Ordered Groups and Topology

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1 Overview of the Field

If the elements of a group can be given a strict total ordering which is invariant under multiplication on the left, the group is said to be left-orderable. Left-orderable groups are also right-orderable, by a possibly different ordering, but if there is an ordering invariant on both sides, one calls the group orderable. For countable groups, left-orderability is equivalent to admitting an effective action on the real line by orientation preserving homeomorphisms.

The study of orderable groups has a long history, dating back to the nineteenth century. Steady progress in understanding orderable groups was made in the twentieth century by the work of many mathematicians, including such notables as O. Hölder, A. A. Vinogradov and W. Rudin. They discovered that many interesting groups are orderable, including free groups, torsion-free abelian groups and fundamental groups of many important topological spaces, such as hyperbolic surfaces and complements of certain hyperplane arrangements. The existence of an ordering on a group implies strong algebraic properties: for example left-orderable groups are torsion-free and obey the zero-divisor conjecture of Kaplansky (still unsolved for torsion-free groups in general); roots of elements of orderable groups are unique, and group rings of orderable groups embed in skew-fields.

In recent years, the theory of orderable groups has attracted much broader interest, in large part due to discoveries of deep connections with topology and the dynamics of group actions on the circle and real line. It was shown by T. Farrell in 1976 that the universal covering of a space X embeds in $X \times I$, respecting the projections, exactly when the fundamental group of X is left-orderable. A major breakthrough in the theory of Artin's braid groups was made in the 1990's when P. Dehornoy showed that the braid groups are left-orderable. This has had strong application in braid and knot theory and inspired discoveries that many other groups which arise in topology also have orderability properties. An important open question is whether the Artin groups of finite type, which generalize the braid groups, are also left-orderable. The answer hinges on the left-orderability of the Artin group of type E_8. It is known that right-angled Artin groups enjoy a 2-sided ordering. These groups have strong connections with topology and geometric group theory.

The fundamental groups of many 3-dimensional manifolds – for example the fundamental group of complements of knots or links in the 3-sphere – are left-orderable. Indeed, in the last few years orderability has proven a very useful tool in 3-manifold theory. For example, 3-manifolds which have particularly nice foliations must have left-orderable fundamental groups. Since many 3-manifold groups are NOT left-orderable, this shows they do not support these nice foliations. Orderability also provides an obstruction to the existence of nonzero degree maps between certain manifolds. Algorithms exist to test the orderability of groups, given generators and relations. A striking application of this, by Calegari and Dunfield, is that the hyperbolic 3-manifold of smallest volume (the so-called Weeks manifold) does not support nice foliations, because its group is not left-orderable. Roberts, Shareshian and Stein constructed an infinite family of 3-manifolds which do not support taut foliations, by showing that their groups cannot act effectively on the real line, or indeed any (possibly non-hausdorff) one-dimensional manifold. In particular, the groups cannot be left-ordered.

Besides the contribution of algebra and orderings to topology, there is a new dynamic in the other direction: topology providing applications to algebra. If G is a group, the set $LO(G)$ of all left-orderings on the group has a natural topology, defined by A. Sikora in 2004. Moreover, there is a natural (conjugation) action of G on $LO(G)$ by homoemorphisms. This is the basis for a recent beautiful argument by D. Witte-Morris, showing that left-orderable amenable groups are locally indicable (meaning any nontrivial finitely generated subgroup has the integers as a quotient group). The argument involves a G-invariant measure on $LO(G)$ and the Poincaré recurrence theorem. A classic result of Burns and Hale shows that all locally indicable groups are left-orderable, but G. Bergman showed that the converse does not hold in general. Another recent result using this topology is P. Linnell's theorem that the number of left-orderings of a group must be either finite or uncountable. By contrast, there exist groups which have a countably infinite number of (two-sided) orderings.

2 Recent Developments and Open Problems

For many groups G the structure of $LO(G)$, and the subspace $O(G)$ of two-sided orderings, is not known at this time. In general we know that these spaces are compact and totally disconnected, but their exact structure is known for only a few families of groups, for example finitely-generated (non-cyclic) torsion-free abelian groups (for which the spaces are homeomorphic to Cantor sets). It was recently shown that if G is a countable (nonabelian) free group, then $LO(G)$ is also a Cantor set. A stronger result, recently announced by A. Clay, is that there exists a left-ordering of the free group G whose orbit under the G-action is actually dense in $LO(G)$. The structure of $O(G)$ is not known for free groups. Another recent surprise regarding the braid groups B_n is that there exist orderings (constructed by Dubrovin and Dubrovina) which are isolated points in $LO(B_n)$, whereas Dehornoy's ordering is not isolated, and is in fact a limit point of the orbit of the D-D ordering. Another fascinating recent discovery, by A. Navas and C. Rivas, is that for the famous Thompson's group F, the space $O(F)$ consists of a Cantor set, plus four isolated points (the structure of $LO(F)$ is unknown).

Many questions dealing with algebraic and analytic properties of orderable groups have been with us for a long time. A few major ones have been answered recently in the works of Witte-Morris, V. Bludov, A. Glass and others – there exist solvable orderable groups that can not be embedded in divisible orderable groups; necessary and sufficient conditions for a free product of two left-orderable groups with amalgamated subgroup to be left-orderable; existence of finitely presented orderable groups with insolvable word problem. In view of recent result obtained by Alexey Muranov for simple groups, one can now hope to get answers to the following: Are there simple orderable groups in which there is no bound on the number of commutators required to express every element of the group as a product of bounded number of commutators?

Returning to applications of orderable groups to topology, there is very recent evidence of connections between the structure of Heegaard-Floer and similar homology theories and the orderability (or non-orderability) of fundamental groups of 3-manifolds constructed by surgery on knots and links. Ozsváth and Szabó define L-spaces to be 3-manifolds with trivial rational homology and HF homology as simple as possible. It was shown by Boyer, Gordon and Watson that the 3-dimensional Seifert fibred spaces which are L-spaces are exactly those whose fundamental group is NOT left-orderable. It would be remarkable if this is true more generally. The interplay of ordered group theory and the topology of manifolds (and structures on them, such as foliations, fibrations, contact structures) is a new and promising area of research.

3 Scientific Progress Made

3.1 Orderability and 3-manifolds

One of the main themes of the workshop centred on the relationships between L-spaces, left-orderability, and co-oriented taut foliations in 3-dimensional topology. CAMERON GORDON's talk surveyed the evidence for his conjecture with STEVEN BOYER and LIAM WATSON that an irreducible, rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. In particular, he described their work which shows that the conjecture holds for non-hyperbolic geometric manifolds as well as for several infinite families of manifolds whose members are generically hyperbolic, such as the 2-fold branched covers of non-split alternating links. NATHAN DUNFIELD reported on his continuing investigation of the conjecture via computer experimentation on small-volume hyperbolic 3-manifolds. Specifically, he has shown that for the $11{,}031$ such manifolds in the Hodgson-Weeks census, at least 27% are L-spaces and at least 2% have left-orderable fundamental groups. So far, these two subsets are disjoint, consistent with the conjecture.

If the fundamental group of an irreducible, rational homology 3-sphere is left-orderable, so is that of any manifold which finitely covers it. This leads to the following question of BOYER-GORDON-WATSON: If a 3-manifold is finitely covered by an L-space, is it an L-space? TYE LIDMAN spoke about his work with CIPRIAN MANOLESCU on this question. They study it from the point of view of Seiberg-Witten theory and show that the answer is yes if the manifolds' Heegaard-Floer groups with integer coefficients are torsion-free.

The existence of an L-space surgery on a knot in the 3-sphere places strong constraints on its other surgeries. In the context of the above conjecture, it is natural to ask whether non-left-orderable surgery on a knot places analogous constraints on the non-left-orderability of the fundamental groups of its other surgeries. In his talk, LIAM WATSON discussed joint work with ADAM CLAY on this question. They show, for instance, that surgeries on cables of knots admitting a non-left-orderable surgery behave analogously, in this regard, to surgeries on cables of knots admitting an L-space surgery.

The connections between L-spaces and left-orderability with the existence of co-oriented taut foliations in 3-dimensional topology was the subject of three talks. It is easily seen that the fundamental group of a manifold admitting an $\mathbb{R}$-covered, co-oriented foliation is left-orderable. RACHEL ROBERTS explained why the converse is false through the use of examples, constructed with SERGIO FENLEY, of irreducible, rational homology 3-spheres which have left-orderable fundamental group but which do not contain $\mathbb{R}$-covered foliations. This shows that the relationship between the existence of co-oriented taut foliations and left-orderability is more subtle than one might naively have hoped for.

STEVEN BOYER described his work with MICHEL BOILEAU which shows that graph manifold integer homology 3-spheres, other than the S^3 and the Poincaré homology 3-sphere, admit co-oriented taut foliations. This combines with work of EFTAKHARY and of CLAY-LIDMAN-WATSON to show that the notions of not being an L-space, of having a left-orderable fundamental group, and of admitting a co-oriented taut foliation, coincide for such manifolds.

ADAM CLAY reported on various aspects of his work with STEVEN BOYER and with LIAM WATSON. He introduced the notion of slope-detectability for 3-manifolds with torus boundary components from two points of view: that of left-orders and that of taut foliations. He outlined how recent group-theoretic work of BLUDOV, GLASS and CHISWELL allows for the development of left-order gluing conditions for the union of such manifolds. He then described how this can be used to study the connection between taut foliations in rational homology 3-sphere graph manifolds and the left-orderability of their fundamental groups.

TETSUYA ITO, in joint work with K. KAWAMURO, introduced a new application of group orderings to *contact structures* on 3-dimensional manifolds. Using Nielsen-Thurston theory, it was proved by ROURKE and WIEST that the mapping class group $MCG(\Sigma)$ of an orientable surface Σ with nonempty boundary is left-orderable. On the other hand, contact structures on a 3-manifold M correspond to open book structures (Σ, ϕ), where the monodromy ϕ belongs to $MCG(\Sigma)$. Ito and Kawamura show that if ϕ is sufficiently large with respect to the Nielsen-Thurston ordering, then the corresponding contact structure on M will be a *tight* structure, i.e. not *overtwisted*.

3.2 Space of orders

The space of left orders $LO(G)$ was proved by ADAM SIKORA in 2004 to be a compact Hausdorff space. The concept had been known for some time, including to algebraic geometers. However it was not until 2006 when DAVE MORRIS used $LO(G)$ to prove that amenable left-ordered groups are locally indicable that it attracted much attention. Since then there has been a spate of papers studying $LO(G)$, dealing with questions such as when it is a Cantor set.

In his talk, P. DEHORNOY showed how to construct concrete examples of ordered groups G such that the space $LO(G)$ of all left-invariant orderings on G has isolated points, which amounts to constructing finitely generated monoids whose left-divisibility relation turns out to be a linear ordering. To this end, he uses the subword reversing method, a combinatorial approach that, in good cases, enables one to analyze presented monoids, in particular to establish cancellativity results and existence of common multiples. As an application, he explained how to obtain a short proof of earlier results by A. NAVAS and T. ITO about the presented groups $\langle x, y \mid x^m = y^n \rangle$. It is worth noting that this approach gives one more proof of the orderability of Artin's braid group B_3.

THOMAS KOBERDA showed that the automorphism group of a residually torsion-free nilpotent group G acts faithfully on $LO(G)$. In the case G is Gromov-hyperbolic, he explained this theorem in terms of the geometry of G.

CRISTÓBAL RIVAS considered dynamical techniques for studying $LO(G)$, that is by considering G as a subgroup of the orientation preserving homeomorphisms of the real line. He use this to show that if G is a nontrivial free product of groups, then $LO(G)$ has no isolated points.

PETER LINNELL extended the topology on $LO(G)$ to the space of locally invariant orders $LIO(G)$. A locally invariant order is a strict partial order $<$ on the group G with the property either $gx > x$ or $g^{-1}x > x$ for all $g, x \in G$ with $g \neq 1$. DAVE MORRIS suggested an improved topology and this led to some new results on $LIO(G)$. Motivation for studying locally invariant orders is that if G has such an order then its group ring kG has only trivial units, and residually finite word hyperbolic groups always have subgroups of finite index which have locally invariant orders; this was proved by DELZANT.

Following the workshop, as a direct result of mutual discussions, DAVE MORRIS and PETER LINNELL proved a new theorem concerning locally invariant orders: Let G be an amenable group. If G has a LIO, then G is locally indicable.

3.3 Braids and knot theory

Artin's braid groups make an important example of orderable groups, and there remain many open questions involving braid orders, and, more generally, orders on mapping class groups, which are natural extensions of Artin's braid groups. In his talk, L. PARIS gave an account of a recent work joint with J. FROMENTIN where the authors describe a simple and efficient algorithm for finding what is known as a σ-definite representative of a braid, namely an expression in the standard Artin generators σ_i with the property that the generator with maximal index occurs only positively or only negatively. The result is directly connected with the question of comparing braids with respect to the standard Dehornoy ordering and it improves on earlier work by S. BURCKEL and P. DEHORNOY and relies on the tricky observation that applying the known methods to a braid and its inverse simultaneously enables one to avoid the difficult case.

W. MENASCO concentrated on another aspect of the standard ordering of braids, namely the connections between the so-called Dehornoy floor of a braid β, which characterizes its position with respect to the successive powers of the central element Δ_n^2, and the knot theoretical properties of the closure of β. Starting from a review of earlier work by A. MALYUTIN and S. NETSVETAEV, he explained the unifying notion of a braid block strand diagram and the general result, due to T. ITO, that, for each such diagram $\mathcal{D}$, there exists a number $n_{\mathcal{D}}$ such that, for every braid β whose floor is at least $m_{\mathcal{D}}$, the closure of β cannot be carried by $\mathcal{D}$. Roughly speaking, this says that, if the absolute value of the floor is large, then the properties of the closure of β can be read simply. Then, W. MENASCO discussed how this approach might lead to a solution of an old conjecture by V. JONES claiming that, for braids with minimal braid index, the algebraic length might be an invariant of the closure.

D. ROLFSEN's talk focused on bi-orderability of knot groups, which are known to be locally indicable, and therefore left-orderable. In joint work with A. CLAY he has shown that if K is a knot in the sphere S^3

and $\pi_1(S^3 \setminus K)$ is bi-orderable then the Alexander polynomial of K has a real positive root. As an application of this result, they showed that a surgery on a knot whose group is bi-orderable cannot produce an L-space in the sense of Ozsváth and Szabó. This is a partial converse to his result with B. PERRON that if all roots of Alexander polynomial are positive real then $\pi_1(S^3 \setminus K)$ is bi-orderable.

3.4 Algebraic and dynamic aspects

A. NAVAS, D. MORRIS, J. PRZYTYCKI, D. ROLFSEN and T. KOBERDA discussed orderability of groups using a variety of methods drawn from algebra, analysis and the theory of dynamical systems.

An algebraic criterion for a bi-ordering of a finitely-generated nontrivial group to be preserved by an automorphism of the group was proved by D. ROLFSEN. If $\phi : G \to G$ preserves a bi-ordering, then the induced linear transformation $\phi_* : H_1(G;\mathbb{Q}) \to H_1(G;\mathbb{Q})$ must have a positive real eigenvalue. This is in a sense complementary to the result of T. KOBERDA that for groups G which are residually torsion-free nilpotent (and therefore bi-orderable) every automorphism has some bi-ordering which is *not* preserved.

In his talk, D. MORRIS explained the theory of amenable groups leading to the following result: among amenable groups, a group is left-orderable if and only if it is locally indicable. He discussed also related results, including one stating that a solvable left-orderable group has a recurrent left-order (and, hence, a Conradian one). As an exciting result of interactions at our workshop, Morris proved a new result using dynamical methods: If G is a left-orderable group, and H is a locally nilpotent subgroup of G, then there is a left-invariant order on G, such that the restriction of the order to H is bi-invariant. This is announced in [18]

A. NAVAS discussed his work on left-ordered groups motivated by the question whether one can extend the scope of the above mentioned Morris' result from amenable groups to all groups which do not contain a free non-abelian subgroup. This problem led him to consider random walks on left-ordered groups with probability measures on them. He proved that for any such group there exists an interval in it such that a random walk ”crosses” this interval infinitely many times with probability one.

Replacing the condition of associativity of a group product by its left-distributivity leads to the notion of a left distributive system or a “shelf”. J. PRZYTYCKI discussed a homology theory for shelfs and speculated about its applications to shelf orderability.

4 Abstracts of the Lectures

Speaker: **Steve Boyer** (UQAM)
Title: *Graph manifolds which are integral homology 3-spheres and taut foliations*
Abstract: We show that a graph manifold, which is an integral homology 3-sphere and is neither the 3-sphere nor the Poincare homology sphere, admits a taut foliation which is transverse to the fibers in each Seifert piece. This result gives a new proof that such a manifold has a left-orderable fundamental group and is not an L-space. This is joint work with Michel Boileau.

Speaker: **Adam Clay** (UQAM)
Title: *Left-orderability and foliations*
Abstract: Every left-ordering of $\mathbb{Z}\times\mathbb{Z}$ corresponds to a line in the plane. As such, whenever M is a 3-manifold with torus boundary, we can say that every left-ordering of the fundamental group 'detects' a slope on the boundary. The idea of r-decay is to show via calculation that those slopes on the boundary of a 3-manifold which are not detected by a left-ordering correspond to those slopes for which the surgery manifold doesn't have a nice foliation, or is an L-space. In this talk, I will discuss the extent to which we can make precise the association between left-orderings and foliations, and outline how recent group-theoretic work of Bludov, Glass and Chiswell may allow for the development of 'gluing conditions' for foliations of 3-manifolds with torus boundary components. This is joint work with Liam Watson and Steve Boyer.

Speaker: **Patrick Dehornoy** (Caen)
Title: *The ordered structure of the Klein bottle group and subword reversing*
Abstract: The Klein bottle group has a simple but interesting ordered structure, which is connected with the fact that the group is a group of fractions of a Garside monoid in which divisibility is a linear ordering.

On the other hand, subword reversing is a combinatorial method relevant for investigating presented groups, in particular those that are groups of fractions. We shall explain how to use this tool in the (easy) case of the Klein bottle group and its ordered structure, with the aim of subsequently applying it to more complex examples.

Speaker: **Nathan Dunfield** (Illinois, Champaign-Urbana)
Title: *L-spaces and left-orderability: an experimental survey*
Abstract: I will discuss the results of some computer experiments on small-volume hyperbolic 3-manifolds. Specifically, for the 11,031 such manifolds in the Hodgson-Weeks census, at least 27% are L-spaces and at least 2% have left-orderable fundamental groups. So far, these two subsets are disjoint, consistent with the conjecture of Boyer-Gordon-Watson that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

Speaker: **Cameron Gordon** (University of Texas)
Title: *L-spaces and left-orderability*
Abstract: We will discuss evidence for the conjecture that a rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. This is joint work with Steve Boyer and Liam Watson.

Speaker: **Tetsuya Ito** (Tokyo)
Title: *Ordering of mapping class groups and contact 3-manifolds*
Abstract: The mapping class group of a surface with non-empty boundaries have a family of left-orderings called Nielsen-Thurston type orderings. We will show that N-T orderings provide a new criterion for tightness of contact 3-manifolds. This relationship between ordering and contact geometry is based on the open book foliation theory, which was developed by the speaker and Keiko Kawamuro.

Speaker: **Thomas Koberda** (Harvard)
Title: *Faithful actions of automorphisms on the space of orderings of a group*
Abstract: I will sketch the ideas which show that the automorphism group of a residually torsion-free nilpotent group G acts faithfully on the space of left-invariant orderings of G. In the case where G is Gromov-hyperbolic, I will explain this theorem in the context of the geometry of G.

Speaker: **Tye Lidman** (UCLA)
Title: *Left-Orderability and a Seiberg-Witten Smith Inequality*
Abstract: If G is left-orderable, then any subgroup of G is automatically left-orderable as well. In terms of covering spaces, if the fundamental group of Y is left-orderable and Y' covers Y, then the fundamental group of Y' is also left-orderable. Boyer-Gordon-Watson have therefore asked the analogous question for L-spaces. However, the obvious methods fail for technical reasons. We study this question from the point of view of Seiberg-Witten theory and present some results in this direction. This is work in progress with Ciprian Manolescu.

Speaker: **Peter Linnell** (Virginia Tech)
Title: *The spaces of left- and locally invariant orders*
Abstract: I will report on separate work with two of my students Kelli Karcher (doctoral) and Li Hao (undergraduate). In the former we are studying the space of left-orders of polycyclic groups. In the latter we are studying the space of locally invariant orders (LIO) of an arbitrary group G.

Recall that the space of left-orders of a group G is the set of all left-orders of G with the topology given by the subbase $\{< \mid g < h\}$ (for $g, h \in G$). Also an LIO of a group G is a strict partial order $<$ such that for all $r, g \in G$ with $r \neq 1$, either $rg > g$ or $r^{-1}g > g$. Then the space of LIO's on G is defined in the same way as the space of left-orders on G, so is the set of LIO's with the topology given by the subbase $\{< \mid g < h\}$.

Speaker: **William W. Menasco** (SUNY-University at Buffalo)
Title: *The Dehornoy floor and the Markov Theorem without Stabilization*
Abstract: The Markov Theorem without Stabilization (MTWS) [Birman-M] tells us that for a fixed braid index n there are a finite number of "modelled" isotopes (dependent only on n) which take any closed n-braid

immediately to a representative of minimal index. Once at minimal index there is a finite number of modelled isotopes (again, dependent only on the value of the braid index) that allows one to jump between conjugacy classes of minimal index. These isotopes which will grow in number as n grows make up the MTWS calculus for closed braids. Connections between the Dehornoy floor and isotopes of the MTWS calculus were first made by T. Ito. In this talk we will expand on these connections by showing a new characterisation of MTWS isotopes for braids. This talk will feature joint work with Doug Lafountain [Aarhus University] and Hiroshi Matsuda [Yamagata University].

Speaker: **Dave Witte Morris** (University of Lethbridge)
Title: *On interactions of amenability with left-orderings*
Abstract: Amenability is a fundamental notion in group theory, as evidenced by the fact that it can be defined in more than a dozen different ways. A few of these different definitions will be discussed, together with some commentary on the theorem that left-orderable amenable groups are locally indicable, and perhaps some speculation on other ways that amenability might be useful in the theory of left-orderings.

Speaker: **Andrés Navas** (USACH, Chile)
Title: *Random walks on left-orderable groups*
Abstract: Given a finitely-generated, left-orderable group endowed with a probability measure supported on a finite system of generators, we are interested on the behavior of typical random products. Among other results, I will sketch the ideas involved in a Polya's like recurrence theorem obtained in collaboration with Deroin, Kleptsyn and Parwani: there exists an interval in the group such that almost every path "crosses" this interval infinitely many times. Potential applications of these ideas will be discussed.

Speaker: **Luis Paris** (Bourgogne)
Title: *A simple and fast method for determining short σ-expressions of braids*
Abstract: Joint work with J. Fromentin. Let $n \in \mathbb{N}$, $n \geq 2$, and $i \in \{1, \dots, n-1\}$. We say that a braid $\beta \in B_n$ is σ_i*-positive* (resp. σ_i*-negative*) if it can be written

$$\beta = \beta_0 \sigma_i \beta_1 \cdots \sigma_i \beta_k \quad (\text{resp. } \beta_0 \sigma_i^{-1} \beta_1 \cdots \sigma_i^{-1} \beta_k)\,,$$

with $k \geq 1$ and $\beta_0, \beta_1, \dots, \beta_k \in B_i$. A celebrated Dehornoy's theorem says that, for any braid $\beta \in B_n \setminus \{1\}$, there exists a unique $i \in \{1, \dots, n-1\}$ such that β is either σ_i-positive, or σ_i-negative, but not both. There are several proofs of this result. Most of them are effective, but the involved algorithms are slow (exponential complexity) and determine σ_i-positive expressions (or σ_i-negative expressions) whose lengths are exponential with respect to the word-length of the original braids. In this talk we present a simple algorithm of quadratic complexity which, given a braid $\beta \in B_n \setminus \{1\}$, determines a σ_i-positive expression (or a σ_i-negative expression) for β, whose length is bounded above by some constant times the word length of β.

Speaker: **Jozef Przytycki** (George Washington University)
Title: *Orderings on Conway algebras, and Tutte algebras; is anything known?*
Abstract: We consider various non-associative binary structures and ask whether they have orderings, and whether orderings lead to some interesting consequences. We concentrate on on entropic property, $(a * b) * (c * d) = (a * c) * (b * d)$, with Conway algebra (including Homflypt polynomial) and Tutte algebra as main examples. Another property of great interest is right self-distributivity, $(a * b) * c = (a * c) * (b * c)$ with quandles, in particular Dehn quandles of surfaces, as premiere examples. We will stress that both structures satisfy "generative property" which we discuss in detail.

Speaker: **Cristóbal Rivas** (ENS-Lyon)
Title: *Left-ordering on free products of groups*
Abstract: Based on the concept of dynamical realization of a left-ordering, we exhibit a dynamical criterion for approximating the giving left-ordering. This criterion is used to show that no left-ordering on a free product of groups is isolated.

Speaker: **Rachel Roberts** (Washington University, St. Louis)
Title: *Manifolds containing no R-covered foliations*

Abstract: We show that there are 3-manifolds which have left-orderable fundamental group but which do not contain R-covered foliations. This is joint work with Sergio Fenley.

Speaker: **Dale Rolfsen** (UBC-Vancouver)
Title: *Ordering Knot Groups*
Abstract: I will discuss orderability of knot groups, that is, fundamental groups of knot complements in 3-space. Howie and Short showed that all knot groups are locally indicable, and therefore left-orderable. In joint work with Bernard Perron and Adam Clay, I'll discuss criteria, involving roots of the Alexander polynomial, determining that certain knot groups are bi-orderable, while many others are not. Sketches of the proofs will be given. Among the applications: if a knot has bi-orderable group, then surgery on that knot cannot produce an L-space in the sense of Ozsváth and Szabó.

Speaker: **Liam Watson** (UCLA)
Title: *Dehn surgery and left-orderability*
Abstract: In light of the conjectured relationship between L-spaces and manifolds with non-left-orderable fundamental group, it is natural to study the behaviour of left-orderable groups in the context of Dehn surgery. This talk will describe some formal properties of Heegaard-Floer homology in this context, and establish analogous behaviour for left-orderable fundamental groups. This is joint work with Adam Clay.

5 Questions and Problems

Two problem sessions were organized during the workshop. Several promising ideas appeared, giving rise to a number of conjectures. Here are some of them.

Question 1

(Andrés Navas)
Give a sequence of hyperbolic manifolds M_n such that $\pi_1(M_n)$ is torsion-free, non left-orderable, and the injectivity radius of M_n tends to infinity as $n \to \infty$.

Discussion

This probably exists. If this exists, then for large enough n $\pi_1(M_n)$ would be non-LO, torsion-free, and would have the unique product property (UPP). To date no such examples are known.

A group Γ is said to have the unique product property if for all finite sets $\{a_1, \cdots, a_n\} \subset \Gamma$ there exists $a = a_i a_j$ such that if $a = a_k a_l$ then $k = i$ and $j = l$. This is also related to the notion of local orderability. A group Γ is locally orderable if there exists a non-left-invariant partial order $<$ of Γ such that for all $f \in \Gamma$ and all $1 \neq g \in \Gamma$, either $gf > f$ or $g^{-1}f > f$. We have

left-orderable $\Rightarrow$ locally orderable $\Rightarrow$ UPP $\Rightarrow$ torsion free.

T. Delzant and Chiswell have also recently proved: Suppose Γ acts by isometries on a δ-Gromov-hyperbolic space X. If for all $x \in X$ and all $1 \neq g \in \Gamma$ we have

$$\mathrm{dist}(gx, x) \geq 6\delta$$

then Γ admits a locally invariant order. As a consequence, if the answer to our question is 'yes,' then we get an example of a locally orderable group that is not left-orderable.

Question 2

(Dave Morris) We have the following theorem: **Theorem**[Howie-Short] If $\Gamma \subset \mathrm{SO}(1,3)$ is a torsion-free lattice (so it's the fundamental group of some finite volume hyperbolic 3-manifold), and there exists a surjective map $\Gamma \to \mathbb{Z}$, then Γ is left-orderable.

Is the same still true for groups $\Gamma \subset \mathrm{SO}(1,n)$? Is there a finite index left-orderable subgroup?

Discussion

This is related to the conjecture that for such a Γ in $\mathrm{SO}(1,n)$ there exists a finite index subgroup $G \subset \Gamma$ such that G surjects onto $\mathbb{Z}$. In this case is G left-orderable?

Question 3

(Jozef Przytycki)

Let $M_L^{(n)}$ denote the n-fold cyclic branched cover of the link L in the manifold M. When is $\pi_1(M_L^{(n)})$ left-orderable?

Discussion

In a talk by Dale Rolfsen in the early 2000's, he showed that $\pi_1(M_{4_1}^{(3)}) = \pi_1(M_L^{(2)})$, where L is the Borromean link, is not left-orderable. Also, Przytycki showed:

- If L is a 2-bridge link with $\dfrac{p}{q} = 2m + \dfrac{1}{2k}$ then $\pi_1(M_{p/q}^{(n)})$ is not LO.
- If $\dfrac{p}{q} = n_1 + \dfrac{1}{1+n_2}$ where n_1, n_2 are odd and $n \leq 3$ then $\pi_1(M_{p/q}^{(n)})$ is not LO.

We may also make the following conjectures:

- $\pi_1(M_{5_2}^{(n)})$ is not LO.
- If L is a 2-bridge link and $|H_1(M_L^{(n)})| < \infty$ then $\pi_1(M_L^{(n)})$ is not LO.

One may also be tempted to conjecture: If L is a hyperbolic link and $|H_1(M_L^{(n)})| < \infty$, then $\pi_1(M_L^{(n)})$ is not LO. This conjecture is not true, however. The even-fold branched covers of the Conway knot were recently shown to have left-orderable fundamental group [Clay, Lidman, Watson].

Question 4

(Dale Rolfsen) Set $\mathrm{LO}(G)$ = the set of all left-orderings of G with the topology defined by Sikora. The subset $\mathrm{O}(G) \subset \mathrm{LO}(G)$ is the space of two-sided orderings (bi-orderings). Let F_n denote the (nonabelian) free group on n generators. Is $\mathrm{O}(F_n)$ a Cantor set or are there isolated bi-orderings? (For the case of left-orderings, $\mathrm{LO}(F_n)$ has no isolated orderings for $n > 2$). If G is the fundamental group of a closed surface of genus > 1 do $\mathrm{LO}(G)$ or $\mathrm{O}(G)$ have isolated points?

Discussion

(Thomas Koberda) Given $\{g_1, \cdots g_n\}$ a finite subset of F_n positive in some bi-ordering, can we find a quotient of F_n and pull back orderings from there? For example, if we set $N_i = F/\gamma_i(F)$, then does there exist $i \gg 0$ such that the semigroup generated by $\{g_1, \cdots, g_n\}$ injects into N_i?

Question 5

(Dave Morris) Suppose G_i are finite index subgroups of G. Then the restriction map $\mathrm{LO}(G) \to \mathrm{LO}(G_i)$ is an injection, and define

$$\mathrm{LO}(G)_{f.i.} = \varinjlim \mathrm{LO}(H),$$

where the limit is over all finite index subgroups $H \subset G$.

- Is this good for anything?
- Conjecture: If Γ is a lattice in $\mathrm{SO}(1,n) \times \mathrm{SO}(1,n)$ (i.e. $\pi_1(M)$ with $\tilde{M} = \mathbb{H}^2$, not $M_1 \times M_2$) then $\Gamma = \pi_1(M)$ is not virtually left-orderable. Can we prove this using $\mathrm{LO}(G)_{f.i.}$?

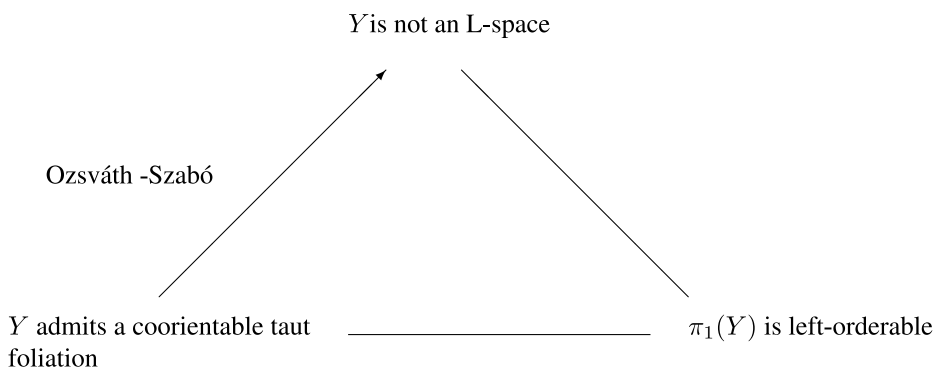
Question 6

(Dale Rolfsen) We know that knot groups are locally indicable and therefore left-orderable. Is there an explicit algebraic or geometric way to understand such an ordering? We now know that the torus knot groups $G = \langle a, b | a^p = b^q \rangle$ have isolated orderings in $LO(G)$. What about other knot groups? If $\pi_1(S^3 \setminus K)$ is bi-orderable and the knot is fibred, we know that the Alexander polynomial $\Delta_K(t)$ has at least two positive real roots. Is this true for non-fibred knots?

Question 7

(Liam Watson)

Let Y be a closed, connected, irreducible, orientable 3-manifold. Fill in the edges of the following triangle with logical implications. The left arrow pointing towards the top is a result of Ozsváth and Szabó. The converse is unknown. The bottom edge of the triangle is understood for some special cases. The connection between left-orderability of the fundamental group and L-spaces is not understood in general. However, we now know that everything in the triangle is equivalent for geometric, non-hyperbolic 3-manifolds.



Question 8

(Patrick Dehornoy)

Our understanding of the canonical braid ordering is still quite incomplete. In particular, the deep property that the restriction of that ordering to the submonoid B_∞^+ is a well-ordering has not been fully exploited. One of the most natural approaches consists in introducing, for every positive braid β, the smallest positive braid $\mu(\beta)$ that is conjugated to β. Computing the function μ would lead to a solution of the Conjugacy Problem of braid groups of an entirely new type. As a first step, experimental evidence support the

Conjecture: One has $\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1\,\mu(\beta)\,\sigma_2^2$ for every β in B_3^+.

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