Problems in Pluripotential Theory

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1 Overview of the Field

Let K be a compact set in $\mathbf{C}^n$. A central object of study in potential theory ($n = 1$) and in pluripotential theory ($n > 1$) is the *pluricomplex Green function*:

$$V_K(\mathbf{z}) := \sup\{\frac{1}{\deg(p)} \log |p(\mathbf{z})| : ||p||_K \leq 1,\ p \text{ (holomorphic) polynomial}\}.$$

The uppersemicontinuous regularization $V_K^*(\mathbf{z}) := \limsup_{\zeta \to \mathbf{z}} V_K(\zeta)$ is either identically $+\infty$, if K is pluripolar; i.e., $K \subset \{\mathbf{z} : u(\mathbf{z}) = -\infty\}$ for some $u \not\equiv -\infty$ which is plurisubharmonic on a neighborhood of K, or else V_K^* is plurisubharmonic in $\mathbf{C}^n$. For $K \subset \mathbf{C}^n$ compact, the *polynomial hull of* K is the set

$$\hat{K}_P := \{\mathbf{z} \in \mathbf{C}^n : |p(\mathbf{z})| \leq ||p||_K \text{ for all polynomials } p\}$$
$$= \{\mathbf{z} \in \mathbf{C}^n : V_K(\mathbf{z}) = 0\}$$

(thus $V_K = V_{\hat{K}_P}$) while the *projective hull of* K (cf. [3]) is the set

$$\hat{K} := \{\mathbf{z} \in \mathbf{C}^n : \exists\, C_{\mathbf{z}} \text{ with } |p(\mathbf{z})| \leq C_{\mathbf{z}}^{\deg p} ||p||_K \text{ for all polynomials } p\}$$
$$= \{\mathbf{z} \in \mathbf{C}^n : V_K(\mathbf{z}) < +\infty\}.$$

Wermer [7] showed that if γ is a real-analytic curve in $\mathbf{C}^n$, then $\hat{\gamma}_P \setminus \gamma$ is a one-dimensional, complex-analytic subvariety of $\mathbf{C}^n \setminus \gamma$. The projective hull is a notion which, *a priori*, is defined for closed subsets K of $\mathbf{P}^n$; if $K \subset \mathbf{P}^n$ is contained in an affine $\mathbf{C}^n \subset \mathbf{P}^n$, then the portion of this more general notion of the projective hull for subsets of $\mathbf{P}^n$ that lies in $\mathbf{C}^n$ coincides with our definition of the projective hull for subsets of $\mathbf{C}^n$. Harvey and Lawson [3] conjectured that if γ is a real-analytic curve in $\mathbf{P}^n$, then $\hat{\gamma} \setminus \gamma$ is a one-dimensional, complex-analytic subvariety of $\mathbf{P}^n \setminus \gamma$.

Clearly the projective hull is interesting only if K is pluripolar. Since there exist C^∞ curves γ in $\mathbf{C}^n$ which are *not* pluripolar [2], the Harvey-Lawson assumption that γ be real-analytic is natural. The projective hull is a subtle object. For example, a fascinating result of Sadullaev [5] implies that if A is a connected, pure $m-$-dimensional complex-analytic subvariety of $\mathbf{C}^n$, $1 \leq m \leq n-1$, and if $K \subset A$ is compact and not pluripolar in A^{reg} (the regular points of A), then $A \subset \hat{K}$ if and only if A is algebraic.

Unwinding the definitions, the condition that $\mathbf{z}_0 \in \hat{K}_P$ says that $|p(\mathbf{z}_0)| \leq ||p||_K$ for all polynomials $p(\mathbf{z})$ while the condition that $\mathbf{z}_0 \in \hat{K}$ says that

$$|p(\mathbf{z}_0)| \leq C_{\mathbf{z}_0}^{\deg p} ||p||_K \tag{1}$$

for all polynomials $p(\mathbf{z})$ where $C_{\mathbf{z}_0} = e^{V_K(\mathbf{z}_0)}$. These growth estimates provides some motivation for the results and questions below.

2 Recent Developments and Open Problems

An old result of Rudin [4] can be paraphrased as follows: let $\Delta := \{z \in \mathbf{C} : |z| < 1\}$ denote the unit disk in $\mathbf{C}$ and let $\phi \in C(\bar{\Delta})$, i.e., ϕ is a continuous, complex-valued function on $\bar{\Delta}$. Consider the vector space

$$\mathcal{M} := \{a + b\phi : a, b \text{ (univariate, holomorphic) polynomials}\}. \tag{2}$$

Suppose for all $z_0 \in \Delta$,

$$|f(z_0)| \leq ||f||_T := \max_{|\zeta|=1} |f(\zeta)| \text{ for all } f \in \mathcal{M}.$$

Then ϕ is holomorphic in Δ. Wermer considered a weak version of this maximum principle hypothesis:

$$\text{For all } z_0 \in \Delta, \text{ there exists } C_{z_0} \text{ such that} |f(z_0)| \leq C_{z_0} ||f||_T \text{ for all } f \in \mathcal{M}. \tag{3}$$

Under the additional assumption that $\phi|_T$ be real-analytic, he reached the same conclusion as Rudin. Note that in the setting of (2), condition (3) becomes

$$|a(z_0) + b(z_0)\phi(z_0)| \leq C_{z_0} ||a + b\phi||_T \text{ for all polynomials } a, b. \tag{4}$$

Now suppose $\phi \in C(\Delta \setminus \{0\})$. We let

$$\gamma := \{(z, \phi(z)) : |z| = 1\}$$

and

$$\Sigma := \{(z, \phi(z)) : 0 < |z| < 1\}.$$

Consider the condition that $\Sigma \subset \hat{\gamma}$. This says that for each $0 < |z_0| < 1$, there exists a constant C_{z_0} with

$$|p(z_0, \phi(z_0))| \leq C_{z_0}^{\deg p} ||p(\cdot, \phi(\cdot))||_T = C_{z_0}^{\deg p} ||p||_\gamma$$

for all polynomials $p = p(z, w)$. Wermer [8] observed that *if ϕ is meromorphic in Δ then $\Sigma \subset \hat{\gamma}$.* He conjectured that the following converse-type result was true: *given $\phi \in C(\bar{\Delta} \setminus \{0\})$ with ϕ real-analytic on T, if $\Sigma \subset \hat{\gamma}$, then ϕ is meromorphic in Δ.* The real-analyticity of ϕ on T was assumed to ensure that γ be pluripolar. Note that (4) is related to this projective hull hypothesis in the sense that (4) is (1) at the point $(z_0, \phi(z_0))$ for polynomials $p(z, w)$ which have degree at most one in w and with C_{z_0} to the first power.

3 Presentation Highlights

Since this was a "Research in teams" assembly there were no formal presentations.

4 Scientific Progress Made

We proved a generalization of the existing version of the Rudin and Wermer results.

Theorem. *Let F be a finite subset of Δ and let $\phi \in C(\bar{\Delta} \setminus F)$. The following are equivalent:*

1. *ϕ is meromorphic on Δ;*

2. *for each* $z_0 \in \Delta \setminus F$ *there exists* C_{z_0} *such that* $|a(z_0)+b(z_0)\phi(z_0)| \leq C_{z_0}||a+b\phi||_T$ *for all polynomials* a, b.

A deep result of Shcherbina [6] states that if $\Omega \subset \mathbf{C}$ is a domain and $f\colon \Omega \to \mathbf{C}$ is a continuous function with $\{(z, f(z)) : z \in \Omega\} \subset \mathbf{C}^2$ pluripolar, then f is holomorphic. Using this, we can show:

Proposition. *Let* $\phi \in C(\bar{\Delta} \setminus \{0\})$. *Suppose* γ *is pluripolar and* $\Sigma \subset \hat{\gamma}$. *Then* ϕ *is holomorphic on* $\Delta \setminus \{0\}$.

A deeper problem is to conclude that ϕ has at worst a pole at the origin. A sufficient condition ensuring this is that $\Sigma = \hat{\gamma} \cap \big((\Delta \setminus \{0\}) \times \mathbf{C}\big)$. This allows us to easily show that V_γ is harmonic on Σ. We suspect this extra hypothesis is unnecessary.

To motivate a future look at some pluripotential-theoretic questions, we considered the classic univariate setting of complex potential theory. Let $\mathcal{M}(K)$ denote the convex set of probability measures supported in a given nonpolar compact set $K \subset \mathbf{C}$. For $\mu \in \mathcal{M}(K)$, let

$$p_\mu(z) := \int_K \log \frac{1}{|\zeta - z|} d\mu(\zeta) \text{ and } I(\mu) := \int_K \int_K \log \frac{1}{|\zeta - z|} d\mu(\zeta) d\mu(z)$$

denote the logarithmic potential and logarithmic energy of μ. Define

$$C_K := \{\mu \in \mathcal{M}(K) : p_\mu \text{ is continuous}\};$$

$$E_K := \{\mu \in \mathcal{M}(K) : I(\mu) < +\infty\};$$

$$P_K := \{\mu \in \mathcal{M}(K) : \mu(P) = 0 \text{ for all polar } P\}.$$

Clearly $C_K \subset E_K \subset P_K \subset \mathcal{M}(K)$. We verified:

Proposition. *Suppose* K *is not polar at each of its points; i.e., for each* $z \in K$ *and each* $r > 0$, $K \cap B(z, r) = \{z' \in K : |z - z'| < r\}$ *is not polar. Then* C_K *is dense in* $\mathcal{M}(K)$ *in the weak-* topology.*

A key element of the proof of the proposition is the following: *if* $K \subset \mathbf{C}$ *is not polar, then there exists a positive measure* μ *with support in* K *such that* p_μ *is continuous.* A deep theorem of Ancona [1] gives a stronger result: *if* $K \subset \mathbf{C}$ *is not polar, then there exists a compact set* $K' \subset \mathbf{C}$ *with* $V_{K'}$ *continuous.* The proof in [1] is difficult; we discussed a different approach to a possible proof beginning with the measure μ.

5 Outcome of the Meeting

We are in the process of writing up and submitting for publication an article which will include our generalization of the Rudin/Wermer/projective hull results. We will continue to work on eliminating the extra hypothesis $\Sigma = \hat{\gamma} \cap \big((\Delta \setminus \{0\}) \times \mathbf{C}\big)$. Future projects may include a possible Rudin/Wermer theorem on the polydisk in $\mathbf{C}^n$, $n > 1$. Questions related to Ancona's theorem [1] and analogues of the subsets C_K, E_K, P_K of $\mathcal{M}(K)$ for $K \subset \mathbf{C}^n$, $n > 1$ will require more thought.

References

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