$\mathcal{H}$-holomorphic maps in symplectic manifolds

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1 Overview of the Field

Symplectic manifolds are the natural domains of the modern mathematical formulation of classical mechanics, and they play a prominent role in many areas of mathematics. Since the introduction of the theory J–holomorphic curves by Gromov in [5], a tremendous amount of progress has been made in the study of these manifolds and the maps which preserve their symplectic structures. This progress has been particularly dramatic in the case of symplectic manifolds of dimension 4. For example, for $S^2 \times S^2$, the symplectic forms are classified (see [5], [14]), their symplectomorphism groups are well understood (see [5], [3]), and their Lagrangian spheres are all known to be symplectically equivalent (see [6]). Ruled symplectic 4–manifolds have also been completely classified (see [11]).

The proofs of these results all have the same starting point; the existence of foliations by J–holomorphic spheres for all tamed almost complex structures. There have been several attempts to establish the existence of such foliations in more general settings. It was recently shown in [7] that such existence statements do not hold for foliations by J–holomorphic spheres of manifolds of dimension greater than 4. For index reasons, one can also not expect to find foliations by J–holomorphic curves of higher genus. To overcome this latter limitation, in the setting of symplectizations of a contact 3–manifolds, H. Hofer proposed in [8] to replace the standard J–holomorphic map equation with a parameterized version where the parameter takes values in the space of harmonic 1–forms on the domain. More precisely, they introduce the notion of an $\mathcal{H}$–holomorphic map which is a map u from a Riemann surface (Σ, j) to the symplectization $(\mathbb{R} \times Z, J)$ of a contact manifold Z such that $\bar{\partial}_J u$ takes values in $\mathcal{H}^{0,1} = \mathcal{H}^{0,1}(u^*\underline{\mathbb{C}}))$, the space of harmonic $(0,1)$–forms on Σ with values in the trivial bundle $u^*(\underline{\mathbb{C}})$, where $\underline{\mathbb{C}} \subset T(\mathbb{R} \times Z)$ is the trivial complex vector bundle generated by the $\mathbb{R}$–factor.

For $\mathcal{H}$–holomorphic maps many new analytic difficulties arise. For example, local intersections of such maps need not be positive, and the space of these maps is in general not compact (see [15]). Despite these difficulties, $\mathcal{H}$–holomorphic maps have been used to obtain foliations of contact 3–manifolds. In particular, in both [1] and [16], it is shown that every contact structure on a 3–manifold admits a contact form and an almost complex structure which support an open book decomposition whose pages are embedded $\mathcal{H}$–holomorphic maps.

The use of parameterized versions of the J–holomorphic map equation is not new. For example, they were used to find non–trivial elements in symplectomorphism groups by O. Buse in [4]. As well, the parameter space introduced in [10] was recently used to compute the Gromov-Witten invariants of Kähler surfaces in [12].

2 Scientific Progress Made

We investigated a generalization of the J–holomorphic map equation for maps into symplectic 4–manifolds. One of our goals is to use these maps to obtain foliations by higher genus surfaces that can be used to generalize some of the classification results obtained for ruled symplectic surfaces. The generalized equations we intend to study are defined in analogy with the $\mathcal{H}$–holomorphic map equation for contact 3–manifolds from [2], as follows.

For a hermitian line bundle (L, ω, J) over a genus g Riemann surface (Σ, j), let $\mathcal{H}^{0,1} = \mathcal{H}^{0,1}(L)$ denote the space of harmonic $(0,1)$–forms with values in L, i.e. sections ξ of $\Lambda^{0,1}T^*\Sigma \otimes L$ satisfying $d^{\nabla}\xi = 0$, where ∇ denotes the induced hermitian connection on L. Suppose that (X, ω) is a symplectic manifold with almost complex structure J. Let $L \subset TX$ be a J–complex line subbundle. A map $u : \Sigma \to X$ is called $\mathcal{H}$–holomorphic (w.r.t. L) if

$$\bar{\partial}_J u \in \mathcal{H}^{0,1}(u^*L) \subset \Gamma(\Lambda^{0,1}T^*\Sigma \otimes u^*TX)$$

Hence, the $\mathcal{H}$–holomorphic map equation for symplectic manifolds is a parameterized version of the J–holomorphic map equation with parameter space given by the finite dimensional vector space $\mathcal{H}^{0,1}$ of twisted harmonic $(0,1)$–forms.

To better understand this definition let (X^4, ω) be a symplectic surface bundle, i.e. X is a fiber bundle $\pi : X \to V$ with connected symplectic fibers of genus g and a symplectic base (V, ω_V) of real dimension 2. Let $F = \ker(d\pi) \subset TX$ be denote the vertical subbundle and let $L = F^{\omega}$ be its symplectic complement. Fix an almost complex structure J so that the splitting $TX = L \oplus F$ is J–invariant.

In this situation X is foliated by J–holomorphic curves in the class of the fiber. Unfortunately the linearized operator for J–holomorphic maps is not surjective. Indeed, the kernel of the linearized operator is 2–dimensional, but the index of this problem is $2(1 - g)$.

On the other hand, each J–holomorphic map u is also $\mathcal{H}$–holomorphic. The bundle u^*L is trivial so $\mathcal{H}^{0,1}(u^*L)$ has dimension $2g$ by Riemann–Roch and u has index $2(1-g)+2g = 2$. Moreover, the linearized operator for $\mathcal{H}$–holomorphic maps is surjective.

During the workshop we investigated the behavior of $\mathcal{H}$-holomorphic maps for more general choices of almost complex structure J and line bundle L. Unlike in the contact manifold case that was studied before, in the current setting the bundle L is not geometrically trivial. This introduces many analytical difficulties that make some of the arguments significantly harder, and render other results false.

Unlike in the contact manifold case we were not able to prove automatic regularity, i.e. surjectivity of the linearized problem, for $\mathcal{H}$-holomorphic maps, however, for generic compatible almost complex structure J all $\mathcal{H}$-holomorphic maps are regular. Moreover, for any fixed compatible almost complex structure J and generic choice of complex line bundle L, all $\mathcal{H}$-holomorphic maps transverse to L are regular.

Another important step when foliating contact manifolds by $\mathcal{H}$-holomorphic maps is that algebraic invariants guarantee that families of maps that start out transverse to L remain transverse to L. We have found counterexamples to show that this is not true any more in our setting.

In order to foliate the image manifold one needs to show that the linearized operator is superregular, that is that its kernel contains a pair of pointwise linearly independent sections that are transverse to the map. In our setting it is the curvature of L that complicates the situation compared the case in contact geometry, and we were not able to prove superregularity of the operator in our setting.

The main difficulty in generalize the results from contact geometry to our setting is given by the curvature of L. In the cases that we are interested in we always assume that L is topologically trivial, so L always admits a flat connection by choosing a complex trivialization. If we assume that we have a canonical choice of flat connection we can significantly improve on the previous theorem.

Examples of cases where such curvature assumptions hold are symplectic mapping tori (see [9]).

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