New Examples with Almost Nonnegative Curvature

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1 Overview of the Field

One of the great unsolved problems of Riemannian geometry is to determine the structure of collapse with a lower curvature bound. An apparently simpler, but still intractable problem, is to determine which closed manifolds collapse to a point with a lower curvature bound. Such manifolds are called almost nonnegatively curved. While many examples of such manifolds are known, few topological obstructions to this class exist and we are far from obtaining a complete classification. Our project is to construct new examples with almost nonnegative curvature, with an eye towards achieving a better understanding of this important class of manifolds.

2 Recent Developments and Open Problems

For our Research in Teams meeting, we proposed to study the following two conjectures. The first generalizes a result of Dyatlov from [4].

Conjecture 1 *Let M be a closed Riemannian G–manifold so that $(M/G, \mathrm{dist}_{\mathrm{orb}})$ has the property that for the image of any stratum, S/G, in the quotient space, M/G, and any $x \in S/G$, $\Sigma_x^{\perp} S/G$ is a join of circles or constant curvature spheres. Then M/G is homeomorphic to a smooth manifold $\bar{M}$ that admits a sequence $\{g_i\}$ smooth Riemannian metrics with curvature $\geq k - \left|O\left(\frac{1}{i}\right)\right|$ so that*

$$\left(\bar{M}, g_i\right) \overset{\mathrm{GH}}{\longrightarrow} (M/G, \mathrm{dist}_{\mathrm{orb}}).$$

Although it is possible that this result could be used for constructing new examples of Riemannian manifolds with positive sectional curvature, it is clear that the number of possible constructions is severely limited by Wilking's connectivity principle [10] and the strong restriction imposed on the space of directions by Conjecture 1. Rather we proposed to use a parametrized version of Theorem 1 below to prove the following.

Conjecture 2 *Let bP_{n+1} be the cyclic group of n-dimensional exotic spheres that bound parallelizable $(n+1)$-dimensional manifolds. For all n there is a generator of bP_{n+1} that admits a family of almost nonnegatively curved metrics. Moreover, when $n \equiv 3 \bmod 4$, the family is invariant under a cohomogeneity four $O(n)$-action.*

The group bP_{n+1} is trivial if n is even and has order 1 or 2 for $n \equiv 1 \bmod 4$ [7]. However, the order of bP_{n+1} grows faster than exponentially in n for $n \equiv 3 \bmod 4$. When $n \equiv 1 \bmod 4$, the generator of bP_{n+1} is a Kervaire sphere (see [6]). It is known to admit a cohomogenity one action. Hence it admits almost nonnegative curvature by work of Schwachhöfer and Tuschmann [9].

Our plan to prove Conjecture 2 begins with Brieskorn's observation ([3]) that in dimensions $n \equiv 3 \mod 4$ all elements of bP_{n+1} can be realized by the varieties, $\mathrm{Br}(6k-1, 3, 2, \ldots, 2)$, described by the following set of complex equations:

$$\begin{aligned} u^{6k-1} + v^3 + z_1^2 + \cdots + z_n^2 &= 0, \\ |u|^2 + |v|^2 + |z_1|^2 + \cdots + |z_n|^2 &= 1. \end{aligned} \tag{1}$$

We have already shown that:

Theorem 1 *Except when $k = 1$, Br$(6k-1, 3, 2, \ldots, 2)$ does not admit a family of metrics that are invariant under the $(S^1 \times O(n))$-action that leaves Display 1 invariant.*

When $k = 1$, the quotient, $Br(5, 3, 2, ..., 2)/O(n)$ is homeomorphic to D^4, with singular stratum ∂D^4 containing $K(3, 5)$, the $(3, 5)$–knot. (see [2], [5]).

We observe, by using Theorem B in [8], that in order to prove Conjecture 2, it is sufficient to show that D^4 admits a family of almost non-negatively curved Alexandrov metrics which has the following characteristics:

1. For $x \in K(3, 5) \subset D^4$, the space of unit normal directions is isometric to the spherical suspension of the constant curvature 4 sphere.
2. For $x \in \partial D^4 \setminus K(3, 5)$ the space of directions is isometric to the constant curvature 1 hemisphere.
3. The interior of D^4 is a smooth Riemannian manifold.

3 Outcome of the Meeting

As Conjecture 1 is part of our plan to prove Conjecture 2, we focused most of our attention at the workshop on obtaining the proof of Conjecture 1. Prior to the meeting most of the details of the proof of Conjecture 1 were written down or had at least been discussed in one way or another. Although our pre-conference efforts were productive, the results were far from publishable form, as different parts of the project had been tackled by different subsets of our group, and the arguments did not hang together well.

We exploited the luxury of being physically together at BANFF to devise a coherent organizational plan. This had at least two positive and unanticipated benefits, which we outline here below.

First, it became clear that it would be considerably easier to approach the proof of Conjecture 1 in the context of certain abstract Alexandrov spaces that we have tentatively decided to call "quotient-like". This led to a long discussion of just what analytic properties characterize those Alexandrov spaces that are quotients of Riemannian manifolds. It seems that we are close to a complete understanding of this characterization, which would be exciting in its own right; in particular, it could lead to further applications of the lifting theorems of [8].

Second, in the quest for this characterization, we were led to the observation that methods of [8] can be applied to infinitely many of the spaces in [1], thus leading us to new and unanticipated examples with almost nonnegative curvature.

In summary, we feel that the week spent at Banff International Research Station during the Research in Teams workshop substantially improved our understanding of the general context of the problems under consideration and we feel that we made significant progress towards solving Conjectures 1 and 2.

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