical formalism to Computational Algebraic Geometry by B. Sturmfels. By now there exist distinct points of view on Tropical Geometry from different areas of Mathematics.

2 Recent Developments and Open Problems

Tropical Geometry already proved to be useful in quite distinct areas of Mathematics. By now it has applications in Real Algebraic Geometry, Enumerative Geometry, Mirror Symmetry, Symplectic Geometry and Computational/Combinatorial Geometry. The list of its applications keeps growing. E.g. most recently, Tropical Geometry had a brand-new appearance in the Statistical Physic work of R. Kenyon and A. Okounkov where they studied mathematical model for dimers accumulation. Currently there are several research groups around the globe that are doing active research in Tropical Geometry from somewhat different points of view.

The main developments so far concern applications of tropical curves. This is currently the most well-understood case of tropical varieties. Many open problems concern higher-dimensional case, in particular, tropical surfaces.

Many new developments in Tropical Geometry were presented during the workshop. These developments are described in the following section.

3 Presentation Highlights

Hannah Markwig gave a talk entitled ”*The j-invariant of a plane tropical cubic*” (joint work with Eric Katz and Thomas Markwig). Several results relate the j-invariant of an elliptic curve to the cycle length of a tropical elliptic curve. E. Katz, H. Markwig, and T. Markwig proved the following theorem which can be seen as one of the justifications of the fact that the cycle length is the tropical counterpart of the j-invariant. Given a plane cubic over the field of Puiseux series such that the tropicalization of the cubic has a cycle (and is dual to a triangulation), the cycle length is equal to the negative of the valuation of the j-invariant. As a corollary, one obtains that the tropicalization of a cubic whose j-invariant has positive (negative in another widely-used convention) valuation does not have a cycle. Possible generalizations of the theorem (for example, for smooth elliptic curves in other toric surfaces) and connections with bad reduction of elliptic curves over discrete valuation rings are subjects to study.

Takeo Nishinou gave a talk entitled ”*Counting problems in tropical geometry*”. The talk was devoted to a tropical count of holomorphic discs with certain Lagrangian boundary condition in a toric variety. This work is presented in [**?**] and can be viewed as a relative version of the count of tropical closed curves. In particular the Lagrangian was assumed to be presented by a fiber in the toric fibration. Accordingly, the corresponding tropical curve “with boundary” was allowed to have a 1-valent vertex at a finite distance (i.e. of sedentarity 0).

Eric Katz gave a talk entitled ”*Equivariant cohomology and localization in tropical geometry*” (joint work with Sam Payne, see [**?**]). E. Katz and S. Payne use localization to describe the restriction map from equivariant Chow cohomology to ordinary Chow cohomology for complete toric varieties in terms of piecewise polynomial functions and Minkowski weights. They computed examples showing that this map is not surjective in general, and that its kernel is not always generated in degree one. They prove a localization formula for mixed volumes of lattice polytopes and, more generally, a Bott residue formula for toric vector bundles.

Askold Khovanskii gave a talk entitled ”*Elimination theory and Newton polyhedra*” (joint work with A. Esterov, see [**?**]). The goal of elimination theory is to describe, for an algebraic variety $X \subset \mathbf{C}^n$ and a projection $\pi : \mathbf{C}^n \to \mathbf{C}^m$, the defining equations of $\pi(X)$ in terms of the equations of X. Let a variety $X \subset (\mathbf{C}^*)^n$ be defined by equations $f_1 = \dots f_k = 0$ with given Newton polytopes and generic coefficients. Assume that $\pi(X) \subset (\mathbf{C}^*)^m$ is a hypersurface given by an equation $g = 0$. A. Esterov and A. Khovanskii describe the Newton polytope and the leading coefficients (that is, the coefficients of monomials which are on the boundary of the Newton polytope) of the Laurent polynomial g in terms of the Newton polytopes and the leading coefficients of the Laurent polynomials $f_1, \dots, f_k$. Several problems related to Newton polytopes and tropical geometry are particular cases of this version of elimination theory. This work is directly related to the subject of the paper [**?**].

Eugenii Shustin gave a talk entitled ”*Recursive formulas for Welschinger invariants of real Del Pezzo surfaces*”. The Welschinger invariants are designed to bound from below the number of real rational curves passing through a given generic real collection of points in a real variety. In some cases these invariants can be calculated using Mikhalkin's approach which deals with a corresponding count of tropical curves. As is known, in certain situations (for example, in the case of generic collections of real points on a toric Del Pezzo surface equipped with the tautological real structure, see [**?**], or in the case of generic collections of points in the three dimensional real projective space, see [**?**]), there is a logarithmic equivalence between the Welschinger invariants and the corresponding genus zero Gromov-Witten invariants. I. Itenberg, V. Kharlamov, and E. Shustin consider generic collections of real points on the projective plane blown up at 4 real points in general position and, using appropriate tropical Caporaso-Harris type formulas, prove that the logarithmic equivalence of the Welschinger and Gromov-Witten invariants holds in this situation as well. More precisely, they prove the following statement. *Let D be an ample divisor on the projective plane blown up at 4 real points in general position; then the Welschinger invariant W_D is positive, and*

$$\log W_{nD} = \log GW_{nD} + O(n),$$

where GW_{nD} stand for genus zero Gromov-Witten invariants. The proof is based on a new version of the correspondence theorem.

The talk of Valery Alexeev was of a survey nature and dealt with several classical algebro-geometric points of view on tropical geometry. These ways include passing to one-parametric deformations as well as the log-geometry. A special emphasis was made on the case of moduli spaces of curves and Abelian varieties. Interestingly enough, in the recently constructed compactification of the moduli space of Abeian varieties (due to the speaker [**?**]) tropical Abelian varieties appear naturally as the boundary strata. This comes as a special case of a more general principle that exhibits tropical geometry as the boundary of complex geometry. Here the boundary can be interpreted either as the limit in the one-parametric families or as the log-geometry boundary.

Richard Kenyon gave an elegant talk presenting an application of tropical geometry to the geometry of statistical models, in particular, of the dimer model. The talk is based on his joint work with Andrei Okounkov [**?**]. They considered the dimer model on a planar hexagonal lattice with the edges weighted by a periodic (more precisely, doubly periodic according to the lattice) function. For each (finite) size of the mesh of the lattice we have several configurations with different probabilities. However, in the limit when this size goes to zero there is a unique configuration with 100% probability.

In the absence of boundary conditions this configuration is parameterized by a certain Harnack curve ([**?**]). (More precisely its so-called height function coincides with the so-called Ronkin function of the corresponding amoeba). The degree of the curve is determined by the period of the weights while the coefficients are determined by the weights themselves.

In the presence of the boundary condition the situation is much more interesting, particularly in the case when the boundary is a broken line with the three possible slopes in the real plane. The limiting configuration has the so-called frozen boundary which comes as the log-front of the Harnack curve responsible for the periodic weights and another curve, responsible for the boundary conditions, see [**?**]. The degree of the second curve depends on the number of chains in the broken line boundary. The geometry of the second curve is in a sense antipodal with respect to the geometry of the first curve.

There is also a "temprerature" parameter in this statistical model. Ironically, tropical configurations in this terminology correspond to the zero temperature (the most regular case). The limiting configuration is easy to find here and then one can trace the change of the situation when we increase the temperature.

An interesting application to classical geometry concerns finding a rational curve of degree d inscribed in a $3d$-gon in the plane that comes as a projection of a closed broken line in the 3-space. Here we assume that the sides of this $3d$-god are parallel to particular 3 directions in the plane that are projections of the coordinate axes with the kernel of the projection parallel to the vector $(1, 1, 1)$. By a projective duality consideration there are as many curves tangent to the lines extending the sides of the 3-gon as the number of rational curves of degree d passing via $3d - 1$ points, which is a rather large number. Nevertheless, there is a unique curve that is geometrically inscribed to the polygon in the case when the polygon is *tilable*, in other words if there exists a corresponding tropical inscribed curve.

The talk of Kristian Kennaway was devoted to the aspects of Tropical Geometry motivated by Physics, see [**?**]. The mathematical construction presented starts from the consideration of the so-called *alga* of a planar curve, i.e. its image in the argument torus under the argument map. It can be shown that in many cases the resulting set has a certain 3-valent graph as its deformation retract. The complement of this graph has d^2 hexagons and the graph itself has $2d^2$ vertices and $3d^2$ edges. One can do a twist along a ribbon in the neighborhood of each edge. This transformation produces a surface (usually of positive genus) with $3d$ boundary components. If we glue the boundary components with disks we get a surface of genus $\frac{(d-1)(d-2)}{2}$ that may be considered as an adjoint curve. Its geometry is intrinsically related to the geometry of the original curve.

The talk of Nikolai Mnev was devoted to combinatorial description of the Grassmannian variety $G_{n,k}$ [**?**], particularly to the relatively little-understood case $k > 2$. If $k = 2$ the case is well-understood thanks to the relation (up to a torus action or, more precisely, the Chow quotient in the sense of Kapranov) of $G_{n,2}$ to the moduli space $\overline{\mathcal{M}}_{0,n}$. In the general case one needs much stronger combinatorial tools.

Brett Parker presented his theory of "Exploded Fibration" [**?**]. This theory may be considered as a partial tropicalization. Namely, in this theory we do not have to tropicalize a variety completely, but may leave a part (or several parts) of it as is. We still have to tropicalize the junctions between such parts and this tropicalization works as usual and allows one to compute the curves in the initial variety if we know the curves in the non-tropicalized parts.

Mikael Passare gave a survey of the theory of *co-amoebas* (or *algae*) [**?**], i.e. the argument projections of complex algebraic and analytic varieties. It shares many features with the theory of amoebae, but it gives an essentially new point of view on complex varieties. As it was shown in the talk an elegant application of this theory gives a new computation of the value of the Riemann ζ-function at 2, see [**?**]. Namely, as it was shown in the work of Passare and Rullgård [**?**] the area of the amoeba of a plane curve can be computed in terms of the so-called *Monge-Ampère measure*. The latter can be computed by means of the area of the Newton polygon of the curve (the proportionality coefficient is π^2, where the square is responsible for dimension 2 of the ambient plane). The $\zeta(2)$ computation comes from integration of the Taylor expansion of the function that gives one of the three arcs in the boundary of the amoeba of the line. The area under (or raher over as the corresponding function is negative) this curve is one third of the total area of the amoeba of the line or $\frac{\pi^2}{6}$.

The area of the algae coincides with the area of the amoeba, so we have the same estimate for the area of alga. This area accumulates the easiest in the case of Harnack curves, see [**?**], [**?**], [**?**], [**?**] as this is the case when the argument map is birationally (from the point of view of real geometry) a covering of degree d^2. In the talk many other interesting examples were considered, particularly the amoebae and algae of hypergeometric functions of many variables.

The talk of Andrei Losev presented a quantum mechanics theory based on tropical geometry. He argued that passing from complex geometry to tropical geometry may be interpreted as passing from the quantum field theory to quantum mechanics. In the presented mathematical theory the states of the particles are enhanced with the slopes (corresponding to the slopes of the corresponding tropical rational functions on the tropical line) and the operators of changing the slopes were introduced. The corresponding correlators then can be interpreted as tropical Gromov-Witten invariants.

Andrei Zelevinsky discussed tropical aspects of cluster algebras. He presented three different points of view on cluster algebras, introduced in his joint papers with Sergey Fomin. One starts (in a simplest case) with a skew-symmetric integer-valued matrix and associated group of variables. The operation of mutation changes the matrix as well as the variables. The mutation is a birational operation on variables. Cluster algebra is an associative algebra generated by all mutated variables. The relation to tropical geometry comes out from the observation that the whole theory can be developed over an arbitary semifield. Zelevinsky also discussed the problem of constructing "canonical bases" in cluster algebras and relation of this problem to recent papers by Fock and Goncharov on Langlands duality for cluster varieties.

The talk of Yan Soibelman was devoted to his joint project with Maxim Kontsevich in which they study Donaldson-Thomas invariants for 3-dimensional Calabi-Yau varieties (possibly non-commutative). The whole subject can be roughly described as a counting problem of stable objects in a Calabi-Yau category endowed with a stability structure (e.g. counting of special Lagrangian submanifolds in a $3d$ Calabi-Yau manifold). As the stability structure changes (e.g. as we move in a complexified Kähler cone toward infinity) the (properly defined) number of stable objects can change as one crosses some "wall" of real codimension one. This change of numbers is described by new "wall-crossing formulas". The moduli space of stability structures resembles tropical hyperkahler variety (a.k.a. skeleton of the corresponding maximally degenerate Calabi-Yau variety introduced in the earlier paper by Konstevich and Soibelman). Wall-crossing formulas for non-commutative $3d$ Calabi-Yau varieties generated by a spherical collection (e.g. vanishing Lagrangian spherical cycles in the geometric case) are described by quivers with potentials, and the wall-crossing formulas give rise to cluster transformations.

Vladimir Berkovich discussed his approach to tropical geometry treated as geometry of analytic spaces over the field $\mathbf{F}_1$ of one element. The latter can be thought of as a monoid with two elements $0, 1$ with operation of multiplication. Then $\mathbf{F}_1$-algebras are commutative monoids. One can mimick Berkovich approach to the theory of analytic spaces, which is based on the notion of Berkovich spectrum of a commutative Banach ring. The spectrum is a compact Hausdorff space of bounded multiplicative seminorms. In the non-archimedean case one can glue spectra of affinoid algebras into more complcated spaces (analytic spaces). In the case of the theory over $\mathbf{F}_1$ there is are analogs of affinoid algebras, and their spectra are PL-spaces equipped with sheaves of affine functions, i.e. tropical spaces. Gluing procedure and hence the global theory of such $\mathbf{F}_1$-analytic spaces had not been developed by the time of the workshop.

Talk by Yong-Geun Oh was devoted to Seidel long exact sequence for Floer homology. Floer homology for a pair of Lagrangian submanifolds are not always well-defined. Aim of his talk was to construct certain exact triangle in the derived Fukaya category of a general Calabi-Yau manifold (previously Seidel considered the case of exact Lagrangian submanifolds). More precisely, let L be an exact Lagrangian sphere (parameterized, i.e. the diffeomorphism with S^n is chosen) in a compact sympectic manifold with contact boundary. With such data one can associate the Dehn twist τ_L. Then for any two exact Lagrangian submanifolds L_0, L_1 Seidel constructed an exact triangle of Floer homology groups with the vertices $HF(\tau_L(L_0), L_1)$, $HF(L_0, L_1)$, $HF(L, L_1) \otimes HF(L_0, L)$. Y.-G. Oh described which technical difficulties one should overcome (and how) in order to generalize this result to arbitrary Calabi-Yau manifolds.

4 Scientific Progress and Outcome of the Meeting

The main outcome of the meeting was in new collaborations of people from different areas of science and different background ranging from Combinatorics to String Theory. The same tropical phenomena appear in different areas in very recent research, so it is very important to set up a uniform terminology and language and it is not less important for researchers in one area to learn the developments that appear in other areas. Grigory Mikhalkin gave an introductory talk based on [**?**] and [**?**] with the definition of tropical varieties and other basic notions (such as tropical modifications). The talks were scheduled thematically each day and were followed by informal discussions that allowed experts in one particular area to understand tropical developments in other areas. (The relevant areas include Combinatorics, Algebraic Geometry, Symplectic Geometry, Complex Analysis and Physics, many of these areas can be considered as the parent fields of Tropical Geometry.) It was crucial for the informal discussions that they were guided by such prominent experts in these fields as Eliashberg, Hori, Khovanski, Viro and Zelevinski. In addition, recent developments in the areas bordering Tropical Geometry (Amoebae, Algae, Patchworking, etc) were also discussed formally and informally. Several cross-area collaborations started during the conference.

5 Schedule of the Meeting

Monday, March 5, 2007

7:00–9:00 Breakfast

9:00–9:15 Introduction and Welcome to BIRS by BIRS Station Manager, Max Bell 159

9:15–10:15 A. Zelevinsky, *Cluster algebras: tropical aspects*.

10:15–10:30 Coffee Break, 2nd floor lounge, Corbett Hall

10:30–11:30 Y. Soibelman, *Donaldson-Thomas invariants, cluster transformation and tropical geometry*.

11:30–13:00 Lunch

13:00–14:00 Guided Tour of The Banff Centre; meet in the 2nd floor lounge, Corbett Hall.

14:00	Group Photo; meet on the front steps of Corbett Hall.
14:30–15:30	H. Markwig, *The j-invariant of a tropical elliptic curve of degree 3*.
15:30–16:00	Coffee Break, 2nd floor lounge, Corbett Hall.
16:00–17:00	V. Alekseev, *Tropical geometry and degenerations*.
17:30–19:30	Dinner

Tuesday, March 6, 2007

7:00–9:00	Breakfast
9:00–10:00	R. Kenyon, *Inscribing curves in polygons*.
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	K. Kennaway, *Geometry of the dimer models*.
11:30–13:30	Lunch
14:00–15:00	M. Abouzaid, *Homological mirror symmetry and tropical geometry*.
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall.
15:30–16:30	T. Nishinou, *Counting problems in tropical geometry*.
17:30–19:30	Dinner

Wednesday, March 7, 2007

7:00–9:00	Breakfast
9:00–10:00	N. Mnev, *Combinatorial models of PL Grassmannians*.
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	A. Khovanskii, *Elimination theory and Newton polyhedra*.
11:30–13:30	Lunch
	Free afternoon
17:30–19:30	Dinner

Thursday, March 8, 2007

7:00–9:00	Breakfast
9:00–10:00	V. Berkovich, *Analytic geometry over the field of one element*.
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	E. Shustin, *Recursive formulas for Welschinger invariants of real Del Pezzo surfaces*.
11:30–13:30	Lunch
14:00–15:00	A. Losev, *Tropical mirror and quantum mechanics*.
15:00–15:20	Coffee Break, 2nd floor lounge, Corbett Hall.
15:20–16:20	M. Passare, *Some remarks on amoebas and coamoebas*.

16:30–17:30 E. Katz, *Equivariant cohomology and localization in tropical geometry*.

17:30–19:30 Dinner

Friday, March 9, 2007

7:00–9:00 Breakfast

9:00–10:00 Yong-Geun Oh, *Seidel's long exact sequence of Floer cohomology on Calabi-Yau manifolds*.

10:00–10:30 Coffee Break, 2nd floor lounge, Corbett Hall

10:30–11:30 B. Parker, *Tropical geometry and the exploded category*.

11:30–13:30 Lunch

Checkout by 12 noon.

References

[1] V. Alexeev, *Limits of stable pairs*. Preprint arXiv: math/0607684.

[2] E. Brugallé and G. Mikhalkin, *Enumeration of curves via floor diagrams*. Preprint arXiv: math.CO/0706.0083.

[3] A. Esterov and A. Khovanskii, *Elimination theory and Newton polytopes*. Preprint arXiv: math/0611107.

[4] B. Feng, Y.-H. He, K.D. Kennaway, and C. Vafa, *Dimer Models from Mirror Symmetry and Quivering Amoebae*. Preprint http://arxiv.org/abs/hep-th/0511287.

[5] E. Frenkel and A. Losev, *Mirror symmetry in two steps: A-I-B*, Commun.Math.Phys. 269 (2007) 39-86.

[6] I. Itenberg, V. Kharlamov, and E. Shustin, *Logarithmic equivalence of Welschinger and Gromov-Witten invariants*, Russian Math. Surveys **59** (2004), no. 6, 1093–1116.

[7] E. Katz and S. Payne, *Piecewise polynomials, Minkowski weights, and localization on toric varieties*. Preprint arXiv: math.AG/0703672.

[8] R. Kenyon and A. Okounkov, *Planar dimers and Harnack curves*. Preprint arXiv: math/0311062.

[9] R. Kenyon and A. Okounkov, *Limit shapes and the complex burgers equation*, Preprint arXiv:math-ph/0507007.

[10] G. Mikhalkin, *Real algebraic curves, moment map and amoebas,* Ann. of Math. (2) **151** (2000), no. 1, 309-326.

[11] G. Mikhalkin, *Tropical Geometry and its application*, to appear in Proceedings of the ICM 2006 Madrid, 25 pages, http://arxiv.org/abs/math.AG/0601041.

[12] G. Mikhalkin, *What is a tropical curve?*, Notices of the American Mathematical Society, April 2007.

[13] G. Mikhalkin and A. Okounkov, *Geometry of planar log-fronts*, Preprint http://arxiv.org/abs/math.AG/0608018.

[14] G. Mikhalkin and H. Rullgård, *Amoebas of maximal area,* Internat. Math. Res. Notices **9** 2001:9, 441–451.

[15] N. Mnev and G. Ziegler, *Combinatorial models for the finite-dimensional Grassmannians*. Discrete and Computational Geometry, 1993, Springer.

[16] T. Nishinou, *Disc counting on toric varieties via tropical curves*. Preprint arXiv: math/0610660.

[17] B. Parker, *Exploded fibrations*. Preprint arXiv: 0705.2408.

[18] M. Passare, *How to compute* $\sum 1/n^2$ *by solving triangles*. Preprint http://arxiv.org/abs/math/0701039.

[19] M. Passare and H. Rullgård, *Amoebas, Monge-Ampère measures and triangulations of the Newton polytope,* Duke Math. J. **121** (2004), no. 3, 481-507.

[20] M. Passare and A. Tsikh, *Amoebas: their spines and their contours*, Contemp. Math. **377** (2005) 275-288.

[21] B. Sturmfels and J. Tevelev, *Elimination Theory for Tropical Varieties*. Preprint arXiv math.AG/0704.3471.

[22] O. Viro, *Dequantization of real algebraic geometry on logarithmic paper*, Plenary talk at the 3rd ECM, Barcelona, July 10-14, 2000.