Tau functions of integrable systems and their applications

Marco Bertola (SISSA, Italy),
Alexander R. Its (IUPUI), Dmitry Korotkin (Concordia University)

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1 Overview of the Field

The notion of tau-function was coined in the early days of the theory of integrable systems about 40 years ago in works of Jimbo-Miwa-Ueno [JMU81], Segal-Wilson [SeWi1985], Sato[Sa1989] and many others. In the case of celebrated Kadomtcev-Petviashvili (KP) equation the notion of tau-function proposed by Segal-Wilson can be viewed as an extension of the notion of the multi-dimensional theta-function. In works of Jimbo, Miwa and their coworkers the tau-function was introduced as a partition function of so-called holonomic quantum fields on a punctured Riemann sphere; the Jimbo-Miwa tau-function was later proven by Malgrange [Ma1983] to play a primary role in the theory of Riemann-Hilbert problems whose solvability is governed by vanishing properties of the tau-function.

Surprisingly enough, tau-functions of the most celebrated integrable systems like Korteveg de Vries (KdV), KP and Toda equations were proven to appear in fundamental combinatorial models, in particular related to random matrices and geometry of moduli spaces. The most famous example is Witten's conjecture about intersection numbers of ψ-classes on moduli spaces of Riemann surfaces; as it was proved by Kontsevich in 1991, their generating function is a partition function of a random matrix model and, moreover, is a tau-function of the KdV equation. Furthermore, as it was later discovered by Okounkov and Pandharipande [OkPa2005], generating functions of Hurwitz numbers are special tau-functions of Toda's equation. These results were extended to intersection numbers of an arbitrary combination of ψ and κ-classes (Kazarian), to Gromov-Witten invariants (Pandharipande et al) etc.

Tau-functions play a central role in Dubrovin's theory of Frobenius manifolds which formalizes the Witten-Dijgraaf-Verlinde-Verlinde equation of associativity arising in topological field theory. From the point of view of symplectic geometry tau-functions play the role of generating functions of commuting Hamiltonians of a completely integrable system.

2 Recent Developments and Open Problems

Among recent developments in the theory of tau-functions and their applications are:

– Series of works initiated by conjecture of Gamayun, Iorgov and Lisovyi [GaIoLi2012] who discovered that the coefficient in expansion of Jimbo-Miwa tau-function of Painlevé IV equation coincide with conformal blocks of certain conformal fiels theories.

– The analysis of Jimbo-Miwa tau function allowed to obtain a non-trivial new relation in the Picard group of the space of admissible covers and the moduli space of holomorphic differentials on algebraic curves (Korotkin-Zograf)[KoZo2011] .

– Appearance of tau-functions in the holomorphic factorization formulas for determinant of Laplacian on flat Riemann surfaces with conical singularities (Kokotov, Korotkin), and in the theory of Lyapunov exponents of Teichmüller flow on moduli spaces (Eskin-Kontsevich-Zorich) [EsKoZo2011].

3 Presentation Highlights

3.1 Tau-functions and geometry of moduli spaces.

The moduli spaces of interest to the workshop are associated to algebraic curves: these are moduli spaces of Riemann surfaces, both pointed and non-pointed, Hurwitz spaces, moduli spaces of abelian, quadratic and n-differentials. Generating functions for interesting quantities related to these moduli spaces (Hurwitz numbers, volumes, intersection numbers of various tautological classes etc) coincide, in many instances, with tau-functions of various integrable systems (KdV, Toda, KP). Within this area we found the talks of Chaya Norton who spoke on the symplectic geometry of moduli spaces of projective connections; the symplectic structures that were discovered are related to the geometry of the "double covering" curve, which is associated in a natural way to any quadratic differential (a well known fact in Teichmüller theory). Here tau functions generate the symplectic transformations between different sets of Darboux coordinates, called "homological". Giulio Ruzza discussed the isomonodromic approach to the computation of intersection numbers in the moduli space of open pointed curves; the approach is similar to the original one by Kontsevich [Kon92] starting from an appropriate matrix model with external source [Ale15a]. Tau functions also appeared as generating functions of generalized Hurwitz numbers, counting the branched covers of the Riemann sphere with various weights; in his talk, John Harnad explained a far reaching generalization of the result by Okounkov and Pandharipande [OkPa2005] that identified the generating function for “double Hurwitz numbers” as a particular tau function of the Kadomtsev-Petviashvili (KP) hierarchy. These weighted enumerations can be also represented graphically by a diagram called “constellation”. The approach that was explained in the talk relies upon the topological recursion algorithm, pioneered by Chekhov (who was one of the speakers as well), Eynard and Orantin. The algorithm hinges on the notion of “quantum spectral curve”; this is essentially a differential equation whose symbol (the “spectral curve”) serves as initial datum for the topological recursion. John explained how, using this approach, they can compute these weighted numbers in terms of the classical objects of the theory of integrable systems.

Within this area we can also list Dimitri Zvonkine’s talk; he reported a new result on explicit constructions of classes in the cohomology of $\mathcal{M}_{g,n}$ which are not “tautological”, i.e. not a combination of the Ψ-classes. A particularly enticing example is the “simplest” non-tautological class, which appears in the moduli space of genus one surfaces (elliptic curve) with 11 marked points. The main reason is due to a clever interpretation of the classical cusp form (of weight 12) or modular discriminant provided by the twenty-fourth power of the Dedekind η function.

3.2 Tau-functions and asymptotics of correlation functions

Correlation functions of various exactly solvable quantum mechanical and statistical models are tau functions associated with isomonodromy deformations of various linear systems. The main analytic issue in these applications of tau functions is their asymptotic behaviour near the relevant critical points. Of a particular importance are the connection formulae between different critical expansions and the evaluation of the constant factors appearing in these asymptotic formulae. These factors, very often, contain the most important information of the models in question. The first rigorous solution of a “constant problem” for Painlevé equations (a special Painlevé III transcendent appearing in the Ising model) has been obtained in the work of Tracy in 1991. Other constant problems have been studied in the works by Basor-Tracy (1992), Krasovsky (2004), Ehrhardt (2006), and in several papers of Its, Deift, Lisovyy and their collaborators starting from 2005. The tau functions that appear in all these works correspond to very special families of Painlevé functions and their extension to the general family of Painlevé transcendents has been a long standing problem in the field . The first rigorous results concerning the general two-parameter families of solutions of Painlevé equations have been obtained only recently in works of Its, Lisovyy, Tikhyy and Prokhorov (2013-2015), and they are based on the combination of the (also recently discovered) conformal block representation of isomonodromic tau functions and on the extension of the original Jimbo-Miwa definition of tau-function given by Malgrange and Bertola. One of the important conceptual outcomes of these results is the realization that the constant factors in the asymptotics of tau functions are the generating functions of certain symplectomorphisms.

Within this scope we found the talks of Gavrylenko, Prokhorov. In his talk Pavlo Gavrylenko explained that the standard principal minor expansion of a Fredholm determinant can be naturally interpreted as multiple sums over partitions (or, equivalently, Maya diagrams). Isomonodromic tau functions of general Schlesinger systems have been expressed as Fredholm determinants and the above expansion provides an explicit combinatorial formula for the general isomonodromic tau function. Interestingly, this approach can be identified with the dual Nekrasov partition function. Gavrylenko also explained how to represent $N \times N$ isomonodromic tau functions as a vacuum expectation value in a vertex-operator-algebra of an N-fermionic CFT; using the combinatorial interpretation alluded above it was explained the terms in the expansion of the combinatorial formula can be identified series over conformal blocks at $c = N - 1$ of a W_N algebra.

The talk by Andrei Prokhorov discussed the relationship between the isomonodromic tau functions of the classical Painlevé equations (in particular Painlevé II and IV) with corresponding classical actions in the Hamiltonian theory (Okamoto) for all Painlevé equations. The main outcome of these relations are certain differential identities which enter in the solution of the "constant" problem in the asymptotic analysis of tau functions. General tau functions display analogous differential identities. He then explained that it is possible to identify the Hamiltonian structure for the isomonodromic deformations corresponding to Painlevé equations and conjecture a generalization thereof to more general cases of isomonodromic deformations.

3.3 Tau functions, Fredholm and Töplitz determinants

One of the prime source of tau functions, both in the integrable systems setting and the isomonodromic settings has been determinants of finite or infinite dimensional operators. This was already explained by Segal-Wilson as a general setup to understand the Sato description of the KP hierarchy. In this optics we can frame both Lisovyy and Basor's talks.

Lisovyy reviewed his recent breakthrough results on the general construction of tau functions for Riemann–Hilbert problems. The method relies upon the identification of the "Widom constant" with a tau function (an identification which was originally found by M. Cafasso [Caf2008]). The Widom constant is the expression appearing in the formulation of the Strong Szegö Theorem for either scalar or matrix-valued symbols on the unit circle. This constant is a Fredholm determinant involving the Malgrange-Bertola forms for both the direct and dual Riemann–Hilbert problem. In those cases where either the dual or the direct problem admits an explicit solution, then the Widom constant can be identified with the isomonodromic tau function. This is precisely the setting of Lisovyy's talk, especially in the case of the sixth Painlevé equation. The resulting expansion in the standard Fourier basis yields interesting combinatorics that have been deeply exploited in the context of conformal blocks (see also the talk by Gavrylenko above). The approach extends so far also to Painlevé V and III.

Of similar nature was the contribution by Estelle Basor; she recalled in a very informative way the original result by Widom whereby the large-size asymptotics of determinants of finite–size (block) Toeplitz matrices leads to the strong Szegö theorem (or Szegö-Widom theorem for the block case). As mentioned before, this constant can be described as a determinant of certain operator that is a trace–class perturbation of the identity operator, and therefore admitting a Fredholm determinant expansion. While in the scalar case the computation is explicit in terms of the Fourier coefficients of the logarithm of the symbol, in the non-scalar (matrix) case it is not possible to express concisely the constant. Estelle focused on classes of nontrivial examples of 2×2 matrix symbols where one can still provide an explicit formula; this relies on particular factorization properties of the symbols that are called "root subgroup factorizations". In these cases the Widom constant's computation reduces to finite dimensional linear algebra.

3.4 Integrable structures of Quantum Field Theory (CFT)

In a 2012 paper by Iorgov-Gamayun-Lisovyi it was conjectured that the isomonodromic Jimbo-Miwa tau-function is closely related to conformal blocks of certain $c = 1$ conformal field theories. These conjectural formulae have been extended to other classes of theories and other tau-functions (Bershtein-Shchechkin, Marshakov-Gavrylenko and others) and, in the case of PVI, proven by Gavrilenko and Lisovyy. Since the original definition of the Jimbo-Miwa tau-function was inspired by the theory of holonomic quantum fields, this new development is naturally closing the circle of ideas originating from quantum field theory which lie behind the notion of Jimbo-Miwa tau-function.

Shchechkin reviewed his recent result on the proof of a particular power series expansion for the discrete q–Painlevé III tau function. This proof settles a conjecture on a q–deformation of the result by Gamayun-Iorgov-Lisovyy that interpreted the coefficients of the expansion as conformal blocks. The idea of the proof is to exploit its equivalence with bilinear relations on q-Virasoro conformal blocks.

The two talks of Marshakov and Bershtein were naturally linked to each other; Marshakov discussed the interpretation of certain dimer models on bipartite graphs and their partition function in terms of cluster algebras. In this context particularly crafted sequences of mutations of the cluster algebra can be viewed as a discrete integrable dynamical system of Painlevé type; moreover he explained how to find their Lax representation.

Bershtein's talk was the natural continuation of the previous talk, where he constructed special solutions of the discrete equations obtained by deautonomization of certain discrete flows in cluster algebras. These solutions can be written in terms of Riemann Theta functions.

Erik Tonni discussed some analytical results describing the entanglement of disjoint intervals in the context of two dimensional conformal field theories (CFT). In particular, he considered the Renyi entropies and the moments of the partial transpose, which provide respectively the entanglement entropy and the logarithmic negativity through some replica limits. These analytic expressions are obtained as the partition function of the CFT model on some particular singular higher genus Riemann surface constructed through the replica method. He presented explicit expressions in terms of Riemann theta functions for simple models like the compactified free boson and the Ising model and also numerical calculations on different lattice models which give evidence of the correctness of the analytic results.

The talk by Kirill Krasnov was devoted to the discussion of the recently discovered colour/kinematics duality in Yang-Mills theory (Bern, Carrasco and Johansson, 2008). This duality states that the Yang-Mills amplitudes at tree level can be represented as "squares" of the Lie algebraic colour structure. This property is intriguing because it suggests that he Yang-Mills theory has some hidden structure which is invisible in its Lagrangian formulation. The duality is by now well understood in the the self-dual (integrable) sector of the theory. Its role in the full theory is not yet completely known; however it is very encouraging that the famous structure Drinfeld double of the Lie algebra of vector fields plays a central role there.

3.5 Tau-functions in the theory of topological recursion and cluster algebras

An alternative method to explore the asymptotic behaviour of matrix integrals (and also more general problems) was developed by physicists and is called the method of topological recursion of Chekhov-Eynard-Orantin which allows to resolve recursively the so-called "loop" equations. Remarkably, various terms arising in this expansion coincide with tau-functions from the theory of Frobenius manifolds, and also encode important combinatorial quantities like Hurwitz numbers, Gromov-Witten invariants, Hodge integrals etc.

Within this area we can list the talk of John Harnad (already reviewed), David Baraglia, Leonid Chekhov, David Baraglia nicely reviewed the famous construction of Hitchin of an integrable system on the co-tangent space of stable vector bundles on a Riemann surface. This is an algebraically integrable system (meaning that the tori are actually Abelian varieties). He explained how the construction works starting from a spectral curve and how these data are precisely the necessary information to initiate the scheme of topological recursion of Chekhov-Eynard-Orantin. However there is no easy interpretation of the outcome of this procedure in terms of geometric invariant;

indeed their meaning is not entirely clear. Nonetheless Baraglia explained that a possible avenue is the theory of special Kahler geometry for the genus zero invariants and Bergman tau functions for the genus one invariants. The interpretation of the higher genera invariants remains a mystery and open problem.

Leonid Chekhov discussed the quantization of the Goldman bracket in the context of SL_k-representations on the moduli spaces $\Sigma_{g,s,n}$ of Riemann surfaces of genus g with s holes and n bordered cusps on the boundaries of the holes. His approach used a certain coordinatization introduced by Fock and Goncharov [FoGa2006] for Teichmüller spaces.

3.6 Tau functions in random matrix theory and random processes.

The theory of random matrices and the asymptotical expansion of matrix integrals is an important topic in combinatorics and physics. From a mathematical point of view most matrix integrals can be viewed as tau-functions of various integrable models; this turns out to be true both for finite matrix and as the matrix size tends to infinity.

Rigorous asymptotics analysis of such matrix integrals is based on Deift-Zhou method of steepest descent applied to an appropriate Riemann-Hilbert problem. The approach allows to obtain at the same time the asymptotic of the tau function (or “partition function”) as well as the strong asymptotics of the corresponding orthogonal polynomials, and hence the method is of interest for the two communities of mathematical physicists on one side as well as the community of approximation theory, mostly focused on the asymptotics of the orthogonal polynomials proper. Additionally, the recurrence coefficients of the orthogonal polynomials provide solutions for the Toda equations and also, when viewed in terms of the index, discrete Painlevé equations.

Within this circle of ideas we had the talks of Percy Deift, Tamara Grava, Peter , Anton Dzhamay, Craig Tracy.

Deift tackled a specific problem of the large-n behaviour of the recurrence coefficients for orthogonal polynomials in a $L^2((-1,1), \ln \frac{2k/(1-x)}{d} x)$. These weights (for $k > 1$) are at the center of a conjecture by Alphonse Magnus, which was thus proved. The problem, while specific, has interesting connotations because its solution requires a slightly different technique from the more standard nonlinear steepest descent. The technical reason is that a crucial step (known as the “parametrix”) near the singularity $x = 1$ of the measure cannot be explicitly constructed in terms of known functions. Instead Deift showed how to cleverly use an approximation scheme in terms of Legendre polynomials.

Grava revisited the problem of the asymptotic expansion of the partition function of the one–matrix model when the asymptotic eigenvalue distribution has two connected components (”two-cuts”). While in the one-cut regime it is known that the partition function and all the associated quantities (orthogonal polynomials, recurrence coefficients etc.) admit a regular asymptotic expansion in inverse powers of the size N of the matrix model, when there are two-cuts or more we also find oscillatory terms at every higher order of approximation. The partition function can be approximated using appropriate Theta functions but also an important contribution of Grava and co-authors has been the explicit correct computation of the ”constant problem”, i.e. the overall multiplicative constant (usually dependent on the size N alone) leading the asymptotic behaviour. This is a notoriously difficult problem (see also the talks of Prokhorov and Lisovyy).

More directly related to the classical theory of integrable systems was the contribution of Peter Clarkson, where he explained how the τ function of certain Painlevé equations can be expressed as the Hankel determinant of so–called “semiclassical” weights. His talk provides a concrete and specific realization of an existing framework that relates these Hankel determinants to more general isomonodromic tau functions [BEH2006]. Moreover he showed how the recurrence coefficients of the corresponding orthogonal polynomials can be interpreted as solving specific instances of discrete Painlevé equations.

Anton Dzhamay showed the connection of certain gap-formation probabilities with discrete Painlevé equations (and also with biorthogonal polynomials). In fact these gap probabilities are also tau-functions of the appropriate type (KP or isomonodromic, depending on the model).

It is well-known that gap probabilities of some discrete probabilistic models can be computed using discrete Painlevé equations. However, choosing correct Painlevé coordinates and matching the model with the standard Painlevé dynamics is a highly non-trivial problem. In this talk we show how this can be done using the geometric tools of Sakai's theory for the model of boxed plane partitions with generalized weights suggested by Borodin, Gorin, and Rains. An important feature of our result is its consistency with the degeneration schemes for the weights matching the degeneration scheme for discrete Painlevé equations.

Random point processes where the correlation are determinants (known therefore as determinantal random point processes) are well known to be related to Fredholm determinants (and tau functions) because the gap-formation probabilities and generating functions of occupation numbers are expressible as such; however there are processes as the Asymmetric Exclusion Process (ASEP) where Fredholm determinants still make a prominent appearance even if the correlations are not determinants; this was the main message of the talk by Craig Tracy. The ASEP is a discrete point process on the line where a particle can jump randomly to the nearest left/right unoccupied site. The probability that the m-th particle from the left is at site x at time t was a breakthrough computation of Tracy and Widom of a few years back. These formulas were expressed in general as sums of multiple integrals and, for the case of step initial condition, as an integral involving a Fredholm determinant. Tracy reported on his more recent work where they obtained a generalization to the case where the m-th particle is the left-most one in a contiguous block of L–particles. These explicit formulæ allow to compute asymptotic behaviour for large time in the Kardar-Parisi-Zhang (KPZ) regime when the initial condition consist of "step-like" data, i.e. there are no particle to the right of the origin and all sites to the left are occupied.

3.7 Tau functions, Integrable Systems and vector bundles.

Jacques Hurtubise explored in his talk the deformation theory of bundles. The key to understanding the development of singularities in the solution of isomondromy problems on the Riemann sphere is whether the underlying bundle is trivial or not: for isomonodromy problems over curves of higher genus, the analogous notion is that of stability of the bundle. Hurtubise then showed that isomonodromy deformations provide a natural framework for the study of transverse deformations away from the unstable loci, whether the connection in question is regular, or has regular or irregular singularities.

The talk of Alexander Bobenko focused on his particular brand of integrable discrete systems, which have found wide applications in computer graphics and even, to some extent, architecture. He presented a procedure which allows one to integrate explicitly the class of (checkerboard) incircular nets. This class of privileged congruences of lines in the plane is known to admit a great variety of geometric properties. The parametrisation obtained in this manner is reminiscent of that associated with elliptic billiards. Connections with discrete confocal coordinate systems and the fundamental QRT maps of integrable systems theory were made.

Simonetta Abenda presented recent results (together with P.G. Grinevich) on the association between points of the totally positive Grassmannian and real algebraic-geometric data in the spirit of Krichever that give algebro–geometric solutions to the Kadomtsev-Petviashvili II (KP) equation. These solutions correspond to a certain finite dimensional reduction of Sato's Grassmannian and their asymptotic behavior is known to be classified in terms of the combinatorial structure of the totally non-negative part of real Grassmannians $Gr^{TNN}(k,n)$. Abenda and Grinevich used Postnikov's classification of totally nonnegative Grassmannians which associates points of the positive Grassmannian with network graphs. They then made the explicit the correspondence with the datum of a rational degeneration of an M–curve of genus g together with a real, non-special divisor of degree g.

The construction of algebro–geometric solution of integrable hierarchies poses the challenge of effective numerical evaluations of canonical data on Riemann surfaces; in this vein Christian Klein discussed a purely numerical approach to compact Riemann surfaces starting from plane algebraic curves and introduced the necessary notions and approximation algorithms used to carry out numerical computations (mostly using Matlab).

In Atsushi Nakayashiki's talk we heard on a higher genus generalization of the classical Weierstrass σ–function. These multi-variate sigma functions have a series expansion whose coefficients are polynomials of coefficients of the defining equation of the curve which has the very desirable property that it behaves smoothly under degeneration of the curve. In his talk he focused on the degeneration of a hyperelliptic curve of genus g to a curve of genus $g-1$. Using the tau function formalism in the context of Sato's Grassmannian, he represented the limiting sigma function in terms of a sum of lower genus sigma functions.

Another application of isomonodromic deformations to the theory of Frobenius manifolds was presented by Davide Guzzetti. In this context it is necessary to consider particular degenerations of isomonodromic systems where the coefficients of the differential equation become resonant (eigenvalues coincide). The system admits a well–defined limit (relevant then for the application to Frobenius manifold theory) only for certain initial data, which he identified and studied.

4 Scientific Progress and Outcome of the Meeting

Talks and discussions which took place during the meeting have sewn the seeds of further developments and advances in all the areas presented at the meeting. We only name a few of them.

In the past years it has become clear that there are still unexplored and deep connections between the now-classical theory of isomonodromic deformations and the symplectic geometry of Painlevé equations; the talks of Prokhorov and Lisovyy are a manifestation of this relationship. In fact, one of the outstanding problems in the area is the computation of the "connection constants"; the ratio of the asymptotic expansions of the tau function near its singular points is a function only of the monodromy data (a "constant" relative to the isomonodromic parameters). Its computation poses significant challenges and result are only recently becoming available in papers of Its, Lisovyy and collaborators. This constant appears also to be a generating function of changes of Darboux coordinates. A novel interpretation that emerged during the discussion is that of this constant as transition function of an appropriate line bundle over the monodromy manifold.

As a result of talk of Zvonkine it has become plausible that the Bergman tau function should be a useful tool in construction of new non-tautological classes on moduli spaces of marked Riemann surfaces; until now very few explicit examples of the non-tautological classes are known.

The unexpectedly important role played by the discrete Painlevé equations in conformal field theory was revealed in the talks of Bershtein-Gavrilenko and A.Marshakov; this link allows to expect that rich geometrical structures associated to the discrete Painlev'e equations can be applied to study correlators and conformal blocks of various conformal field theories.

As a result of the talk of Jacques Hurtubise and subsequent discussion involving Baraglia, Bertola, Korotkin, Norton, Harnad, Ruzza, opened the way to generalization of the notion of isomonodromic tau functions to Hitchin integrable systems including their non-autonomous counterpart and higher genus analogues.

Talks by Baraglia and Hurtubise lead also to elucidation of the link between moduli spaces of abelian differentials on Riemann surfaces and spaces of spectral covers of generalized Hitchin systems. In particular, variational formulas on spaces of spectral covers were derived which generalize the celebrated Donagi-Markman cubic to the meromorphic case. A relationship to the topological recursion of Checkhov-Eynard-Orantin was also derived.

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