# New Perspectives in Univariate and Multivariate Orthogonal Polynomials (10w5061)

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## 1 Overview of the Field

Given a finite positive measure $\mu$ on the real line, with infinitely many points in its support, we can define orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfying, for all $m, n \geq 0$,

$$\int p_n p_m d\mu = \delta_{mn}.$$

Here

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

is a polynomial of degree $n$, with positive leading coefficient $\gamma_n$. The $\{p_n\}$ may be generated by the Gram-Schmidt process, applied to the monomials $1, x, x^2, ...$ with inner product

$$(f, g) = \int fg\, d\mu.$$

Orthonormal polynomials have been the subject of investigation for over 150 years. They have applications in areas ranging from statistical physics to combinatorics to signal processing. There are obvious links to special functions and harmonic and numerical analysis.

The notion of an orthogonal polynomial has been greatly generalized in recent decades. While the extension to measures on the plane is obvious, multivariate analogues already present the problem of how to order monomials in higher dimensions. Then there are important generalizations to the case where a single orthogonality relation is replaced by one involving more than one measure, or more than one polynomial.

Active intrinsic topics of study include analytic and algebraic aspects, and asymptotics. Applications to areas like random matrices and numerical analysis have given new insight into orthogonal polynomials, and their generalizations. Some key references are [14], [16], [18], [26], [39], [40], [41], [43].

## 2 Topics Covered by the Workshop

### 2.1 Measures on the Real Line

A classical result of Szegő asserts that when $\mu$ is an absolutely continuous measure supported on $[-1, 1]$, with

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty,$$

then

$$\lim_{n\to\infty} \frac{p_n(z)}{\left(z+\sqrt{z^2-1}\right)^n} = D(z),$$

uniformly for $z$ in closed subsets of $\mathbb{C}\setminus[-1,1]$. Here $z+\sqrt{z^2-1}$ is the conformal map of $\mathbb{C}\setminus[-1,1]$ onto $\mathbb{C}\setminus\{z:|z|\leq 1\}$, and $D(z)$ is the Szegő function for $\mu$. The case where $[-1,1]$ is replaced by finitely many intervals was considered by Harold Widom in a celebrated paper [45]. The case of infinitely many intervals (and more general sets of homogeneous type) was considered more recently by Peherstorffer, Sodin, and Yuditskii. At the workshop, Jacob Christiansen presented asymptotics, and related conjectures, for the case where the support is still more general than a set of homogeneous type.

Lilian Manwah Wong discussed the problem of adding point masses to a given measure on the real line, and comparing the asymptotics of the new orthogonal polynomials, and related quantities, to those for the original measure. Christian Remling discussed reflectionless measures, with applications to extensions of the Denisov-Rakhmanov Theorem relating recurrence coefficients of orthogonal polynomials, and the support of the measure. Vilmos Totik presented new methods for establishing asymptotics for Christoffel functions on the real line (which also work over arcs in the plane), and consequences in approximation theory. Avram Sidi presented asymptotics for coefficients in Legendre expansions, and related quadrature errors, when the underlying functions have certain types of singularities.

Sasha Aptekarev showed how to use sophisticated analysis of recurrence relations, to derive specific types of Plancherel-Rotach asymptotics for discrete orthogonal polynomials such as Meixner polynomials. As a consequence, the local behavior of the reproducing kernels is obtained. Mourad Ismail discussed the J-Matrix method introduced in the 70's to study the spectrum on Schrödinger operators in physics. It is a tridiagonalization technique and Ismail [19] discussed how to make the technique rigorous and apply it to study orthogonal polynomials.

## 2.2 Orthogonal Relations in the Complex Plane

Weighted Bergman polynomials involve an orthogonality relation against an area measure, rather than an arc. Let $G$ be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary is a Jordan curve, let $w$ be a function positive on $G$, and define $\{p_n\}$ by the Hermitian relation

$$\int_G p_n(z)\,\overline{p_m(z)}w(z)\,dA(z)=\delta_{mn},$$

where $A$ stands for area measure. The classical Bergman case is the unweighted case $w=1$. There are many unresolved questions concerning the behavior of the polynomials and their zeros, for example, when the boundary of $G$ is not a smooth curve. Laurent Baratchart presented asymptotics for $\{p_n\}$ when $G$ is the unit disk when weak assumptions are made about $w$, involving its behavior on the circle $|z|=r$ as $r\to 1-$. Erwin Miña-Diaz considered the case where $G$ is the disk, and $w=|h|^2$, for some polynomial $h$.

Nikos Stylianopoulos discussed the case $w=1$, for domains with piecewise analytic boundary - but without cusps. An especially interesting case with cusps is the hypocycloid. Nikos presented joint work with Ed Saff for this region, showing how certain Hessenberg matrices approach Toeplitz matrices associated with Faber polynomials. He also presented a very interesting application to the Arnoldi process for numerical calculation of orthogonal polynomials, showing its stability in comparison to the classical Gram-Schmidt process. A further conclusion is that "finite term" recurrence relations for Bergman polynomials do not hold, except in the essentially "trivial" case where the region is bounded by an ellipse. Thus for most regions, the associated Hessenberg matrices are not banded.

There is a close connection between Padé approximation and orthogonal polynomials. Let $f(z)$ be a function admitting an expansion about $\infty$ in negative powers of $z$:

$$f(z)=\sum_{j=1}^{\infty}f_j z^{-j}$$

The $(n-1,n)$ Padé approximant to $f$ is a rational function $\pi_n$ of type $(n-1,n)$ satisfying, as $z\to\infty$,

$$f(z)-\pi_n(z)=O\left(z^{-2n}\right).$$

In the case where $f$ is a Markov function, the denominator polynomial in $\pi_n$ is an orthogonal polynomial. In the general case, the denominator still satisfies a non-Hermitian orthogonality relation, arising from the matching condition. Maxim Yattselev presented asymptotics for the Padé approximants associated with certain types of elliptic functions,

$$f(z) = \frac{1}{i\pi} \int_{\Delta} \frac{h(t)}{t-z} \frac{dt}{\sqrt{(t-a_0)(t-a_1)(t-a_2)(t-a_3)}_+}.$$

Here $\Delta$ is a collection of three arcs joining the point $a_0$ to the non-collinear points $a_1, a_2, a_3$, taken to have minimal capacity, and $h$ satisfies a suitable Dini condition. Riemann-Hilbert techniques are used to obtain the asymptotics.

Of course, Padé approximants are a special type of rational approximation. Vasiliy Prokhorov discussed results on best rational approximation, derived via elaborations and extensions of the Adamjov-Arov-Krein theory.

## 2.3 Potential Theory in One and More Variables

The link between polynomials and potentials is easily seen from the relation

$$\frac{1}{n} \log \left| \prod_{j=1}^{n} (z - z_j) \right| = \int \log |z - t| \, d\nu(t), \tag{1}$$

where $\nu$ places mass $\frac{1}{n}$ at each of the $z_j$. The function

$$U^{\nu}(z) = \int \log |z - t|^{-1} d\nu(t)$$

is the potential associated with the measure $\nu$.

Joe Ullman was a pioneer in using potential theory to analyze asymptotics of orthogonal polynomials for compactly supported measures. It was Hrushikesh Mhaskar, Evgenii Rakhmanov, and Ed Saff who developed its use for the case of measures with non-compact support, and for varying measures [36]. Thus if $d\mu(x) = e^{-x^2} dx$ is the Hermite weight, one looks for a probability measure $\nu$ with compact support, such that

$$U^{\nu}(x) = -x^2 + \text{constant}, \; x \in \text{supp}[\nu].$$

More generally, if the field $x^2$ is replaced by $Q(x)$, one replaces $x^2$ by $Q(x)$ in this last identity. The measure $\nu$ is called an equilibrium measure for the external field $Q$. The support of $Q$, and the properties of $\nu$, are crucial in analyzing orthogonal polynomials. At the conference, Benko and Dragnev gave new conditions for convexity of $\nu'$ using an elaboration of the iterated balayage algorithm, which they call ping pong balayage.

Another powerful application of potential theory was given by Igor Pritsker. He derived discrepancy estimates, in the spirit of the Erdős-Turán theorem, but instead involving discrete energies. As a consequence a classic problem of means of zeros of integer polynomials was analyzed, and surprising restrictions were given for growth of integer polynomials in the disk.

Potential theory in the multivariate case is far more challenging than the univariate case. There is no longer such a simple relationship between polynomials and potentials like (1). A fascinating account of recent developments was given by Norman Levenberg [25]. He showed how the complex Monge-Ampere operator arises in both weighted and non-weighted multivariate potential theory. Concepts of weighted transfinite diameter, and Fekete points, and $L_2$ approximations to equilibrium measures were discussed. Using deep recent results of Berman and Boucksom [3], [4] from a more abstract setting, it was shown that appropriate discrete approximations to equilibrium measures converge weakly, as the number of points grows to infinity. In particular, this is the case for Fekete points.

Tom Bloom showed how the same tools of pluripotential theory can be applied to discuss large deviations for random matrices, and give alternative tools and insights to those typically used in the theory of random matrices.

## 2.4 Universality Limits and Riemann-Hilbert Problems

It was the physicist Eugene Wigner who in the 1950's first used eigenvalues of random matrices to model the interactions of neutrons for heavy nuclei. One classical setting can be described as follows: let $\mathcal{M}(n)$ denote the space of $n$ by $n$ Hermitian matrices $M = (m_{ij})_{1 \le i,j \le n}$. Consider a probability distribution on $\mathcal{M}(n)$,

$$P^{(n)}(M) = cw(M)\left(\prod\nolimits_{j=1}^{n} dm_{jj}\right)\left(\prod\nolimits_{j<k} d(\operatorname{Re} m_{jk})\, d(\operatorname{Im} m_{jk})\right).$$

Here $w(M)$ is a function defined on $\mathcal{M}(n)$, and $c$ is a normalizing constant. One important case is $w(M) = \exp(-2n \operatorname{tr} Q(M))$, involving the trace tr, for appropriate functions $Q$ defined on $\mathcal{M}(n)$. In particular, the choice $Q(M) = M^2$, leads to the Gaussian unitary ensemble, apart from scaling, that was considered by Wigner. One may identify $P^{(n)}$ above with a probability density on the eigenvalues $x_1 \le x_2 \le ... \le x_n$ of $M$,

$$P^{(n)}(x_1, x_2, ..., x_n) = c\left(\prod_{j=1}^{m} w(x_j)\right)\left(\prod\nolimits_{i<j}(x_i - x_j)^2\right).$$

See [10, p. 102 ff.]. Again, $c$ is a normalizing constant.

It is at this stage that orthogonal polynomials arise [10]. Let $\mu$ and $\{p_n\}$ be as above. The $n$th normalized reproducing kernel for $\mu$ is

$$\widetilde{K}_n(x, y) = \mu'(x)^{1/2}\mu'(y)^{1/2}\sum_{j=0}^{n-1} p_j(x)\, p_j(y).$$

When $\mu'(x) = e^{-2nQ(x)}dx$, there is the basic formula for the probability distribution $P^{(n)}$ [10, p.112]:

$$P^{(n)}(x_1, x_2, ..., x_n) = \frac{1}{n!}\det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \le i,j \le n}.$$

One may use this to compute a host of statistical quantities - for example the $m-$point correlation function for $M(n)$ [10, p. 112]:

$$\begin{aligned} R_m(x_1, x_2, .., x_m) &= \frac{n!}{(n-m)!}\int ... \int P^{(n)}(x_1, x_2, ..., x_n)\, dx_{m+1}\, dx_{m+2} ... dx_n \\ &= \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \le i,j \le m}. \end{aligned}$$

The *universality limit in the bulk* asserts that for fixed $m \ge 2$, and $\xi$ in the interior of the support of $\{\mu\}$, and real $a_1, a_2, ..., a_m$, we have

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{\tilde{K}_n(\xi,\xi)^m} R_m\left(\xi + \frac{a_1}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi,\xi)}, ..., \xi + \frac{a_m}{\tilde{K}_n(\xi,\xi)}\right) \\ = \ &\det\left(\frac{\sin\pi(a_i - a_j)}{\pi(a_i - a_j)}\right)_{1 \le i,j \le m}. \end{aligned}$$

Of course, when $a_i = a_j$, we interpret $\frac{\sin\pi(a_i - a_j)}{\pi(a_i - a_j)}$ as 1. Because $m$ is fixed in this limit, this reduces to the case $m = 2$, namely

$$\lim_{n\to\infty}\frac{\tilde{K}_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \frac{\sin\pi(a-b)}{\pi(a-b)}. \qquad (2)$$

There are a variety of methods to establish (2). The deepest methods are the Riemann-Hilbert methods, which yield far more than universality [10], [11]. The whole topic of universality limits was dramatically advanced by Riemann-Hilbert experts, and they also communicated the topic to others, including those using more classical techniques to analyze orthogonal polynomials. We note that there are several settings, other than that described above, for universality limits for random matrices [42].

How general is (2), that is what restrictions are need on $\mu$? Here is a

**Conjecture**
*Let $\mu$ be a measure with compact support. Then for a.e. $\xi \in \{\mu' > 0\}$, we have (2).*

Here, of course, $\{\mu' > 0\} = \{\xi : \mu'(\xi) > 0\}$. The most general pointwise result to date towards this conjecture is due to Vili Totik [44]. He showed that if $\mu$ is a regular (in the sense of Ullman, Stahl, and Totik) measure with compact support, and $(c, d)$ is an interval such that

$$\int_c^d \log \mu' > -\infty,$$

then, indeed, (2) holds for a.e. $\xi \in (c, d)$. Barry Simon established a similar result when the support has finitely many intervals. Another recent development, presented by Doron Lubinsky at the workshop [28], is that without local or global regularity, universality holds in measure.

One cannot in general expect that universality with the sinc kernel holds at points where $\mu'(x) = 0$. For example, at the edges of the support of $\mu$, when the support consists of finitely many intervals, one instead obtains the Bessel kernel. At interior points where $\mu'$ has a jump discontinuity, Martinez et al discovered that one obtains a new, non-classical kernel. This suggests that universality with the since kernel is associated with points where $\mu'$ exists and is positive. A very interesting result of Breuer, presented for the first time at the Banff conference, was that there are measures $\mu$ with support $[-1, 1]$, that are purely singularly continuous, and yet universality with the sinc kernel holds at each point of $(-1, 1)$. This surprising result is obtained by sparsely perturbing the recursion relation of classical Chebyshev polynomials.

Of course universality goes way beyond measures with compact support, or even varying measures. This was powerfully illustrated by the talk of Arno Kuijlaars. In modelling the Brownian motion of particles that start at time $t = 0$ from a finite number of given points, and end at time $t = 1$ at a finite number of points, while following non-intersecting paths, one is led to mixed type multiple orthogonal polynomials. In analyzing the asymptotics of these, one use Riemann-Hilbert problems of larger size, such as $4 \times 4$ matrices for the case of two start and end points. In contrast, classical orthogonal polynomials require only $2 \times 2$ matrices. Kuijlaars illustrated the depth of techniques required for the analysis, and the new universality phenomena that arise, often described using solutions of Painlevé equations.

## 2.5 Sobolev Orthogonal Polynomials

Sobolev orthogonal polynomials are polynomials whose orthogonality relation involves derivatives. Thus we might search for polynomials $\{p_n\}$ that satisfy, for example,

$$\int p_n(x) p_m(x) d\mu(x) + \int p_n'(x) p_m'(x) d\nu(x) = \delta_{mn},$$

where $\mu$ and $\nu$ are positive measures. Higher derivatives could also be involved. They arise in a number of applications, and have received substantial attention in recent decades [1], [30]. An obvious question is how the measures $\mu$ and $\nu$ interact. In many standard cases, the dominant term is provided by the derivative term, and $p_n'$ behaves roughly like an orthogonal polynomial for the measure $\nu$. In other cases, however, there is interaction between the two terms.

At the workshop, Paco Marcellan considered the case when $\mu$ has unbounded support, while $\nu$ is a Dirac delta, or sum thereof. Issues such as zeros, asymptotics, comparison to the compact case were considered. A multivariate version of these was discussed by Miguel Pinar, with the derivative being replaced by a gradient.

## 2.6 Multiple Orthogonal Polynomials

Given measures $\{\mu_j\}_{j=1}^p$ on the real line, and a $p$-tuple of integers $(n_1, n_2, ..., n_p)$, the type II multiple orthogonal polynomial $P$ is a monic polynomial of degree $n_1 + n_2 + ... + n_p$ such that for $j = 1, 2, ..., p$, and $0 \leq k \leq n_j - 1$,

$$\int P(x)\, x^k d\mu_j(x) = 0.$$

The dual type I polynomials $A_1, A_2, ..., A_p$ are determined by the conditions

$$\int x^k \left( \sum_{j=1}^{p} A_j d\mu_j(x) \right) = 0,$$

for $0 \leq k \leq n_1 + n_2 + ... + n_p - 2$, with degree$(A_j) \leq n_j - 1$.

Multiple orthogonal polynomials have connections to rational approximation in the complex plane, to diophantine approximation in number theory, and to random matrix ensembles. Bill Lopez presented powerful results on Nikishin systems for two intervals, finding probability measures, and associated multiple orthogonal polynomials that satisfy a recurrence relation of order 4. Walter Van Assche showed how potential theory, Riemann-Hilbert (and other) methods can be used to analyze asymptotics of multiple orthogonal polynomials. Arno Kuijlaars exhibited the use of multiple orthogonal polynomials in non-intersecting Brownian motions.

# 3 Multivariate polynomials

From the orthogonality relation it follows that any family of orthogonal polynomials on the real line satisfies a three term recurrence relation:

$$a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) = x p_n(x).$$

The classical orthogonal polynomials (Jacobi, Hermite, Laguerre, Bessel) which appear in numerous applications in mathematics and physics are characterized by the fact that they are eigenfunctions of a differential operator, which is independent of the degree $n$. In other words the classical orthogonal polynomials are characterized by a bispectral problem [13] since they satisfy a second-order difference equation in the degree variable $n$ and a differential equation in the variable $x$. The construction of bispectral orthogonal polynomials in higher dimensions brought different new tools from combinatorics, representation theory and integrable systems into this old classical area.

## 3.1 Orthogonal polynomials associated with root systems

One possible extension of the above theory to orthogonal polynomials of more than one variable is related to the theory of symmetric functions and the corresponding Macdonald-Koornwinder polynomials [29, 22]. These polynomials were introduced as the unique eigenfunctions of certain remarkable commuting symmetric difference operators. Each family depends on a root system and several free parameters. Special cases lead to classical families of symmetric functions such as Schur functions and characters of corresponding Lie groups, Hall-Littlewood functions, Jack polynomials, or more generally, the multivariate Jacobi polynomials due to Heckman and Opdam [17]. The bispectrality in this case is closely related to the Macdonald conjectures which were established with the theory of double affine Hecke algebras [8]. Recently, there has been a major development in this field leading to biorthogonal elliptic functions generalizing Macdonald-Koorwinder polynomials [33]. In particular, one needs to work with generalized eigenvalue problems which require several new techniques. The latest progress in this beautiful theory was described by Eric Rains who outlined the main ingredients of the construction and the crucial properties. Tom Koornwinder studied the nonsymmetric Askey-Wilson polynomials as vector-valued polynomials. As a particular new result made possible by this approach he obtained positive definiteness of the inner product in the orthogonality relations, under certain constraints on the parameters.

## 3.2 Orthogonal polynomials in $\mathbb{R}^d$

Yuan Xu discussed a discrete Fourier analysis on the fundamental domain of $A_d$ lattice that tiles the Euclidean space by translation [27]. In particular, Chebyshev polynomials can be defined using symmetric and antisymmetric sums of exponentials. One of the interesting outcomes of this theory is the construction of Gaussian cubatures, which exist very rarely in higher dimension.

Bispectral properties of orthogonal polynomials within the usual framework [14] of orthogonal polynomials in $\mathbb{R}^d$ attracted a lot of attention recently. Interesting examples of such polynomials go back to the multivariate Hahn and Krawtchouk polynomials in the pioneering works of Karlin and McGregor [20] and Milch [31] related to growth birth and death processes. A probabilistic model that involves cumulative Bernoulli trials led Hoare and Rahman to a new family of 2D Krawtchouk polynomials. In his talk, Mizan Rahman derived a 5-term recurrence relation, thus showing that these polynomials possess the bispectral property. He also indicated possible extensions to 3 or more variables. Paul Terwilliger explained how the recurrence formulas for the same polynomials can be derived using the Lie algebra $\mathfrak{sl}_3$. George Gasper considered general methods for the derivation of second-order partial difference equations. Alberto Grünbaum illustrated with examples the interaction between orthogonal polynomials and random walks. Plamen Iliev discussed a new characterization of the commutative algebras of ordinary differential operators that have orthogonal polynomials as eigenfunctions, which leads to multivariate extensions. Luc Vinet showed that the $d$-orthogonal Charlier and Hermite polynomials appear naturally as matrix elements of nonunitary transformations corresponding to automorphisms of the Heisenberg-Weyl algebra, thus establishing duality, recurrence, and difference equations.

Greg Knese described recent results [21] on polynomials orthogonal on the bi and poly circle and their relation to bounded analytic functions on the polydisk. Important in this work is a Christoffel-Darboux like formula which in the bivariate case can be related to stable polynomials, Bernstein-Szegő measures and gives a new proof of Ando's celebrated theorem in operator theory. Geronimo [6] discussed a new proof of Gasper's theorem on the positivity of sums of triple products on Jacobi polynomials. This theorem plays an important role in setting up a convolution structure for Jacobi polynomials. The new techniques are based on a correlation operator which was discovered by Carlen, Carvahlo, and Loss in their solution of the spectral gap problem in the Kac model. The correlation operator is an operator on the N-sphere looking for its eigenfunction expansion in various angular momentum sectors leads to Gasper's Theorem and to the Koornwinder-Schwartz product formulas for the biangle This is an extension of Gasper's theorem to the bivariate case.

# 4 Connections with integrable systems and algebraic geometry

One of the landmarks in the modern theory of integrable systems is the work of Sato-Sato [37] which assigns a solution to the Kadomtsev-Petviashvili (KP) hierarchy to each point of a certain infinite dimensional Grassmannian. The construction uses the so called $\tau$-function, which defines a Baker-Akhiezer function via the formula:

$$\psi(t,z) = \frac{\tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \dots)}{\tau(t)} \exp\left(\sum_{k=1}^{\infty} t_k z^k\right),$$

where $t = (t_1, t_2, \dots)$ are the KP flows. Important examples of $\tau$-functions are the Schur functions, the Riemann $\theta$-function (appropriately evaluated and multiplied by a quadratic exponential factor in the time variables) and the partition function of the two-dimensional gravity [23, 38, 46]. John Harnad reviewed this construction with an emphasis on the algebro-geometric solutions of Krichever [24]. He discussed the subtle question of determining the Plücker coordinates appearing in the expansion of the $\tau$-function as an infinite linear combination of Schur functions.

An interesting link between Krichever's work and the polynomials associated with root systems was discovered in [7], by constructing a mutivariate Baker-Akhiezer function for specific values of the free parameters in the Macdonald-Koornwinder operators. In particular, this approach can be used to prove the bispectrality uniformly for all root systems as well as for certain deformations where other techniques (e.g. Hecke algebras) do not seem to be applicable. Oleg Chalykh explained the main ingredients of this connection and derived new orthogonality relations for the Baker-Akhiezer functions.

# 5 Outcome of the Meeting

The conference led to several unusual interactions: between researchers in the abstract special function side, and those on the analysis side; between those studying orthogonal polynomials of a single variable, and those studying many variables; between those studying multivariate polynomials from a real angle, and those studying from a multivariate complex angle; and between those applying potential theory in one variable, and practitioners of the multivariate theory. In addition, there were numerous interactions within individual topics.

Several recent doctorates expressed the belief that their research horizons expanded. Participants agreed that they learnt a lot about the broader field. This was especially the case for univariate researchers, who learnt a lot about multivariate potential theory, and the general multivariate settings.

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