DIALGEBRAS, LEIBNIZ ALGEBRAS, AND QUASI-JORDAN ALGEBRAS

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29 April to 6 May 2012

1 Background

The primary motivation for this workshop is the recent discovery of many new varieties of nonassociative structures that can be regarded as "noncommutative" analogues of classical structures. This development originated in the work of Loday [6, 7] in the early 1990's on Leibniz algebras (which satisfy the Jacobi identity but are not necessarily anticommutative) and associative dialgebras (with two associative operations); the connection between these two structures closely resembles that between Lie algebras and associative algebras, but also presents many new features. Ten years later, Liu [5] introduced alternative dialgebras, the natural analogue of alternative algebras in the setting of dialgebras. Shortly after that, Raúl Felipe (one of the participants in this workshop) and one of his Ph.D. students [10] initiated the study of quasi-Jordan algebras (also known as Jordan dialgebras), which are related to Jordan algebras as Leibniz algebras are to Lie algebras. Around the same time, Kolesnikov [4] developed a general method for passing from a variety of nonassociative algebras defined by polynomial identities to the corresponding variety of dialgebras. This method has recently been simplified and formalized by three of the participants in this workshop [2] in the so-called KP algorithm.

2 The Cayley-Dickson Process

An important topic in classical algebra is the theory of composition algebras and their close connection with quadratic forms and the eight-square theorem. This leads to the construction, starting from the real numbers, of the complex numbers, quaternions, and octonions, through a doubling process which originated in the works of Cayley and Dickson. For the early history of these developments, see Dickson [3], and for the completion of the classical theory, see Albert [1] and Schafer [9]. The original idea of Raúl Felipe and his son, Raúl Felipe-Sosa, was to create an analogue of this theory in the setting of dialgebras. Their initial work on this project was done in 2011, and was extended during the visit of Murray Bremner to Havana in December 2011, who also used the computer algebra system Maple to construct examples of dialgebras analogous to the classical quaternions and octonions. The main difficulty at this point was to find the appropriate conditions under which the double of an associative dialgebra would necessarily be an alternative dialgebra. This topic is also related to ongoing work, by Juana Sánchez-Ortega and her colleagues in Málaga, on dialgebra versions of the Moufang identities and a generalization of Artin's theorem (every alternative algebra on two generators is associative) to the setting of dialgebras.

3 The Results of Our Workshop

We began with a review of our earlier calculations and a discussion of possible approaches to resolving the difficulty mentioned above. The fundamental problem seemed to be to determine an appropriate analogue in the setting of dialgebras of the classical conditions that xx^* and $x + x^*$ belong to the base field for any element x of an algebra with involution $x \mapsto x^*$. After a number of false starts, we have now found the correct hypotheses, which involve conditions relating the symmetric elements in a dialgebra with involution to the Leibniz bracket on the dialgebra. We are confident that this approach will also allow us to prove that the double of a flexible dialgebra is again flexible. A closely related problem, although not strictly related to the Cayley-Dickson process, is to prove that every alternative dialgebra becomes a Jordan dialgebra by means of the quasi-Jordan product (also called the antidicommutator or Jordan diproduct).

We also discussed the little-known paper by Loday and Pirashvili [8], which was brought to our attention by Michael Kinyon. This work provides a different interpretation of associative dialgebras using the so-called infinitesimal tensor product on the category of linear maps. We expect that this will provide another approach to the problem of generalizing the Cayley-Dickson process to dialgebras, and will also lead to a much simpler justification of the KP algorithm for converting varieties of algebras into varieties of dialgebras (the current proof of its correctness relies heavily on a great deal of machinery from the theory of operads).

4 Future Publications and Collaborations

The purpose of this Research in Teams workshop was to develop existing collaborations and to lay the foundations for future collaborations among the participating researchers in the area of nonassociative algebra. In this regard it has already been very successful. We expect to obtain at least three journal articles resulting from our collaboration at BIRS:

1. *The Cayley-Dickson process for dialgebras*.
2. *Varieties of dialgebras and the tensor category of linear maps*.
3. *Artin's theorem for alternative dialgebras*.

In closing, we mention that three of us are co-organizers of a Research School to be held in February 2013 on topics closely related to this BIRS workshop; for details, see

`http://www.cimat.mx/Eventos/associative_and_nonassociative/`

This event is supported financially by CIMPA, the International Center for Pure and Applied Mathematics,

`http://www.cimpa-icpam.org/`

and will take place at the CIMAT (Centro de Investigación en Matemáticas) in Guanajuato, Mexico. This school will introduce graduate students and postdoctoral researchers, primarily from the developing countries of Latin America, to contemporary developments in associative and nonassociative algebras and dialgebras, in both theoretical and algorithmic aspects.

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