Free Probability, Extensions, and Applications

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1 Overview of the Field

Free probability theory is a line of research that parallels aspects of classical probability, in a highly non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out from attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost twenty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability and the theory of large deviations, algebraic combinatorics.

Since free probability relates to random matrices, its tools and techniques have a significance for questions about random matrices. Consequently, various communities which use random matrices in their models have started to use notions and results from free probability for their calculations. A prominent example of this kind are the investigations of electrical engineers on multiuser communication systems. Furthermore, there are presently quite a number of attempts to extend the methods and the applicability of free probability in various directions.

The workshop put special emphasis on applications of free probability and the attempts to extend its framework. In the following we will first give a brief general description of some of the main results and directions of free probability. Then we will address the applications of free probability in applied fields and point out some of the directions for extensions of the framework of free probability.

2 Recent Developments

2.1 Some of the fundamental results of free probability

Recently, Haagerup and Schulz [17, 18] achieved a crucial break-through on the famous invariant subspace problem; by relying on free probability techniques and ideas, they showed that every operator in a II_1 factor whose Brown measure is not concentrated in one point has non-trivial invariant subspaces.

Free probability provides new ideas and techniques for investigating random multi-matrix models. Recently, in this direction, Haagerup and Thorbjornsen [19] obtained a sweeping generalization to several matrices for a number of results concerning the largest eigenvalue of a Gaussian random matrix, see also [17].

A celebrated application of free entropy was Voiculescus proof [31] of the fact that the free group von Neumann algebras do not have Cartan subalgebras (thus solving a longstanding problem); many other powerful applications of free entropy to questions on operator algebras are due to Ge, Stefan, Jung, and Shlyakhtenko. Recently, Connes and Shlyakthenko [8] introduced a kind of L^2-Betti number for von Neumann algebras. There exist close relations between this theory and free entropy.

The study of free group factors via free probability techniques has also had an important application to subfactor theory: not every set of data for a subfactor inclusion can be realised in the hyperfinite factor; however, the recent work of Shlyakhtenko, Ueda, and Popa has shown that this is possible using free group factors. Thus the connection of free group factors to free probability and random matrices appears to be also related to a key role of this class of factors within von Neumann algebras.

First elements of a stochastic calculus and a stochastic analysis for free Brownian motion were developed by Kmmerer and Speicher, and by Biane and Speicher [6]. These are expected to be crucial tools for getting a deeper understanding of random multi-matrix models. A groundbreaking result in this direction, on the fluctuations of two non-commuting Gaussian random matrices, is due to Cabanal-Duvillard [7].

In relation with probability, the work of Ben-Arous, Guionnet, and Cabanal- Duvillard ([1, 9]) shows that Voiculescus free entropy is useful in the study of the rate functions of large deviation principles for special random matrices. A significant progress on one of the most intriguing problems in this context (the unification problem for the two approaches to free entropy) was recently achieved by Biane, Capitaine, and Guionnet [4].

Free probability has a significant combinatorial facet. Speicher [27] showed that the transition from classical to free probability consists in replacing all partitions by non-crossing partitions. The combinatorial machinery of free probability was used by Nica and Speicher to obtain new results in free harmonic analysis (see [23]); and by Biane to derive new results on the asymptotics of representations of symmetric groups; it gives rise to new interesting conjectures in that area, in particular in connection with the Kerov polynomials, see [3, 26, 15].

Other important recent directions currently pursued in free probability are free Wasserstein distance and free log Sobolev inequalities, processes with free increments and analytic subordination, properties of cyclic derivatives and automorphisms of the free group factors, duality for coalgebras of free difference quotients and more general ”free analysis” questions.

2.2 Applications of free probability in applied fields

Through the quite unexpected relation between free probability theory and random matrices (which was discovered by Voiculescu in [29]), the tools and techniques from free probability theory have a significance for questions about random matrices. The power of those tools from free probability (in particular, the R-transform and S-transform, and the notion of free cumulants) has become more and more apparent in recent years to various communities which use random matrices in their models.

One active line of research in this spirit is the use of the free probability machinery in investigations of electrical engineers for multi-user wireless communication systems. This direction was initiated by Tse; the relevance of free probability for such problems has been pointed out to the engineering community in the survey article of Tulino and Verdu [28]. Whereas simple models have been treated very succesfully, more realistic/complicated models still have to be solved and present new challenges for free probability theory.

Statistics is another field where random matrices (in particular, Wishart matrices and their variants) are of eminent interest and where the interest in free probability techniques has risen during the last years. A present point of contact between statistics and free probability are results on linear statistics or eigenvalue fluctuations, which are closely related to the new concept of second order freeness [10]. Other statistical topics of interest, like the statistics of the largest eigenvalue, are still not within the realm of free probability but could serve as inspiration for further developments.

Another interesting link between applied fields and free probability is provided by the implementation by Edelman and Rao (see [12]) of the tools of free probability on a computer by using symbolic software (building on MATLAB). On one side, this has made available the tools of free probability to a much wider audience and, on the other side, presents the challenge of incorporating new developments into this ’free calculator’.

Random matrices are also used in various models in theoretical physics. An interesting recent contribution to this connection was made by Guionnet and Maurel-Segala [16] who could give rigorous proofs (and connect them with the concept of free entropy) of some expansions for multi-matrix models, which had appeared before on a formal level in the physical literature.

2.3 Extensions of free probability

One of the main focus of the workshop was on various extensions of the framework of free probability theory. The motivation for such extensions comes from various sources, both theoretical and applied. We put much effort into having a good balance of people who are confronted with concrete problems and those who might develop the tools for solving them.

We describe below some of the present ideas for extensions of free probability.

Voiculescu introduced from the very beginning a more general version of free probability: operator valued free probability theory, where the scalars are replaced by more general algebras and the role of the underlying state is taken over by a conditional expectation. This extension of free probability has a much wider applicability but still shares many of the features with usual free probability. Some of the basic results of free probability theory were extended right away to the operator valued version [30, 27]. However, the more advanced theory and also the detailed examination of fundamental examples has only started recently, and we expect a surge of new investigations in this direction. Further progress on operator-valued free probability seems to be instrumental both for pure and for applied questions. E.g., on the pure side, recent work of Junge [20, 21] on realizations of the Hilbertian operator space OH relies on generalizing norm inequalities for sums of free variables to the operator-valued context. Further results in this direction promise a deeper understanding of OH, which is a central object in the theory of operator spaces. On the applied side, operator-valued versions of free probability have turned out to provide new techniques for dealing with more complicated types of random matrices; see, in particular, the results of Shlyakhtenko [25] on Gaussian band matrices. Recently, this was taken up and extended in [24], who considered block matrices which are of relevance in wireless communication MIMO models.

A few years ago, Biane, Goodman, and Nica [5] introduced a type B version of free probability theory. The usual, type A, free probability theory relates to the permutation groups. Replacing those by corresponding type B Coxeter groups leads to type B free probability. There exists a nice combinatorial theory of this version, however, random matrix models for this extension are still missing. Models used in applied problems might provide some inspiration for this.

Whereas free probability deals with operators on Hilbert spaces, corresponding to square random matrices, Benaych-Georges [2] introduced recently a version of free probability which deals with operators between different Hilbert spaces and relates to rectangular random matrices. The latter show up, in particular, in the context of Wishart matrices, and thus rectangular free probability promises to provide the right framework for some applied problems in wireless communication and statistics.

In the random matrix literature there has been a lot of interest in linear statistics or global fluctuations of random matrix eigenvalues (see, e.g, [11]). In order to extend the framework of free probability to be able to deal with such problems, Mingo and Speicher [22] introduced the concept of 'second order freeness'. They developed, with Collins and Sniady [10], an extensive combinatorial theory for this concept and are presently applying, with Edelman and Rao [13], their theory to statistical eigen-inference problems.

Other extensions of the framework which are presently under investigation are, e.g., q-deformations of the free Fock space, free extreme values, or monotone convolution.

3 Presentation Highlights

Since the workshop brought together researchers from quite different communities there were a couple of survey talks which presented aspects of free probability and random matrix theory from various points of views. Particular emphasis in those talks was given to pointing out open problems and possible future directions.

- Voiculescu gave an introduction to free probability with special emphasis on the recent developments and questions around the coalgebra structure and operator-valued resolvents.

- Bercovici gave a survey covering limit theorems in free probability, and regularity issues for free convolutions. He discussed in particular the prospects of extending these results to operator-valued variables, and to other variants of noncommutative probability theory.
- Verdu gave a survey on the questions in wirleless communications, and the use of free probability theory and random matrix theory in such models. This was complemented by a talk of Tulino who described in more detail specific models.
- Letac showed the use of random matrices in statistics by presenting results on the computation of moments for beta distributions, and applications of such tools.
- Mingo gave an introduction to the calculation of unitary matrix integrals in terms of the so-called Weingarten function, and presented the main results about the asymptotics of the Weingarten function.
- Speicher gave a survey on the questions, ideas, and results behind the notion of higher (in particular, second) order freeness.
- Nowak showed how in physics a quaternion valued extension of free probability theory is used to treat new examples of non-normal random matrix ensembles.
- Silverstein outlined recent work on spectral properties of random matrices, in particular in connection with the ”spiked population model”, with is of relevance in statistics, and in connection with some random matrix models which appear in the modeling of MIMO systems in wireless communications.
- Edelman gave a survey on computational aspects of random matrices and free probability.
- Nica gave an introduction to recent developments on the combinatorial side of free probability theory.

The other talks presented more specific results; but in each case the speaker provided some background and motivation for the considered question.

The other talks can roughly be divided into two groups. The first group consisted of mathematial results about free probability and random matrices:

- Capitaine and Donati-Martin talked about the asymptotics spectrum of deformed Wigner matrices, with particular emphasis on the behaviour of the largest or smallest eigenvalue.
- Dykema presented calculations about the microstates and the free entropy dimension in amalgamated free products over a hyperfinite subalgebra.
- Hiai discussed the notions of free pressure and microstates free entropy from the point of view of hypothesis testing. In particular, he considered a free analogue of Stein’s lemma.
- Kargin defined the notion of Lyapunov exponents for a sequence of free operators and showed how they can be computed using the S-transform. He also discussed the relations with corresponding results on the product of large random matrices.
- Kemp presented results about the resolvent behaviour of a prominent class of non-normal, so-called R-diagaonal, operators.
- Shlyakhtenko discussed the relation between planar algebras, subfactors and random matrices and free probability. In particular, he showed how random matrix constructions of planar algebras can be used to give a new proof of a theorem of Popa on realization of subfactors.
- Anshelevich presented his results on polynomial families (Appell, Sheffer, Meixner) in the context of Boolean probability.
- Collins introduced the notion and presented the analytic and combinatorial aspects of free Bessel laws and explained its connection with quantum groups and random matrices.

The second group treated concrete problems in wireless communications:

- Moustakas discussed the application of diagrammatic methods developed in physics to the calculation of the asymptotic eigenvalue spectrum of specific random matrix models. He also discussed the use of the replica method in such a context.
- Müller considered the problem of minimizing random quadratic forms and explained the relevance of this problem in transmitter processing of wireless communications.
- Ryan presented the use and implementation of free probability methods for an example of channel capacity estimation in MIMO systems.

4 Open problems and possible future directions

There was also scheduled a special discussion meeting on Thursday afternoon in which open problems and possible future directions of free probability theory were discussed. In the following we list some of the major relevant problems which were pointed out in this discussion meeting and in numerous other discussions between the participants throughout the workshop.

- What is the relevance of free entropy in applied fields. Free entropy is clearly a major notion in free probability (with many technical problems still open), but there was also the feeling that similar to classical entropy it should play a fundamental role in statistics and information theory. In this context, it seems that one also needs a more general notion of relative free entropy.
- The use (and the success) of the replica method in many calculations has no mathematical justification. Can free probability justify the replica method, or at least provide other tools for deriving results which are obtained by the replica method.
- The need for a better understanding of the analytic theory of operator-valued freeness and resolvents is not only mathematical apparent, but there are also many applied quite successful methods (as quaternion extension of free probability, or deterministic equivalents of Girko for random matrix models) which might benefit from such a theory.
- Tulino and Verdu presented calculations on the Gaussian Erasure Channel; some of the results are the same as one would obtain if one assumes freeness between involved operators; however, no freeness is apparent. Does this hint at a new instance of asymptotic freeness in this model or is there something else going on?
- The recent results of Belinschi and Nica revealed a deeper connection between free and boolean convolution. This deserves further clarification.
- Whereas complex random models are the most basic in mathematics and engineering, in statistics one is usually only interested in real random matrices. Whereas for questions about the eigenvalue distribution the distinction between complex and real does not play a role, it becomes significant for finer questions, like the fluctuations of eigenvalues. This asks for an extension of the theory of second order freeness from the complex to the real case.
- Whereas engineers have no problems in applying asymptotic freeness results for unitarily invariant ensembles it has become apparent that they do not have the same confidence in the analogous results for Wigner matrices. The main reason for this is the lack of precise statements on this in the literature. This has to be remedied in the future.

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