Report on the 5-day workshop at BIRS (18w5002)
Around Quantum Chaos (in conjunction with 2018 ICMP in Montreal)
Organizers: D. Jakobson, S. Nonnenmacher, S. Zelditch

1. Introduction

The workshop was focussing on various aspects of "quantum chaos", namely the study of quantum or wave systems, the classical limit of which exhibits some chaotic behaviour. More generally, the talks concerned the high frequency and semiclassical limit of wave propagation in various geometric settings.

Quantum Chaos aims to understand spectra and eigenstates of quantum systems, and relate them to the properties of the corresponding classical dynamical system. The Correspondence Principle in Quantum Mechanics predicts that certain features of the classical system manifest themselves in the semiclassical limit of its quantization. For example, the properties of high energy eigenvalues and eigenfunctions of the Laplacian Δ on a Riemannian manifold should reflect the properties of the corresponding geodesic flow. Other examples include the billiard flow, for spectral problems on manifolds with boundary (Gérard–Leichtnam, Zelditch–Zworski); the frame flow, for the Dirac operator and the Hodge Laplacian (Jakobson–Strohmaier–Zelditch); and branching billiard flows, for systems where the operator coefficients have jump discontinuities (Colin de Verdière, Jakobson–Safarov–Strohmaier).

The workshop gathered specialists studying such problems for various systems, including Riemannian manifolds (with ergodic, completely integrable or mixed geodesic flows), arithmetic manifolds, quantum graphs, as well as random systems. The spectral theory of such systems is studied using a variety of different methods, including microlocal analysis, number theory, the theory of elliptic PDE, probabilistic methods and graph theory. The goal of the workshop was to review the latest progress in the field (including the work related to several fascinating *universality conjectures*), and to facilitate interactions between specialists of different subdomains.

The workshop was organized in conjunction with 2018 ICMP in Montreal that was held the following week. Many workshop participants also took part in the ICMP.

2. Participants

The workshop attracted altogether 34 participants, among whom 6 graduate students, 3 postdocs and about 5 recently hired academics. A vast majority of the participants stayed the whole week. The origin of the participants was essentially Western Europe and North America.

3. Detailed report

Monday was devoted to the analysis of high frequency eigenfunctions of Laplace-Beltrami or Schrödinger operators on Riemannian manifolds. The structure of these eigenfunctions can be described in various ways: through their L^p norms (how fast do the norms grow with the frequency?), through their restrictions on submanifolds, through their microlocal concentration properties (that is, their localization properties on the manifold and in Fourier space), through the structure of their nodal set. The interplay between these different aspects has proved very fruitful. Some of the talks focussed on situations of *Quantum chaos*, that is when the geodesic flow on the manifold is chaotic, leading to a certain amount of delocalization for the eigenfunctions.

The first talk was given by Jeff Galkowski. The title was *Concentration of eigenfunctions: sup norms and averages.* Jeff related the microlocal concentration of Laplace eigenfunctions (that is, their localization in phase space) to their sup-norms and their averages on submanifold. In particular, he presented a unified picture for sup-norms and submanifold averages, which characterizes the concentration of those eigenfunctions with maximal growth. He then exploited this characterization to derive geometric conditions under which maximal growth cannot occur. Moreover, he obtained quantitative gains on possible growth in a variety of geometric settings. The talk was mostly based on joint works with Y.Canzani and J.Toth.

The second talk was given by Semyon Dyatlov, who spoke on *Lower bounds on eigenfunctions and fractal uncertainty principle.* The talk concerned the following problem. Let (M, g) be a compact Riemannian manifold and $\Omega \subset M$ a nonempty open set. Take an L^2 normalized eigenfunction u_λ of the Laplacian on M with eigenvalue λ^2. What lower bounds can we get on the mass $m_\Omega(u_\lambda) = \int_\Omega |u_\lambda|^2$? There are two well-known bounds for a general manifold M:

(a) $m_\Omega(u_\lambda) \geq c\, e^{-c\lambda}$, following from unique continuation estimates,
(b) $m_\Omega(u_\lambda) \geq c$, assuming that Ω intersects every sufficiently long geodesic (this is known as the *geometric control condition*).

(in both cases $c > 0$ is independent of λ). In general one cannot improve the bound (a) for an arbitrary domain Ω, as illustrated by the example of the "Gaussian beam" eigenstates of the round sphere, which are exponentially concentrated along the equator.

Semyon presented a recent result which establishes the lower bound (b) for *any* choice of Ω if M is a compact surface of constant negative curvature. This bound has numerous applications, such as a control for the Schrödinger equation, the exponential decay of damped waves (no matter how small the support of the damping is), and the full support property for the semiclassical measures associated with the eigenstates. The proof uses the chaotic nature of the geodesic flow on M. The key new ingredient is a recently established *Fractal Uncertainty Principle*, which states that no function can be simultaneously localized close to fractal sets in both position and frequency. This talk was based on joint works with Joshua Zahl, Jean Bourgain and Long Jin.

The first talk of the afternoon was given by John Toth, joint work with X. Wu. It was titled *Reverse Agmon estimates for Schrödinger eigenfunctions.* Let (M, g) be a compact, Riemannian manifold and $V \in C^\infty(M; \mathbb{R})$. Given a regular energy level $E > \min V$, John considered L^2-normalized eigenfunctions, u_h of the semiclassical Schrödinger operator $P(h) = -h^2\Delta_g + V$ with eigenenergies $E(h) = E + o(1)$ as $h \to 0^+$. The well-known Agmon-Lithner estimates are exponential decay estimates (i.e. upper bounds) for eigenfunctions in the forbidden region $\{V(x) > E\}$. The decay rate is given in terms of the Agmon distance function d_E associated with the degenerate Agmon metric $(V - E)_+\, g$ supported in the forbidden region.

The main result is a partial converse to the Agmon estimates (ie. exponential *lower* bounds for the eigenfunctions) in terms of Agmon distance in the forbidden region, under a control assumption on the L^2 mass of the eigenfunction in the allowed region $\{V(x) < E\}$ arbitrarily close to the caustic $\{V(x) = E\}$. John explained this reverse Agmon estimate, and then gave some applications to the restrictions of the eigenfunctions on hypersurfaces situated in the forbidden region, along with corresponding nodal intersection estimates.

The next talk was given by Lior Silberman (joint work with S. Eswarathasan), entitled *Scarring of quasimodes on hyperbolic manifolds.* For M be a compact hyperbolic manifold, the entropy bounds of Anantharaman *et al.* restrict the possible invariant measures on T^1M that can be obtained as semiclassical measures of sequences of Laplace eigenfunctions. On the other hand, if one relaxes the strict eigenfunction condition, but authorizes approximate eigenfunctions ("quasimodes") with a certain precision, the associated semiclassical measures may be less constrained. The precision threshold beyond which the entropy bounds relax corresponds to so-called "logarithmic quasimodes" (the precision depends inverse logarithmically with the approximate eigenvalue). It is thus relevant to construct quasimodes of this precision, which may be associated with more singular semiclassical measures. Generalizing works of Brooks and Eswarathasan–Nonnenmacher (who constructed logarithmic quasimodes concentrated on closed hyperbolic geodesics), Silberman considered hyperbolic manifolds M admitting a totally geodesic submanifold, and then constructed logarithmic quasimodes concentrating on this submanifold.

Etienne Le Masson gave the last talk on Monday, titled *Quantum Chaos in the Benjamini-Schramm limit,* based on joint works with T. Sahlsten, M. Abert and N. Bergeron. One of the fundamental problems in quantum chaos is to understand how high-frequency waves behave in chaotic environments. A famous but vague conjecture of Michael Berry predicts that they should look on small scales like Gaussian random waves. Etienne showed how a notion of convergence for sequences of manifolds, called the Benjamini-Schramm convergence, can provide a satisfying formulation of this conjecture. The Benjamini-Schramm convergence includes the high-frequency limit as a special case, but provides a more general framework. Based on this formulation, he expanded the scope and considered a case where the frequencies stays bounded but instead the size of the manifold increases. He formulated the corresponding random wave conjecture and presented some results to support it, including a quantum ergodicity theorem.

Tuesday was concerned with the study of nodal sets and critical sets of high frequency eigenfunctions, a topic which has experienced a boom in the last 10-15 years. Examples included the Laplace-Beltrami operator on manifolds or on quantum graphs, as well as the perturbations of the Harmonic oscillator on $\mathbb{R}^n$, where the geometry of the nodal set drastically differs between the allowed vs. forbidden regions of space.

The first talk was by Boris Hanin, titled *Nodal Sets and Eigenvalues for Small Radial Pertrubations of the Harmonic Oscillator.* In this talk, he presented some recent results (joint with T. Beck) about the behavior near infinity of the nodal sets of eigenfunctions for a small radial perturbation of the harmonic oscillator. For the unperturbed oscillator $P(h) = -h^2\Delta + |x|^2$, the separation of variables allows to write eigenfunctions as products of Laguerre functions and spherical harmonics.

For a fixed energy $E_0 > 0$ and small semiclassical parameter $h > 0$, the eigenfunctions with energies $E = h(n + d/2) \approx E_0$ may be composed of spherical harmonics of degrees (= angular momentum) up to h^{-1}. Imposing a radial perturbation, the energy E eigenspace breaks into different energies with different angular momenta. The radial rate of growth for the eigenfunctions is an increasing function of their energy E. The authors' results give precise information about which angular momenta give the largest energies after perturbation, and are able to draw quantitative estimate for the nodal set of these eigenfunctions deep into the forbidden region.

The next talk was by Robert Chang, titled *Log-scale equidistribution of nodal sets in Grauert tubes.* Let M be a compact real analytic negatively curved manifold. It admits a complexification in which the metric induces a pluri-subharmonic function $\sqrt{\rho}$ whose sublevel sets are strictly pseudo-convex domains M_τ, known as Grauert tubes. The Laplace eigenfunctions on M analytically continue to the Grauert tubes, and their complex nodal sets are complex hypersurfaces in M_τ. S.Zelditch proved that the normalized currents of integration over the complex nodal sets tend to a single weak limit $dd^c\sqrt{\rho}$ along a density one subsequence of eigenvalues. In this talk, Robert discussed a joint work with S.Zelditch, in which they showed that the weak convergence result holds on small scale, namely, on logarithmically shrinking Kaehler balls whose centers lie in $M_\tau \setminus M$. The main technique is a Poisson-FBI transform relating quantum ergodicity on Kaehler balls to quantum ergodicity on the real domain. Similar small-scale quantum ergodicity results were obtained in the Riemannian setting by Hezari-Riviere and Han, and in the ample line bundle setting by Chang-Zelditch.

In the next talk, Graham Cox talked about *Nodal deficiency, spectral flow, and the Dirichlet-to-Neumann map.* Courant's nodal domain theorem provides a natural generalization of SturmLiouville theory to higher dimensions; however, the result is in general not sharp. It was recently shown that the nodal deficiency of an eigenfunction is encoded in the spectrum of the Dirichlet-to-Neumann operators for the eigenfunction's positive and negative nodal domains. While originally derived using symplectic methods, this result can also be understood through the spectral flow for a family of boundary conditions imposed on the nodal set. In the talk Graham described this flow for a Schrödinger operator with separable potential on a rectangular domain, and describe a mechanism by which low energy eigenfunctions do or do not contribute to the nodal deficiency. Operators on non-rectangular domains and quantum graphs were also discussed. The talk represents joint work with G. Berkolaiko and J. Marzuola.

Gregory Berkolaiko gave a talk on *Nodal statistics of graph eigenfunctions,* based on joint work with L. Alon and R. Band. He started by reviewing the notion of "quantum graph", its eigenfunctions and the problem of counting the number of their zeros. The nodal surplus of the n-th eigenfunction is defined as the number of its zeros minus $(n-1)$, the latter being the "baseline" nodal count of Sturm-Liouville theory. It appears from numerics that the distribution of the nodal surplus of large graphs has a universal form: it approaches a Gaussian as the number of cycles grows. Gregory discussed recent progress towards proving this conjecture. When the graph is composed of two or more blocks separated by bridges, he proposed a way to define a "local nodal surplus" of a given block. Since the eigenfunction index n has no local meaning, the local nodal surplus has to be defined in an indirect way via the nodal-magnetic theorem of Berkolaiko, Colin de Verdière and Weyand. By studying the symmetry properties of the distribution of the local nodal surpluses the authors showed that for graphs with disjoint cycles the distribution of (total) nodal surplus is binomial.

Ram Band gave the next talk titled *Neumann domains on manifolds and graphs,* based on joint works with L. Alon, M. Bersudsky, S. Egger, D. Fajman and A. Taylor. The nodal set of a Laplace eigenfunction forms a partition of the underlying manifold or graph. Another natural partition is based on the gradient vector field of the eigenfunction (on a manifold) or on the extremal points of the eigenfunction (on a graph). The submanifolds (or subgraphs) of this partition are called Neumann domains. Ram presented the main results concerning these Neumann domains. He compared the situation for manifolds and graphs and related the Neumann domain results in each case to the corresponding nodal domains.

The last talk on Tuesday was given by Junehyuk Jung, titled *Boundedness of the number of nodal domains of eigenfunctions.* The asymptotic number of nodal domains of eigenfunctions is related with the dynamics of the geodesic flow on the manifold. For instance, if a surface with boundary has an ergodic geodesic flow, then for any given Dirichlet eigenbasis, one can find a density one subsequence of eigenfunctions, for which the number of nodal domains tends to $+\infty$ with the frequency. In this talk, Junehyuk discussed what happens to the unit circle bundle over a manifold. When equipped with a metric which makes the Laplacian commute with the circular action on the fibers, the geodesic flow never is ergodic. Recently he and S. Zelditch proved that among such metrics the following property is generic: for any given orthonormal eigenbasis one can find a subsequence of density one where the number of nodal domains is identically 2. This highlights how the underlying dynamics can impact the nodal counting. Junehyuk sketched the proof in the case of a compact surface of genus $\neq 1$, and presented an explicit orthonormal eigenbasis on the 3-torus where *all* the eigenfunctions only featured two nodal domains.

On **Wednesday** morning, the talks dealt with the study of resonances, both in the case of hyperbolic dynamical systems (Pollicott-Ruelle resonances) and in quantum disordered systems (Anderson model). The dynamical resonances were investigated using tools from microlocal analysis, thus establishing a direct connection with quantum chaos.

Gabriel Rivière gave a talk titled *Witten Laplacians and Pollicott-Ruelle spectrum,* based on joint work with N.V. Dang. Given a smooth Morse function and a Riemannian metric on a compact manifold M, Witten defined a twisted Laplacian, which is nowadays referred to as the Witten Laplacian. In light of the recent developpement towards the spectral analysis of hyperbolic dynamical systems, Gabriel discussed some well-known properties and some new ones of these Witten Laplacians. For instance, he explained that the spectrum of this operator converges, in the semiclassical limit, to a certain Pollicott-Ruelle spectrum, describing the decay of correlations for the geodesic flow on M.

The next talk by Frederic Faure was titled *Some properties of hyperbolic dynamics from micro-local analysis,* joint work with M. Tsujii. In a uniformly hyperbolic system (Anosov system), each trajectory is strongly unstable and its behavior is unpredictable. A smooth probability distribution evolves also in a complicated way since it acquires higher and higher oscillations. Nevertheless, using microlocal analysis, this evolution is predictable in the sense of distributions. It is similar to a quantum scattering problem in the cotangent space, as treated by Helffer and Sjöstrand using escape functions in 1986. In the talk Frederic used wave packet formalism (or FBI transform) and explained how to derive some spectral properties of the dynamics: the existence of the intrinsic discrete spectrum of Ruelle-Pollicott resonances, informations about their distribution (fractal Weyl law, band structure), or estimates on the wave front set of the metastable states associated with the resonances.

The last talk on Wednesday (followed by a free afternoon) was delivered by Martin Vogel, titled *Resonances for large random systems.* There have been many works studying resonances generated by compactly supported potentials and by potentials which decay sufficiently fast at infinity. However, in the case of random potentials there are only very few results. Martin gave an overview of some recent results in this direction by J. Sjöstrand, A. Drouot and F. Klopp. In particular he discussed his results obtained in collaboration with F. Klopp on the distribution of resonances close to the real axis, and their link to the eigenstates of a full random Schrödinger operator in the localized regime.

The first four talks on **Thursday** concerned various forms of random wave models, which were initially proposed as models describing the statistical properties of quantum chaotic eigenfunctions. These models are now investigated by both probabilists and mathematical physicists, making connections with percolation theory and real algebraic geometry.

Igor Wigman spoke on *Russo-Seymour-Welsh estimates for the Kostlan ensemble of random polynomials,* based on joint work with D. Beliaev and S. Muirhead. Beginning with the predictions of Bogomolny-Schmit for the random monochromatic plane wave, in recent years deep connections have emerged between the level sets of smooth Gaussian random fields and percolation. In classical percolation theory a key input into the analysis of global connectivity are scale-independent bounds on crossing probabilities in the critical regime, known as Russo-Seymour-Welsh (RSW) estimates. Similarly, establishing RSW-type estimates for the nodal sets of Gaussian random fields is a major step towards a rigorous understanding of these connections. The Kostlan ensemble is an important model of Gaussian homogeneous random polynomials. The nodal set of this ensemble is a natural model for a 'typical' real projective hypersurface, whose understanding can be considered as a statistical version of Hilbert's 16th problem. In the talk, Igor established RSW-type estimates for the nodal sets of the Kostlan ensemble in dimension two, providing a rigorous relation between random algebraic curves and percolation. The estimates are uniform with respect to the degree of the polynomials, and are valid on all relevant scales; this result resolves an open question raised recently by Beffara–Gayet. More generally, the arguments yield RSW estimates for a wide class of Gaussian ensembles of smooth random functions on the sphere or the flat torus.

Yaiza Canzani spoke on *Local universality for zeros and critical points of monochromatic random waves,* based on joint work with B. Hanin. In the talk she discussed the asymptotic behavior of zeros and critical points for monochromatic random waves on compact smooth Riemannian manifolds, as the energy of the waves grows to infinity.

Damien Gayet gave a talk titled *Percolation of random nodal lines.* If a real smooth function is given at random on the plane, what is the probability that its vanishing locus has a large connected component? Damien explained some recent answers he obtained with Vincent Beffara to this question, for some natural models coming from algebraic geometry and spectral analysis.

Melissa Tacy gave a talk titled *Does it matter what we randomize?* The behaviour of quantum chaotic states of billiard systems is believed to be well described by Berry's random plane wave model $u = \sum_j c_j e^{i\lambda x\cdot\xi_j}$, where the c_j are Gaussian random variables. However, in $\mathbb{R}^n$ there are many other candidate waves over which we could randomize. Some are easier to adapt to manifolds than others. In this talk Melissa discussed when (in $\mathbb{R}^n$) we can replace the plane wave $e^{i\lambda x\cdot\xi_j}$ with other waves and how those can be adapted to manifolds.

The last two talks on Thursday as well as the first one on Friday were concerned with wave propagation or wave control in presence of nonsmooth coefficients: control of the Schrödinger operator by an arbitray L^2 control function; propagation of singularities on surfaces with conical singularities, or in presence of a potential with a conormal singularity along a hypersurface.

Nicolas Burq spoke about *Rough controls for the Schrödinger equation on the torus.* He presented some results on the exact controllability of the Schrödinger equation on the torus. In a general setting, these questions are well understood for wave equations with continuous localization functions, while for Schrödinger one only has partial results. For rough localization functions, Nicolas first presented some partial results for waves. Then he showed how one can take benefit from the particular simplicity of the geodesic flow on the torus to get (for continuous localization functions) strong results (works by Haraux, Jaffard, Burq-Zworski, Anantharaman-Macia). Finally, for general localization functions (typically characteristic functions of measurable sets) he showed how one can go further, by taking benefit from dispersive properties (on the 2 dimensional torus), to show that in this setting the Schrdinger equation is exactly controllable by any L^2 (non trivial) localization function (and in particular by the characteristic function of any set with positive measure).

Luc Hillairet spoke on *The wave trace on a flat surface with conical singularities.* In joint work with A. Ford and A. Hassell, he studies the contribution to the wave trace of diffractive periodic orbits on Euclidean surfaces with conical singularities. Using a new description of the propagator near the so-called geometrically diffractive rays, he is able to compute the leading term of any kind of diffractive periodic orbit.

Jared Wunsch spoke on *Diffraction of semiclassical singularities by conormal potentials,* joint work with O. Gannot. Consider a semiclassical Schrödinger operator $P = -h^2\Delta + V$, where V has a conormal singularity along a hypersurface. The singular structure of V affects the propagation of semiclassical singularities for solutions to $Pu = Eu$, and in particular there is a 'diffraction' of the wavefront set by the interface: singularities are reflected as well as transmitted as they cross the interface transversely. The reflected wave, however, is more regular, with the improvement depending on the regularity of the interface; moreover singularities cannot (for high enough regularity) glide along the interface.

The last conference talk on **Friday** by Semyon Klevtsov described the mathematical theory of the Quantum Hall Effect, more specifically the study of Laughlin states, the fundamental Ansätze of this theory. It was titled *Geometry and large N asymptotics in Laughlin states.* Laughlin states are N-particle wave functions, successfully describing the fractional quantum Hall effect (QHE) for plateaux with simple fractions. It was understood early on that much can be learned about QHE when Laughlin states are considered on a Riemann surface. Mathematically, it is interesting to know how the Laughlin states depend on the Riemannian metric, the magnetic potential, the complex structure moduli, the singularities – for a large number of particles N. Semyon reviewed the results, conjectures and further questions in this area, and their relation to topics such as Coulomb gases/beta-ensembles, Bergman kernels for holomorphic line bundles, Quillen metric, or zeta regularized determinants.

The conference ended around noon on Friday.