SUBORDINATION PROBLEMS RELATED TO FREE PROBABILITY

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1 Overview of the field

Our research project is in the area of noncommutative probability. Noncommutative probability emerged in the early '80s as a very powerful tool for the study of finite operator algebras. The fundamental idea is to view a pair (A, τ), where A is a unital algebra over the complex numbers (usually endowed with a suitable norm topology), and τ is a linear, unit preserving, functional on A, as a noncommutative probability space, in which the role of integration is taken by τ, while A plays the role of a function algebra over a probability space from the commutative case. There are several notions of independence specific only to the noncommutative setup. From an operator algebraic perspective, one can think of a noncommutative independence as a rule on how to extend τ from a family of independent subalgebras to the algebra they generate together. Our project is mainly related to the *free independence.* Free independence was introduced by Voiculescu [10] in the eighties with the intention of studying in a probabilistic framework the free group factors $L(\mathbb{F}_n)$ generated by the left regular representation of the free group with n generators $\mathbb{F}_n$. Since then, free probability became a powerful tool in several other areas of mathematics beyond operator algebras, especially in random matrix theory (the paper [11] initiated this direction of investigation, which since then has known a spectacular growth).

The subordination property (in the sense of Littlewood) for free convolutions, which forms the main subject of our research project, has been first noted in [12] by Voiculescu for free additive convolution: it has been shown in this paper (under an easily removable genericity condition) that the Cauchy-Stieltjes transform

$$G_{\mu_X \boxplus \mu_Y}(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d(\mu_X \boxplus \mu_Y)(t) \quad \Im z > 0$$

of the free additive convolution $\mu_X \boxplus \mu_Y$ of the probability distributions μ_X and μ_Y of the selfadjoint random variables X and Y is subordinated to G_{μ_X}. That is, there exists an analytic self-map of the complex upper half-plane ω so that $G_{\mu_X \boxplus \mu_Y}(z) = G_{\mu_X}(\omega(z))$, $\Im z > 0$. Biane [3] extended this result to a statement about conditional expectations: given two selfadjoint random variables X, Y which are free from each other, there exists a unique analytic self-map of the upper half-plane ω so that:

1. $\lim_{y \to +\infty} \omega(iy)/iy = 1$, and
2. $E_X((z - X - Y)^{-1}) = (\omega(z) - X)^{-1}$, where E_X denotes the conditional expectation from the unital algebra generated by X and Y onto the unital algebra generated by X.

(As E_X is trace-preserving, this is indeed a generalization of Voiculescu's result.) This analytic subordination result, and a similar one for free multiplicative convolution, has been used by many authors (Belinschi, Benaych-Georges, Bercovici, Biane, Chistyakov, Götze, Guionnet, Lenczewski, Nica, Voiculescu, Wang, Williams etc) to study free entropy for single random variables, to prove regularity results and arithmetic properties for free convolutions, as well as existence results and connections with other convolutions from noncommutative probability theory.

Another major step in the understanding of the role of subordination in free probability was made by Voiculescu in [14]. Paralleling on the one hand the construction of free products with amalgamation and on the other classical conditional expectations, Voiculescu defined *freeness with amalgamation* over a unital $*$-subalgebra B. In this context, he showed that if X and Y are free with amalgamation over B, then there exists a unique analytic self-map $\omega : B^+ \to B^+$ so that

$$E_{B\langle X\rangle}((b - X - Y)^{-1}) = (\omega(b) - X)^{-1}, \quad b \in B^+,$$

where B^+ denotes the set of elements of B with strictly positive imaginary part, and $E_{B\langle X\rangle}$ is the conditional expectation from $B\langle X, Y\rangle$ (the $*$-algebra generated by B, X and Y) onto $B\langle X\rangle$. (The result is expressed in terms of fully matricial sets and functions, and the proof is based on the observation that certain conditional expectations involved are co-algebra morphisms; however, we will not go into those details here.) Thus, the analytic subordination plays an important role in operator-valued non-commutative probability.

An aspect of considerable interest in the probabilistic approach to operator algebras is the study of joint distributions of k-tuples of selfadjoint random variables. Unlike in classical probability, however, the distribution of $(a_1, \dots, a_k)$ with respect to τ needs not be a probability measure, as the variables $a_1, \dots, a_k$ need not commute with each other. Thus, we define the distribution of $(a_1, \dots, a_k)$ as simply the values that τ takes on all monomials in $a_1, \dots, a_k$ (we call this sequence the *moment sequence* of $(a_1, \dots, a_k)$.) As for the case when $k = 1$, it is convenient to place the moment sequence in a formal power series in non-commuting variables $z_1, \dots, z_k$:

$$M_{(a_1,\dots,a_k)}(z_1, \dots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{k} \tau(a_{i_1} \cdots a_{i_n}) z_{i_1} \cdots z_{i_n}.$$

If μ is the distribution of $(a_1, \dots, a_k)$, then $M_{(a_1,\dots,a_k)}$ is sometimes denoted by M_μ. By starting from M_μ one defines two other important series, the R-transform R_μ and the η-series η_μ. These series linearize the operation $\boxplus$ of free additive convolution and respectively the operation of Boolean convolution $\uplus$, which is the counterpart of $\boxplus$ in Boolean probability theory.

For the study of sums and products of free k-tuples of random variables, the main tool has been the combinatorial aparatus of free cumulants developed by Nica and Speicher [9]. This tool is based on formal power series in non-commuting variables (as above), and as of this moment no explicit subordination result is known in this context, despite the recent progress realized by Nica in defining a subordination distribution [7].

Operator valued free probability and the study of distributions of k-tuples of free random variables are intimately connected, at least in principle, in the sense that one can rephrase problems from the second in terms of the first. However, despite this fact being known for more than a decade, a precise correspondence of the tools and methods involved is still missing.

2 Some recent developments

Results related to subordination have appeared recently in literature in significant numbers. We would like to mention some of them which are relevant to our project.

1. Generalizing Lenczewski's work [6], where an operator model for the so-called subordination distribution was proposed, Nica [7] has constructed a subordination distribution for k-tuples of selfadjoint random variables, and showed that they satisfy the same arithmetic properties with respect to non-commutative convolutions as in the case when $k = 1$.

2. Recent work of Capitaine, Donati-Martin, Féral and Février [4] uses the subordination property in the investigation of the eigenvalues of spiked perturbations of Wigner matrices.

3. Using Voiculescu's machinery of fully matricial sets and functions, Belinschi, Popa and Vinnikov [1] have recently proved several limit theorems for operator-valued selfadjoint random variables and given a description of the Boolean-to-free Bercovici-Pata bijection in the operator-valued context.
4. In [5], Curran extends the free difference quotient coalgebra approach to analytic subordination to the case of a free compression in free probability. Free compressions with a projection have been shown to correspond in terms of distributions to free convolution powers by Nica and Speicher [8]. The subordination property for scalar-valued random variables has been proved by Belinschi and Bercovici.

3 Scientific progress made

The first two days of the meeting were dedicated to presentations made by the four participants concerning their fields of specialization (Anshelevich on operatorial realizations for distributions, Belinschi on the analytic approach to Voiculescu's theory of fully matricial maps and sets, Février on the use and interpretation of the subordination property in random matrix theory, and Nica on combinatorial aspects of both scalar and operator valued distributions of k-tuples of random variables). For the rest, we have approached several problems, and we can report definite progress on issues.

First, following the realization of subordination distributions for k-tuples by Nica [7], it was a natural question to ask whether this distribution indeed deserves its name: does it satisfy a subordination relation? The following theorem of Anshelevich and Nica answer this question in the affirmative:

Theorem. *Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ be selfadjoint random variables in a non-commutative probability space $(\mathcal{A}, \tau)$. Assume that $\mathcal{B} \subset \mathcal{A}$ is a unital subalgebra so that $X_i \in \mathcal{B}$ and Y_i are free from $\mathcal{B}$ for all $1 \leq i \leq k$. Let $z_1, \ldots, z_k$ be non-commuting indeterminates, and denote by μ the joint distribution of $(X_1, \ldots, X_k)$ and by ν the joint distribution of $(Y_1, \ldots, Y_k)$ Then*

$$\mathbb{E}_{\mathcal{B}}\left[\left(\mathbf{1} - \sum_{i=1}^{k}(X_i + Y_i)z_i\right)^{-1}\right] = \left((\mathbf{1} - \eta^{\nu,\mu}(z_1, \ldots, z_k)) - \sum_{i=1}^{k} X_i z_i\right)^{-1}.$$

Here $\mathbb{E}_{\mathcal{B}}$ denotes the conditional expectation onto $\mathcal{B}$ and $\eta^{\nu,\mu}(z_1, \ldots, z_k))$ denotes the eta-series *of the subordination distribution of ν with respect to μ (as in [7]).*

The second aspect concerns aspects of free additive convolution and free Brownian motions for operator-valued free probability. As in classical probability, the centered free central limits (the semicircular, or Wigner, distributions) are known [10] to be indexed by their variances $t \in (0, +\infty)$. Moreover, if γ_t is the centered semicircular distribution with variance t, then $\gamma_t \boxplus \gamma_s = \gamma_{t+s}$, so that convolution powers $\{\gamma_t = \gamma_1^{\boxplus t} : t > 0\}$ form a convolution semigroup. Voiculescu [13] proved a free central limit theorem for operator-valued distributions, and found the operator-valued semicircular distributions to be naturally indexed by the semigroup of *completely positive* linear maps $\eta \colon B \to B$, so that $\gamma_\eta \boxplus \gamma_{\eta'} = \gamma_{\eta+\eta'}$ (as above, B denotes the algebra of scalars). This fact motivates a natural question (explicitely asked by Hari Bercovici), namely under what conditions one can define convolution powers $\mu^{\boxplus\eta}$ for operator-valued distributions μ and completely positive maps η?

Nica and Speicher found in [8] that any probability measure μ on the real line belongs to a partial free convolution semigroup $\{\mu^{\boxplus t} : t \geq 1\}$. This result, together with the connections of such semigroups to free Brownian motions found in [2], motivated our search for the main result of our recent preprint "Convolution powers in the operator-valued framework" (arxiv:1107.2894v1). We denote by $\Sigma(B)$ the set of B-valued positive conditional expectations from the algebra $B\langle\mathcal{X}\rangle$ freely generated by B and the selfadjoint symbol $\mathcal{X}$ onto B. Then,

Theorem. *If $\mu \in \Sigma(B)$, then the following inclusion holds:*

$$\{\mu^{\boxplus\eta} | \eta \colon B \to B \text{ compeletely positive so that } \eta - 1 \text{ is completely positive}\} \subset \Sigma(B).$$

The set on the left side of the above inclusion is indeed a partial semigroup: $\mu^{\boxplus\eta} \boxplus \mu^{\boxplus\eta'} = \mu^{\boxplus\eta+\eta'}$. If μ is $\boxplus$-infinitely divisible, $\mu^{\boxplus\eta} \in \Sigma(B)$ for any completely positive η. In addition, we obtain among others a correspondence between these partial semigroups and free Brownian motions paralleling the one obtained in [2] for scalar-valued distributions, a subordination formula for operator-valued Cauchy-Stieltjes transforms associated to $\mu^{\boxplus\eta}$ and show that the Cauchy-Stieltjes transform of a free Brownian motion started at a B-valued distribution μ satisfies a B-valued version of the inviscid Burgers equation (the free analogue of the heat equation, as shown in [10]).

4 Outcome of the meeting

The meeting gave us the opportunity to make significant progress in several areas related to the role of subordination in non-commutative probability. At least one paper will be written as a result of our work together during the RIT meeting in Banff.

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