The p-adic Langlands program for non-split groups

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1 Overview of the Field

The Langlands program was originally formulated as a link between number theory and analysis, but over the last 40 years it has grown to link together much of pure mathematics and parts of theoretical physics. Results on the Langlands program have been at the heart of many of the most spectacular developments in number theory in the last 20 years, including the proofs of Fermat's Last Theorem, Serre's Conjecture and the Sato–Tate conjecture. The p-adic Langlands program is an exciting recent generalisation of the Langlands program, which has already led to major results in number theory, in particular the proofs of the two-dimensional Fontaine–Mazur conjecture by Emerton and Kisin ([2],[4]).

2 Recent Developments and Open Problems

The p-adic Langlands program is still at a nascent stage, and at present the local correspondence only exists for the group $\mathrm{GL}_2/\mathbb{Q}_p$, and the global correspondence only for $\mathrm{GL}_2/\mathbb{Q}$. Experience with the classical Langlands program has shown that the full strength of the correspondence is only apparent when one works with multiple groups at once, at which point Langlands' functoriality principle becomes one of the most powerful tools available to number theorists. In particular, it is frequently vital to be able to work with other forms of a given group; for example, when making arguments with modular forms, it is frequently helpful to be able to pass to quaternion algebras, which are non-split forms of GL_2, and the proof of the Sato–Tate conjecture heavily relies on the use of unitary groups, which are non-split forms of GL_n. It is therefore of great interest to explore the p-adic Langlands correspondence for non-split groups, as one anticipates that similar advantages will be gained here, just as in the classical case.

3 Scientific Progress Made

Much of our progress was centered around understanding an inertial version of the existing p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, and understanding the interaction of the p-adic local Langlands correspondence with the Taylor–Wiles–Kisin patching method, with a view towards generalisations to non-split quaternion algebras, and extensions of $\mathbb{Q}_p$.

The classical local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ gives a bijection between Frobenius semi-simple representations of the Weil–Deligne group of $\mathbb{Q}_p$ over any algebraically closed field of characteristic zero, and irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over the same field. Here we will restrict our attention to Frobenius semisimple representations of the Weil group $W_{\mathbb{Q}_p}$ itself (equivalently, Weil–Deligne representations with trivial monodromy operator); this corresponds to omitting the Steinberg representation and its twists from the $\mathrm{GL}_2(\mathbb{Q}_p)$ side of the correspondence.

The Weil group representations are naturally organized into continuous families, by placing two representations into the same component of the family if their restrictions to inertia coincide. (The point is that the image of inertia under a representation of the Weil group is stipulated to be finite, and it cannot vary continuously; but the Frobenius eigenvalues can be made to vary in a family.) Similarly, the irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ can be arranged in families (this is part of the theory of the *Bernstein centre*), with two representations lying in the same family if they share a common minimal type. (If π is any irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, it decomposes into a direct sum of irreducible subrepresentations of $\mathrm{GL}_2(\mathbb{Z}_p)$, and there is a unique such subrepresentation that is minimally ramified, which we call the *minimal type* of π.)

Henniart [3] has shown that there is an *inertial local Langlands correspondence* between those two-dimensional representations of the inertia group $I_{\mathbb{Q}_p}$ which extend to representations of $W_{\mathbb{Q}_p}$, and those representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ which arise as the minimal K-type of a smooth irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation, so that the family of Weil group representations with a fixed restriction to inertia matches via local Langlands with the family of smooth irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations with the corresponding minimal K-type.

Let D_p denote the unique ramified quaternion algebra over $\mathbb{Q}_p$. The local Jacquet–Langlands correspondence induces a bijection between the irreducible smooth representations of $D_p^\times$ (which are necessarily finite-dimensional, since $D_p^\times$ is compact modulo its centre) and those irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ which are *not* principal series representations. It may seem surprising that a finite-dimensional representation of the group $D_p^\times$, which is essentially compact, can carry the same amount of information as an infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, but one can note that the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations in the image of the Jacquet–Langlands correspondence are determined up to a twist by their minimal type (since in the non-principal series case, the family of representations with a fixed minimal type is simply a family of twists), and so one can essentially regard the local Jacquet–Langlands correspondence as matching representations of two compact groups (namely $D_p^\times$ modulo its centre, and $\mathrm{GL}_2(\mathbb{Z}_p)$ modulo its centre).

One project that we emphasised during the workshop is to study analogues of the inertial local Langlands correspondence and of the local Jacquet–Langlands correspondence in the context of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$; recall that this correspondence has been constructed by Colmez and Paškūnas [1, 5], and associates to any continuous representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ (with E a finite extension of $\mathbb{Q}_p$) a corresponding admissible continuous unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E.

Whereas a smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E will decompose as a direct sum of finite-dimensional irreducible $\mathrm{GL}_2(\mathbb{Z}_p)$-subrepresentations, this need not be true of a continuous representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an E-Banach space. Indeed, if $\Pi(\rho)$ is the Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to some continuous $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ via p-adic local Langlands, then $\Pi(\rho)$ does not contain any non-zero finite-dimensional $\mathrm{GL}_2(\mathbb{Z}_p)$-subrepresentation unless ρ is de Rham (up to a twist), and even in this case $\Pi(\rho)$ will not be semisimple as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation. In most cases we actually expect $\Pi(\rho)$ to be topologically irreducible as a $\mathrm{GL}_2(\mathbb{Z}_p)$-representation (even though it is infinite-dimensional!). Thus we cannot expect to define a notion of minimal type in the context of the p-adic local Langlands correspondence.

However, this non-semisimplicity suggests the following alternative approach to phrasing the inertial local Langlands correspondence in the p-adic context. Namely, during the workshop we formulated the following conjecture.

Conjecture 1 If ρ and ρ' are two continuous representations of $G_{\mathbb{Q}_p}$ over E, then there is a natural isomorphism $\mathrm{Hom}_{I_p}(\rho, \rho') \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\Pi(\rho), \Pi(\rho'))$.

Note that this would simply be false in the context of the classical local Langlands correspondence, already in the case when $\rho = \rho'$, since an infinite-dimensional smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is a direct sum of an infinite number of irreducible $\mathrm{GL}_2(\mathbb{Z}_p)$-representations. The reason that it has a chance to be true in the p-adic case is the non-semisimple nature of the $\mathrm{GL}_2(\mathbb{Z}_p)$-action on $\Pi(\rho)$ and $\Pi(\rho')$ that was noted above.

We expect to prove this conjecture by using the description of the p-adic local Langlands correspondence in terms of (φ, Γ)-modules [1].

Establishing a p-adic local Jacquet–Langlands correspondence will be much more difficult then establishing the inertial correspondence, since currently very little is known about the p-adic representation theory of the group $D_p^\times$. Furthermore, the classical local Jacquet–Langlands correspondence is characterized by character identities, and we don't have character theory available in the p-adic context.

Nevertheless, we are hopeful that we can obtain a correspondence. Since so little is known about the p-adic representation theory of $D_p^\times$, it seems safest to use the relationship with Galois representations as an anchor, and so during the workshop we formulated the following conjecture.

Conjecture 2 There is an injection $\rho \hookrightarrow \Pi^{\mathrm{JL}}(\rho)$ from the isomorphism classes of continuous representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ to the isomorphism classes of admissible unitary continuous E-Banach space representations $\Pi^{\mathrm{JL}}(\rho)$ of $D_p^\times$.

Recalling that $\Pi(\rho)$ denotes the E-Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to ρ as in the preceding definition via the p-adic local Langlands correspondence, we would then declare $\Pi(\rho)$ and $\Pi^{\mathrm{JL}}(\rho)$ to be related by the p-adic Jacquet–Langlands correspondence.

Just an in the classical case, it seems strange at first that one might hope to match representations of the essentially compact group $D_p^\times$ with representations of the non-compact group $\mathrm{GL}_2(\mathbb{Q}_p)$, and one of the points of establishing Conjecture 1 is to allay this concern: this conjecture shows that little information about $\Pi(\rho)$ is lost by restricting to the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$. Just as we explained above that in the classical local Jacquet–Langlands correspondence one is more-or-less matching representations of the essentially compact group $D_p^\times$ with representations of the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$, the same will be true in the p-adic setting — except that now the representations will be infinite-dimensional!

A second point to note is that, unlike in the classical local Jacquet–Langlands correspondence, we do not restrict the ρ that we consider. (In the classical Jacquet–Langlands correspondence, omitting principal series representations from the correspondence corresponds to omitting reducible Weil group representations on the other side of the local Langlands correspondence.) Our reason for believing that no such restriction is necessary is as follows: the p-adic local Langlands correspondence is compatible with p-adic interpolation, and we expect that the p-adic Jacquet–Langlands correspondence should be similarly compatible. But on the Galois side, those ρ which are de Rham with distinct Hodge–Tate weights, and whose associated Weil–Deligne representations are irreducible, are Zariski dense in the space of all two-dimensional ρ. Thus if we imagine that there is some way to interpolate the classical Jacquet–Langlands correspondence into a p-adic correspondence, there should be no restriction on the ρ that we consider.

We expect the p-adic Jacquet–Langlands correspondence to be compatible with the classical Jacquet–Langlands correspondence in the following manner, namely that $\Pi^{\mathrm{JL}}(\rho)$ will contain a finite-dimensional subrepresentation of $D_p^\times$ if and only if ρ is de Rham with distinct Hodge–Tate weights (up to a twist), and this finite-dimensional subrepresentation will match with the Weil–Deligne representation associated to ρ (up to a twist by an algebraic representation depending on the Hodge–Tate weights of ρ) via the composition of local Langlands and classical Jacquet–Langlands.

One approach to constructing the p-adic Jacquet–Langlands correspondence that we intend to pursue in future work is global, making use of techniques related to Taylor–Wiles–Kisin patching; note that Taylor–Wiles–Kisin patching is applicable in this context because the group $D_p^\times$ is compact modulo its centre — so this brings out the importance for our strategy of working with (essentially) compact groups, and lends additional importance to proving Conjecture 1. With this in mind, we spent some of the workshop investigating Taylor–Wiles–Kisin patching in relation to the p-adic Langlands correspondence. We now explain what we discovered.

We assume that p is an odd prime, and that k is a finite field of characteristic p. Let F be a totally real field, and $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ a continuous representation which is irreducible and modular, and satisfies the additional assumptions necessary to apply the Taylor–Wiles–Kisin method, namely that $\overline{\rho}_{|G_{F(\zeta_p)}}$ is absolutely irreducible, with a further technical assumption if $p = 5$.

We fix a quaternion algebra D over F, split at exactly one archimedean prime and unramified at the primes above p, so that D determines a family of Shimura curves over F. For an ideal $\mathfrak{n}$ in the ring of integers $\mathcal{O}_F$, prime to p and to the discriminant of D, we let $X_1(\mathfrak{n})$ denote the corresponding Shimura curve of level $\mathfrak{n}$. We assume that $\mathfrak{n}$ is chosen so that $X_1(\mathfrak{n})$ has no elliptic points (i.e. the congruence subgroup that determines it is torsion free). We fix a prime v of F above p, and let $\mathcal{O}_{F_v}$ denote the completion of $\mathcal{O}_F$ at v.

We fix a finite extension E of $\mathbb{Q}_p$ (which will serve as our coefficient field), with ring of integers $\mathcal{O}_E$, uniformizer ϖ_E, and residue field k_E, which we assume contains k (so that we may regard ρ as being defined over k_E). If L is any finitely generated $\mathcal{O}_E$-module equipped with a smooth representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$, then L determines a local system $\widetilde{L}$ on $X_1(\mathfrak{n})$ (actually, one has to take a little care with central characters in order for this to be true, but we suppress that detail in this discussion), and we may consider the cohomology group $H^1(X_1(\mathfrak{n}), \widetilde{L})_{\overline{\rho}}$, where the subscript $\overline{\rho}$ indicates that we complete at the ideal corresponding to $\overline{\rho}$ in the Hecke algebra generated by Hecke operators at primes away from $\mathfrak{n}$, p, and the discriminant of D. (This ideal is either maximal, or else the unit ideal.)

The Taylor–Wiles–Kisin method allows us, by adding carefully chosen auxiliary primes to the level $\mathfrak{n}$, to pass to a limit and "patch" these cohomology groups into a coherent sheaf $M_\infty(L)$ over the local deformation ring $R_v(\overline{\rho})$ which parameterizes framed deformations of $\overline{\rho}_{|G_{F_v}}$ over complete local $\mathcal{O}_E$-algebras, with some auxiliary formal variables adjoined (the *patching variables*). In fact a careful application of the method shows that we may perform this patching compatibly for all choices of L, and so regard $M_\infty(L)$ as a functor from the category of $\mathrm{GL}_2(\mathcal{O}_{F_v})$-modules to the category of coherent sheaves on $\mathrm{Spec}\, R_v(\overline{\rho})[[x_1, \dots, x_n]]$ (where $x_1, \dots, x_n$ are the patching variables).

Suppose for a moment that $F = \mathbb{Q}$, so that p-adic local Langlands and local-global compatibility are available. One can then give a different construction of the patched modules $M_\infty(L)$ which shows that, despite the global nature of their construction, they are in fact of purely local nature. Indeed, one of us [5] has constructed a universal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathrm{Spec}\, R_p(\overline{\rho})$, which we denote by $\mathcal{P}$, and which realizes p-adic local Langlands, in the sense that the fibre of $\mathcal{P}$ over a point corresponding to a continuous lifting $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ of $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ is the dual to the Banach space representation $\Pi(\rho)$ attached to ρ via p-adic local Langlands. (The appearance of a dual here is a purely technical point.) During the workshop we showed, using the local-global compatibility result of [2], together with the compatibility of classical and p-adic local Langlands, that the patched module $M_\infty(L)$ is equal to the basechange from $R_p(\overline{\rho})$ to $R_p(\overline{\rho})[[x_1, \dots, x_n]]$ of the $R_p(\overline{\rho})$-module $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\mathcal{P}, L^\vee)^\vee$ (where again, the appearance of the various duals is purely technical). Heuristically, this expresses the fact that $M_\infty(L)$ is supported at exactly those Galois representations ρ whose associated $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\Pi(\rho)$ contains a non-zero quotient of L as a $\mathrm{GL}_2(\mathbb{Z}_p)$-subrepresentation.

4 Outcome of the Meeting

We are now writing a joint paper that will prove Conjecture 1 and explain the connections between the Taylor–Wiles–Kisin method and the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

References

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