Noncommutative L_p spaces, Operator spaces and Applications

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Overview of the field. Noncommutative L_p-spaces are at the heart of this conference. These spaces have a long history going back to pioneering works by von Neumann, Dixmier and Segal. They are the analogues of the classical Lebesgue spaces of p-integrable functions, where now functions are replaced by operators. These spaces have been investigated for operator algebras with a trace, and then around 1980 generalizations to type III von Neumann algebras have appeared (Kosaki, Haagerup, Terp, Hilsum). These algebras have no trace and therefore the integration theory has to be entirely redone. These generalizations were motivated and made possible by the great progress in operator algebra theory, in particular the Tomita-Takesaki theory and Connes's spectacular results on the classification of type III factors.

Since the early nineties and the arrival of new theories like those of operator spaces and free probability, noncommutative integration is living another period of stimulating new developments. In particular, noncommutative Khintchine and martingale inequalities have opened new perspectives. It is well-known nowadays that the theory of noncommutative L_p-spaces is intimately related with many other fields such as:

- **Operator algebras.** By definition noncommutative L_1-spaces are the preduals of von Neumann algebras. The structure von Neumann algebras is naturally reflected in these spaces. More generally, noncommutative L_p-spaces allow to address many questions related to algebraic structures. For example, in Connes's noncommutative geometry, "Fredholm modules" are defined as elements of a noncommutative L_p-space adapted to the geometry. These different theories form an important field of research in functional analysis and have numerous interactions with other disciplines like algebras, K-theory, mathematical and theoretical physics.

- **Geometry of Banach spaces.** Noncommutative L_p-spaces are a generalization of classical L_p-spaces as well as Schatten classes. They have therefore provided - and continue to provide - many important examples and counterexamples for Banach space theory. Furthermore, their geometrical properties are sometimes crucial for certain problems in mathematical physics (see the works of Lieb and his collaborators).

- **Operator spaces.** This theory is placed at the intersection of the preceding two topics. It is of interest to researchers coming from other subjects such as operator algebras, quantum probability, or Banach spaces. Within this theory noncommutative L_p-spaces are experiencing a strong development. The research in this area is very active and there is a strong international competition. Let us mention the works of Blecher, Junge, Musat, Oikhberg, Ozawa, Randrianantoanina, Rosenthal and Ruan in the United States, Le Merdy, Lust-Piquard, Pisier, Raynaud, Ricard and Xu in France, and Bozejko, Defant, Haagerup, Lindsay and Parcet in the other European countries.

- **Quantum probability.** This field developed from the probabilistic interpretation of quantum mechanics. Its origins can be traced back to Bonn and von Neumann, but it is in the seventies and eighties that it grew into a separate theory through the works of Accardi, Bozejko, Hudson, Meyer, Parthasarathy, von Waldenfels, and many others. Noncommutative L_p-spaces arise naturally in the study of noncommutative martingale and ergodic theories. The work of Biane and Speicher has established a bridge to Voiculescu's free probability and underlines the interactions between this discipline, operator spaces/algebras, and random matrices.

- **Noncommutative harmonic analysis.** Matrix-valued harmonic analysis goes back to the classical works of Wiener-Masani and Helson-Lowdenslager on prediction theory. This led Arveson to introduce his theory of subdiagonal algebras (or noncommutative H_∞-spaces). This line of research continues to be very active, as show the recent works of Blecher-Labuschagne. On the other hand, the investigation of semigroups of completely positive maps on von Neumann algebras has recently shown to be an important tool in the work of Popa-Ozawa and Shlyakhtenko. Noncommutative L_p-spaces allow to formulate problems from classical harmonic analysis in the context of group von Neumann algebras.

- **Quantum information.** The interaction between operator algebras/spaces and quantum information theory is very recent and has brought some intriguing new ideas and concepts. One connection between them is the notion of channels, called completely positive maps in C*-algebras. Completely positive maps have been used in solving long-standing open problems, for example in Kirchbergs works on exact C*-algebras, and more recently in Popas works on rigidity of von Neumann algebras and groups. It is thus natural that quantum information has attracted attention and interest of researchers from operator algebras/spaces. From this point of view, the theory of operator algebras/spaces has many tools to offer. On the other hand, channels are the basic objects used in quantum information to model experiments.

The topics listed above correspond to separate mathematics communities and cultures, which nonetheless present many links and interactions. Noncommutative L_p-spaces have so far not been studied through these strong links to other topics.

Outline of the conference. A wide range of topics have been presented during this conference, including interactions of the theory of operator spaces and quantum probability with the following areas

- Quantum information theory
- Noncommutative harmonic analysis
- The theory of double integrals
- Quantum groups
- Operator algebras in connection with geometric group theory

At the planning stage of this conference it has been by no means clear that the interaction between quantum information theory and operator spaces/operator algebras could be as fruitful as it turns out to be during the conference. With C. King, D. Kribs, A. Winter, and P. Shor well-known researchers in quantum information could be attracted to participate in a conference whose title seem hardly be connected to quantum information theory. However, the Banff researchers center is one of the leading institutions supporting this exciting new area of interaction. Indeed, following up on problem due to A. Winter, and communicated to some fraction of the operator space community during the last workshop on interaction between operator spaces and quantum information, a major problem on tensor products of quantum channels could be solved (see Winter's problem below). The solution to Winter's problem is related to dilation properties of completely positive maps. Quite surprisingly these dilation properties also play an important role in certain aspects of noncommutative analysis. The new results in noncommutative analysis presented in this conference show that certain results from classical analysis are still meaningfull when replacing abelian discrete groups by their noncommutative counterparts. Further tools required in this recent research line are based on quantum probability. Thus, despite of the wide range of problems discussed in this conference, there is in fact a common ground and several connections between some of these topics. Last, but not least, the connection between Banach space and the Hastings' famous solution of the additivity problem also played an important role in this successful conference.

1) Noncommutative analysis and noncommutative probability. Despite its name harmonic analysis is not really the theory of harmonic functions. In fact, classical harmonic analysis is certainly concerned with Fourier analysis, abelian groups, and connection to number theory. In particular, fundamental results in the theory provide the analytic aspects of the Pontrajgin duality, such as Fourier transforms, multipliers, and properties of the heat semigroup in $\mathbb{Z}^n$ and $\mathbb{T}^n$. By now these properties are very well understood, and modern harmonic analysis moved on to different problems. However, by replacing the discrete groups by noncommutative discrete groups, one also has to replace their compact duals by the corresponding noncommutative space. This naturally leads to operator algebras and the corresponding L_p spaces associated with them. In this different setting very little is known. On this conference J. Parcet [13] presented a method, a suitable adaptation of the classical Calderón-Zygmund theory to the setting of finite dimensional cocyles on groups. Through discussions during the conference new mulitplier results for classical groups and Schur multipliers have been found. The talks by U. Haagerup [8] and E. Ricard [12] also concerned Schur-Herz multipliers on discrete groups. They computed the complete bounded norms of radial multipliers.

These results have interesting applications to approximation properties of group von Neumann algebras and their noncommutative L_p spaces. See the paragraph below on Araki-Wood factors and also the recent work of M. de la Salle and V. Loforgue [7].

An important technical tool in the recent works on noncommutative harmonic analysis is the theory of generalized BMO spaces. In classical analysis this space of function with bounded mean oscillation replaces the space of bounded functions as an endpoint for interpolation. A similar role plays the corresponding the Hardy or the weak L_1 space. For BMO spaces this fundamental fact and further applications were presented by T. Mei, based on a series of recent works the first two of which were already published (see [18, 19]). Indeed, Mei invented an intrinsic definition of BMO spaces which seems new even in commutative setting. The interpolation result cannot be obtained from classical methods, but from probabilistic methods related to the dilation problem. In talks delivered by PhD students , S. Avsec gave an illustration of Mei's theory and its application to square functions for finite dimensional cocylces. M. Perrin outlined the theory of noncommutative H_p spaces for martingales with a continuous parameter set. Both talks are based on works in progress. This theory, together with the H_p theory developed by Le Merdy and a subset of the organizers, provides the backbone of the interpolation results.

2) The dilation problem. In operator algebras certain dilations or factorizations results are of fundamental importance. In this conference several talks were motivated by well-known results for positive maps on commutative spaces. The first instance of this result is the construction of a Markov process associated with a positive matrix. More generally Rota could construct a suitable probability measure on the space of pathes of a given set which encodes the transition probability of the underlying map. In the theory of semigroups a similar problem is known as the martingale problem. Here one requires in addition that the measure on the path space is supported on the space of continuous pathes.

A similar question arises for completely positive maps on operator algebras, in particular on matrix algebras. The addition"completely" here means that the map is positivity preserving even after adding an environment. M. Musat presented the surprising result (obtained with U. Haagerup; see [9]) that not all completely positive maps are factorizable. The notion of factorizability was invented by C. Anantharaman-Delaroche who showed that factorizable maps admit a Rota (or Markov) dilation. Due to the work of Haagerup and Musat the condition for a map to be factorizable is now very well understood and many examples are available. Moreover, the link to the notion of "dilation" previously studied by Kümmerer is clarified and the relation between Kümmerer's examples and the new counterexamples is now clear.

Y. Dabrowski [6] (a PhD student) presented a result showing that for certain symmetric semigroups the dilation problem has a positive solution. Using slightly different methods a similar result has been obtained by one of the organizers and his coauthors (work in progress by M. Junge, E. Ricard and D. Shlyakhtenko).

It is probably fair to say that the Markov dilation problem for completely positive maps is now completely clarified, and this conference (and the previous conference on operator spaces at CIRM in Luminy) played an important role.

3) Winter's problem. The starting point of Winter's problem is Birkhoff's classical theorem on the characterization of extreme points of the set of double stochastic matrices. Indeed, these extreme points are given by permutation matrices. In the context of quantum channels the analogue is a characterization of completely positive trace preserving maps. This characterization has been obtained by Choi, and there are recent results by Wolff on this subject. The nice aspect about the classical results is that permutation matrices are given by unitary maps. Winter's problem itself is not about an individual channel, but about the n-fold tensor product. Winter asks whether eventually the cb distance to the convex hull of unitary channels goes to 0 (see [20]; see also the Report of the workshop on Operator structures in quantum information theory at BIRS, February 11-16, 2007). By the results of Haagerup and Musat the answer to this problem is negative. Indeed, those trace preserving completely positive maps which are not factorizable can never be in convex hull of unitaries and even not their tensor powers.

The history of this problem is a nice Banff success story because Winter's problem become known to the operator space community on a prevrious meeting and then has been communicated to U. Haagerup by V. Paulsen, a participant of all the workshops in Banff connecting operator space theory and quantum information. Now, the solution had been presented again in a Banff workshop, and has since been picked up by the quantum information community.

4) Approximation problem for Araki-Wood factors. Araki-Wood factors have been introduced in the late sixties by physicists in the context of calculating thermodynamical limits of large systems. Since then they played a prominent role in operator algebras because they served as a model for Connes' classification and the interesting invariants extracted from this work. More recently, D. Shlyakhtenko studied the free analogue of Araki-Wood factors and their properties. For example, in the context of S. Popa's rigidity theory, it is important to know whether these factors have the approximation property, more precisely the completely contractive approximation property. Following the techniques for the free group this could be established under the additional assumption that the modular group of these factors is almost periodic. The case of continuous spectrum remained open. The solution to this problem by C. Houdayer and E. Ricard [12] had been presented by a talk of Ricard in this conference. The beautiful aspect of the solution is the clever use of Schur multipliers in a suitable larger algebra which then drop down to very useful maps on the factor.

5) Additivity conjecture and connections to Quantum Information. The additivity conjecture for the minimal entropy of quantum channels has recently been solved by M. Hastings [10]. This posed a longstanding problem in Quantum Information Theory. A number of talks were dedicated to this problem. New proofs and a better insight to why these counterexamples are positive were presented. Essentially there seem to be two avenues to simplify or extends Hastings' approach which has its roots in the work of Haydon and Winter [20]. The first approach uses a basic but important observation which encodes a quantum channel, a completely positive trace preserving map, via a subspace of the tensor product of two Hilbert spaces. In this tensor product one may consider different norms induced by the Schatten norms. The way this approach is set up is that for $p = 2$ one finds the range of the partial isometry implementing the channel. However, the minimal entropy might be considered as a limit for p tending to 2 of the structure of these spaces with the inherited Schatten norm. Very small entropy is obtained if the norm for $p > 2$ is almost constant. But this corresponds to an almost isometric embedding of a Hilbert space in a Schatten p-class. Almost isometric embedding of Hilbert spaces in arbitrary Banach spaces have been studied in Banach space theory very intensively. This topic is particularly tied to the seminal work of Dvoretzky based on random techniques. S. Szarek has used and extended many of these techniques in particular in the connection with Gluskin spaces, estimates for eigenvalues of random matrices and many other topics. For fixed $p > 2$ one can find counterexamples (as observed by Haydon and Winter) simply by applying the known results for Dvoretzky's for Schatten p classes. However, Hastings's result even requires an improvement of standard Banach space result with better estimates of the error. In this case the desire to find a conceptual proof of Hastings's result lead to a deeper understanding of problems central to Banach space theory. S. Szarek clearly exposed in this conference his recent works [1, 2] with G. Aubrun and E. Werner on this subject.

B. Collins' talk presented another approach to these results relying on estimates from free probability, based on his joint works with I. Nechita [3, 4, 5]. These results provide new insights in the fine structure of random unitaries and random projections using deep combinatorial tools.

A different family of connections between operator space theory and quantum information theory concerns Bell inequalities and their generalizations. Bell inequalities are in the heart of the classical paper by A. Einstein, B. Podolsky, N. Rosen, and have ever since been used to indicate why quantum mechanics is not compatible with locality. In a talk by C. Palazuelos [15] new results on Bell inequalities and examples of violation were presented which provide asymptotically large violation in high dimension. In a talk by V.B. Scholz [14] the connection between Tsirelson's problem on calculating quantum probabilities with arbitrary commutating POVM's and tensor product structures was shown to follow from a positive solution of Connes' famous embedding problem. If in addition matrix valued coefficients are considered then Tsirelson's conjecture and Connes' embedding problem are indeed equivalent.

In addition to these talks there were additional informal talk on the connection of Entropy and Banach space or Operator space techniques in quantum information theory.

6) Additional Highlights It is a tradition in Operator Space Theory to include speakers from related topics, but not necessarily core subjects. This include beautiful talks of U. Haagerup and M. Bozejko on Schur multipliers and the connection to combinatorial objects.

The broad spectrum of talks was complemented by contributions on logmodular algebras (V. Paulsen), Lieb-Robinson bounds on operator inequalities based on deep results of Hastings (E. Carlen), operator space structure of certain multiplier spaces and Hausdorff-Young inequality in quantum groups (Z-J. Ruan and his student T. Cooney) and the recent solution by F. Sukochev and D. Potapov [16, 17] on bounds for double operator integrals.

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