Global/Local Conjectures in Representation Theory of Finite Groups

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1 Overview of the Field

The Representation Theory of Finite Groups is a thriving subject, with many fascinating and deep open problems, and some recent successes. In 1963 Richard Brauer [4] formulated a list of deep conjectures about ordinary and modular representations of finite groups. These have lead to many new concepts and methods, but basically all of his main conjectures are still unsolved to the present day. A new development was opened up by the mathematician John McKay from Concordia University [17] in 1972 with an observation on degrees of characters for simple groups which was soon named the McKay conjecture and led to a wealth of difficult problems, relating global and local properties of finite groups, that also remain unproved. Profoundly significant are the Alperin Weight Conjecture formulated by J. Alperin [2] in 1986, and the structural explanation of some of the conjectures proposed by M. Broué [5]. These are usually called global/local conjectures, since they propose strong links between the representation theory of a finite group and that of certain of its local subgroups. We all gather that there should be a hidden theory explaining all these global/local phenomena, but yet such a theory still remains to be discovered.

Recently, considerable progress has been made in several directions. Firstly, there are now reductions to simple groups of the original McKay conjecture, the Alperin–McKay blockwise refinement and of the Alperin weight conjecture. Still, more and deeper knowledge of simple groups is needed in order to give a proof of these conjectures. Yet, for the first time the prospect of a complete proof of several of these fascinating conjectures seems in reach.

Another line of successes have been provided by proving some of these statements in some special cases. The difficulty of those give us a hint of the task ahead. For instance, Brauer's $k(GV)$-problem has only been solved after years of inspiring work of many mathematicians.

2 Recent Developments and Open Problems

In the meeting, we concentrated on recent progress on the following famous and longstanding global/local conjectures.

2.1 The McKay Conjecture and refinements

The McKay conjecture is at the center of the representation and character theory of finite groups. It is the origin, together with the Alperin Weight Conjecture, of the more far-reaching Dade conjecture as well as of Broué's conjecture. If p is a prime and G is a finite group with Sylow p-subgroup $P \leq G$, then the McKay conjecture asserts that

$$|\operatorname{Irr}_{p'}(G)| = |\operatorname{Irr}_{p'}(\mathbf{N}_G(P))|\,,$$

where we denote by $\operatorname{Irr}(G)$ the set of irreducible complex characters of G and by $\operatorname{Irr}_{p'}(G)$ its subset consisting of characters having degree not divisible by p.

That is to say, fundamental information on the representation theory of G is encoded in some local subgroup of G, namely the Sylow normalizer.

In fact, J. McKay [17] made his conjecture only for G a simple group and for $p = 2$. It was Isaacs, in his landmark paper [10], who proved the conjecture for any prime p and any group of odd order. Some time later, J. Alperin [1] extended the statement to include Brauer blocks. This generalization is now known as the Alperin–McKay conjecture. T. Wolf [26] and E. C. Dade [7, 8], proved the conjecture for solvable groups. Various other classes of groups have also been considered.

Recently, M. Isaacs and G. Navarro [12], [19] discovered several refinements of the conjecture and these have contributed significantly to a further understanding of the problem. Yet another refinement is due to A. Turull [25] to include p-adic fields and Schur indices. Still more recently, Isaacs, G. Malle and Navarro [13] reduced the McKay conjecture to a question on simple groups. The latter has been solved for important families of simple groups by Malle and B. Späth [16], [23], relying heavily on the Deligne–Lusztig theory, and has led to interesting and difficult questions on automorphism groups of simple groups of Lie type. Ongoing work of M. Broué, P. Fong, and B. Srinivasan aims at proving the McKay conjecture and its recent refinements for finite reductive groups (in cross characteristic). The Isaacs–Malle–Navarro reduction raises the hope that a complete proof of the McKay conjecture may be possible in the not too distant future. This year, at the meeting, Späth has proposed a generalization of this reduction to the blockwise version, the Alperin–McKay conjecture [24].

2.2 The Alperin Weight Conjecture

If the McKay Conjecture counts characters of p'-degree, then the Alperin Weight Conjecture (AWC), as formulated by R. Knörr and G. R. Robinson [15], counts characters with maximal p-part, the so called defect zero characters. Specifically, a character $\chi \in \operatorname{Irr}(G)$ has defect zero if $\chi(1)_p = |G|_p$. Alperin's original formulation [2] is given by the formula

$$l(G) = \sum_{Q} z(\mathbf{N}_G(Q)/Q)\,,$$

where Q runs over representatives of conjugacy classes of p-subgroups in G; furthermore, $z(X)$ is the number of defect zero characters of the group X, and $l(G)$ is the number of p-regular classes of G. By using Möbius inversion, it is then possible to describe $z(G)$ in terms of local subgroup information.

A strong form of AWC for π-separable groups was proved by Isaacs and Navarro in [11]. Quite recently, L. Puig [22] announced a reduction of AWC to a question on simple groups, and at the same time, Navarro and P. H. Tiep published another reduction to a statement for finite quasi-simple groups [21] which they verified for various classes of simple groups.

2.3 The Height Zero Conjecture

The Height Zero Conjecture, another famous problem on blocks, formulated by Brauer [4] in 1963, should lead to interesting questions on characters. If B is a p-block of a finite group with defect group D, then Brauer conjectured that all irreducible characters in B have height zero if and only if D is abelian. In particular, a positive solution would exhibit another connection between local and global invariants, providing, for example, an extremely simple method to detect from a group's character table whether its Sylow p-subgroup is abelian. The p-solvable case of the Height Zero Conjecture is an impressive theorem by D. Gluck and Wolf [9]. Interesting results on the Height Zero Conjecture include work of M. Murai [18], and of T. Berger and Knörr [3]. But perhaps the most significant paper concerning this conjecture in recent years has been the solution of the Brauer's Height Zero Conjecture for 2-blocks of maximal defect by Navarro and Tiep [20]. At the conference, Malle announced work in progress with R. Kessar [14] which solves one direction of the Height Zero Conjecture relying on the above mentioned reduction theorem of Berger–Knörr. Also Tiep announced a very recent result of Navarro and him which would provide the first step towards a proof of the Brauer's Height Zero Conjecture for p-blocks of maximal defect for odd primes p.

2.4 The Broué Conjecture

The Alperin–McKay conjecture asserts that if B is a Brauer block of a finite group with b its First Main Theorem correspondent, then B and b have the same number of height zero characters. The block b is a uniquely determined block of the local subgroup $\mathbf{N}_G(D)$, where D is the defect group of B.

Now, blocks are algebras, and Broué [5] conjectured that the algebras B and b have intimate structural connections that should imply the desired facts on height zero characters and much more. Currently, the Broué Conjecture is only stated for abelian defect group D, and it remains a challenge to find the correct formulation when the defect group is nonabelian. In 2008, J. Chuang and R. Rouquier [6] have given a proof of it for symmetric groups. At the conference, A. Evseev announced a refinement of the Broué Conjecture, and yet another conjecture which implies the Isaacs–Navarro refinement of the McKay conjecture as well as the Alperin–McKay conjecture.

3 Presentation Highlights

Definitely, one of the highlights was Späth's announcement of her reduction for the Alperin–McKay conjecture to simple groups. This goes along the lines of the Isaacs–Malle–Navarro reduction [13] for the McKay conjecture, as well as the recent Navarro–Tiep reduction [21] for the Alperin Weight Conjecture, and states that the Alperin–McKay conjecture holds for every p-block of every finite group, if every finite non-abelian simple group S satisfies a certain, *inductive Alperin–McKay* condition for the prime p (which is a collection of several statements to be verified for S). She also announced that this inductive Alperin–McKay condition is satisfied for the alternating groups Alt_n with $n \geq 8$, and for the simple groups of Lie type in their defining characteristic p.

Coming back to the McKay conjecture, recall that the Isaacs–Malle–Navarro reduction [13] for the McKay conjecture states that the McKay conjecture holds for the prime p for every finite group, if every finite non-abelian simple group S satisfies a certain, *inductive McKay* condition for the prime p. It is known that the inductive McKay condition holds for all alternating groups. It has now been partly proved by Brunat, another young participant of the workshop, and then in general by Späth, building on recent work of Maslowski that the inductive McKay condition holds for the simple groups of Lie type in the same characteristic p. Brunat gave a talk at the workshop about some steps of this proof.

With all the sporadic simple groups and the simple groups of Lie type with exceptional Schur multiplier already handled by Malle, the simple groups of Lie type in cross characteristic $\ell \neq p$ are the remaining case on the way towards a complete proof of the McKay conjecture. Cabanes gave a talk about his recent joint work with Späth proving the inductive McKay condition for exceptional Lie-type groups of types G_2, ${}^3D_4$, ${}^2F_4$, F_4, and E_8, showing how the bijections obtained by Malle [16] and Späth [23] could be chosen to be equivariant under outer automorphisms.

Broué gave an inspiring lecture about his recent work aiming at generalizing some basic tools of the representation theory of finite groups to arbitrary symmetric algebras, including Casimir element, induction-restriction functors, Higman's criterion, and character correspondences. It is expressed the hope that this approach will help explain various mysteries of many recent refinements of basic conjectures in modular representation theory.

One of the main tasks of the modular representation theory is to determine the p-blocks of finite (quasi-) simple groups. Malle gave a talk about his work in progress jointly with R. Kessar to classify the quasi-isolated blocks of finite exceptional groups of Lie type in bad characteristic. Together with previous results of many other mathematicians, this completes the aforementioned task. He also indicated how this important work leads to nice consequences on the characterization of nilpotent blocks for quasi-simple groups, as well as on the completion of proof for the "if" direction of the Brauer Height Zero Conjecture.

The opening lecture of the workshop was given by M. Geck, who spoke about his very recent result on bounding the number of irreducible Brauer characters in any block of a finite group G of Lie type in cross characteristic by a constant depending only on the type of G. This in turns relies on recent deep work of G. Lusztig establishing the so-called cleanness of cuspidal character sheaves. Geck's result is expected to play a role in showing the inductive Alperin weight condition for groups of Lie type in their non-defining characteristic.

One of the key ingredients of the Gluck–Wolf proof [9] of the Brauer Height Zero Conjecture for the p-solvable groups and also of the Navarro–Tiep proof [20] of the Brauer Height Zero Conjecture for 2-blocks of maximal defect is provided by a result of Gluck and Wolf concerning a certain special situation in Clifford theory, which holds for all p-solvable groups and also holds when $p = 2$. In his talk at the conference Tiep announced and explained a very recent result of Navarro and him which generalizes the Gluck–Wolf theorem to odd primes $p > 5$.

Despite works of many mathematicians including recent work of J. Thompson, the structure of finite rational groups is still not well understood. An old conjecture in this area states that if G is rational then so are its Sylow 2-subgroups. So it was a big surprise that Navarro announced his recent joint work with Isaacs leading to a counterexample, a solvable group of order $2^9 \cdot 3$) to this old conjecture! One could think *a posteriori* that one could discover this counterexample by searching the available database of groups of this kind of order. But as Navarro pointed out in his talk, it would take lots and lots of time to complete this search!

Somewhat related to this last topic, Gluck spoke about results concerning his recent conjecture that if a defect group D of a 2-block B of a finite group G is rational and satisfies $D' \leq Z(D)$, then all ordinary characters in B are 2-rational. In particular, he has proved his conjecture in the cases where G is solvable or if $|D| \leq 8$.

A new trend in modular representation theory centers around the idea of *categorification* to explore its connections with Lie theory, particularly through an action of a Lie algebra on the sum of the Grothendieck groups of representation categories of a sequence of groups or algebras, like the symmetric group Sym_n with $n = 1, 2, \ldots$. B. Srinivasan gave a talk about her recent work concerning a similar situation where the Heisenberg algebra acts on the sum of the Grothendieck groups of the categories of unipotent (ordinary) representations of $\mathrm{GL}_n(q)$, $n = 1, 2, \ldots$. She showed in particular that this action is related to the Deligne–Lusztig functors.

An important problem in the modular representation theory is to understand the structure of a p-block with given defect groups D, for instance when D is small. At the workshop, S. Koshitani spoke about his recent joint work with R. Kessar and M. Linckelmann concerning the case when D is elementary abelian of order 8. In particular, they showed that the Alperin Weight Conjecture and a weak version of the Broué conjecture hold for such a 2-block. This is the first result of this type for arbitrary groups with given defect group in more than 30 years.

Another young participant, Kunugi, gave a talk about her joint work with H. Miyachi and Okyuama, in which they used Scott modules to prove the Morita equivalence between principal p-blocks of various general linear groups $\mathrm{GL}_n(q_i)$, with q_i, $i = 1, 2$, two distinct prime powers. The case where the Sylow p-subgroups of $\mathrm{GL}_n(q_i)$ are abelian was known before, and confirmed the Broué conjecture in that case.

Yet another intriguing refinement of the Broué conjecture and (the Isaacs–Navarro refinement of) the McKay conjecture was announced by Evseev. He also showed that these latter refinements would imply the Alperin–McKay conjecture as well. In his talk he offered some evidence for these new conjectures.

C. Eaton's talk described the work of his Ph. D. student P. Ruengrot on a new block invariant — the group of all perfect isometries of the block. G. Hiß spoke about his joint work with N. Naehrig on some general set-up including p-modular Hecke algebras which would produce a bijection between simple modules in the head of a finite-dimensional module Y and simple modules in the socle of the endomorphism algebra of Y.

4 Scientific Progress Made and Outcome of the Meeting

A main result of the meeting was to make clear that several old and famous conjectures in modular representation theory now seem in reach, like the McKay-conjecture, Alperin's weight conjecture and Brauer's height zero conjecture.

Aside from the officially scheduled talks, various informal discussions took place. One of them, which included about half of the participants of the meeting, concentrated on an assessment of the objectives to fulfill in checking the inductive conditions necessary for a proof of McKay's conjecture and Alperin's weight conjecture for all finite simple groups along with a tentative cast of the remaining tasks.

Numerous additional discussions led to further results; we reproduce some of the comments on these outcomes given by participants below. Following discussions of M. Geck, G. Malle and P.H. Tiep, it now seems reasonable to try a new attack on the basic set problem for all finite groups of Lie type in non-defining characteristic. This should play a role in an eventual verification of the inductive Alperin weight condition for these groups. Through this discussion, G. Navarro became interested in a reduction of the basic sets conjecture to simple groups.

Also, through discussions of M. Geck, G. Hiss and G. Malle, there is new progress on their programme to understand modular Harish-Chandra series in the non-linear prime case for finite classical groups.

Discussion of C. Eaton and S. Koshitani at this meeting resulted in a collaboration to investigate Alperin's weight conjecture and also Broue's abelian defect group conjecture for blocks with defect group of order p^2. S. Koshitani and B. Späth discussed much the Dade–Glauberman–Nagao correspondence, which plays an important role in the Navarro–Tiep reduction of the Alperin weight conjecture. They also discussed a recent preprint "About a minimal counterexample for the Alperin-McKay conjecture" by M. Murai related to Späth's reduction project. G. Hiss and S. Koshitani talked on Donovan's conjecture for groups with elementary abelian Sylow subgroup of order p^2, and on the classification problem of blocks of finite groups up to Morita equivalence under a mild hypothesis.

G. Navarro discussed and solved with B. Späth the problem of preservation of congruences in the Dade–Glauberman–Nagao correspondence for the reduction of the Alperin–McKay conjecture. This is expected to lead to a reduction of the Isaacs–Navarro refinement of the Alperin–McKay conjecture to simple groups. As a consequence of the talk of A. Evseev, G. Navarro was able to prove the Sylow self-normalizing case of Evseev's new conjecture for p-solvable groups.

M. Cabanes and B. Späth worked on their ongoing project to prove that the family of finite projective linear groups $\mathrm{PSL}_n(q)$ satisfies the inductive McKay condition, and investigated possibilities to generalize older results on McKay's conjecture towards the inductive Alperin–McKay condition. Späth and B. Srinivasan discussed Maslowski's recent Ph.D. thesis which provided the first half in the recent proof of the McKay conjecture for groups of Lie type in their defining characteristic and which is related to a project of Srinivasan to parametrize some semisimple conjugacy classes in certain classical groups.

M. Broué, P. Fong and B. Srinivasan continued work on their joint project concerning Dade's conjecture for groups of Lie type. P. Fong and B. Späth discussed a joint project, begun two years ago, on Harish-Chandra induction and Galois actions.

C. Eaton had a useful discussion with Kunugi on some unpublished work which resolved one of the questions posed in his talk. Also, he had the opportunity to continue his collaboration with Koshitani.

M. Cabanes also discussed very concrete points with B. Srinivasan about questions on Deligne–Lusztig constructions revisited by Bonnafé–Rouquier in a modular framework; he discussed irreducible characters of twisted group algebras with E. Dade, fusion in blocks with dihedral defect with C. Eaton.

During the discussions of A. Evseev, G. Malle, B. Späth and S. Koshitani, it was realized that it may be feasible (and easier than previously thought) to check the Evseev's refinement of the McKay conjecture in several new cases (e.g. for some groups of Lie type in non-defining characteristic).

In closing we would like to emphasize that all participants expressed the opinion that the meeting was very successful and stimulating for their research. They also underlined the fact that, due to the quite focussed topic and the informal atmosphere with the audience asking clarifying questions or making insightful comments, they profited much more from the individual talks than at most other conferences, with a broader theme.

As an example, we cite from a comment by B. Srinivasan: “In my view this was a very successful workshop. The participants were a judicious mix of senior, mid-level and junior researchers. I was able to learn from every lecture and have discussions with most of the participants [...] which led to new insights and ideas for my continuing research. It was also interesting to meet some of the junior participants for the first time and to learn what they are working on.”

M. Cabanes told us: “The week in Banff was for me very fruitful. I had there an opportunity, unique in the past two years, of discussing subjects of interest to me with the best experts in the field. This was achieved through and around talks (including mine) but also by informal discussions occurring at all the moments in the day.”

P. Fong told us: “Banff was a good meeting. The intimate setting gave a directness and immediacy to mathematical discussions. The talks ranged from the delicate and difficult nailing-down of individual cases for inductive proofs to bold, engaging questions such as Evseev’s. We all went home with lots to think about.”

D. Gluck commented: “I thought the conference was a nice blend of coherence and diversity.”

And G. Hiß said: “I learned a lot form the talks give at the workshop, which all where excellent, without exception.”

In all, this conference lived up to its promises of a quiet, inspiring and very comfortable place to make mathematics.

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