Stability of multidimensional waves

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1 Workshop Participants

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- Yuri Latushkin, The University of Missouri, Department of Mathematics, Columbia MO.
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2 Overview of the subject area of the workshop

Traveling waves are solutions of reaction diffusion equations that preserve their shape while moving in a preferred direction. Traveling waves are basic coherent structures in partial differential equations and they often serve as building blocks for complex patterns. Traveling fronts and pulses are abundant in nature and human activities. They arise in applied problems from different fields: optical communication, combustion theory, biomathematics, chemistry, population dynamics, to name a few. Stability of the wave which describes their resilience under perturbations therefore is important.

In this project we are focused on fronts and pulses which are waves asymptotic to spatially equilibria states of the system. These asymptotic states are called rest states of the wave. Stability theory of the traveling fronts and pulses in reaction-diffusion equations is a vast and current subject [3, 4, 6, 7]. Stability analysis is a multi-step process that involves determining the location of the spectrum of the linearization of the underlying system about the front or pulse. The dynamics near the asymptotic rest states of the wave is responsible for the location of the essential spectrum.

Planar traveling waves in the reaction-diffusion systems has been well studied, but the stability theory theory in the multidimensional cases is not as well developed as for the waves in systems posed on the one-dimensional physical space. In 1997, T. Kapitula in [2] proved one of the most important results for multidimensional, planar waves in equi-diffusive reaction-diffusion systems. The result reduces the problem of investigating the stability of a multidimensional planar front to the stability of the associated one-dimensional front. The result implies that if the one-dimensional front is stable, then small perturbations to the planar front decay algebraically.

In 2016, this result was extended to the cases when the one-dimensional front associated to the planar one is marginally stable. More precisely, its has essential spectrum that extends to the imaginary axes. For these cases, exponential weights technique by Sattinger was combined with Kapitula's approach to prove a result similar to Kapitula's result under assumptions on the nonlinear reaction term in the system and under the same assumption on the diffusion matrix - the diffusion matrix has to be an identity.

The main objective of this collaborative effort is to further generalize these two results. Two main goals is to (1) weaken the assumptions on the nonlinearity in [1] ; (2) to allow for a diagonal diffusion matrix which is not a multiple of an identity. We proposed to work on specific models first and then to work on generalization of the results.

3 Progress made during the workshop

During the workshop the participants have identified a specific model that exhibits properties relevant to the topic of the workshop. A significant progress was made during the workshop and details of the analysis of the existence of a planar wave and some partial stability results were worked out. In particular, the participants developed an idea that is the key to overcome issues related to the stability analysis described above is to consider waves supported by the system the dynamics of which is governed by a scalar equation. The existence properties and the stability of the wave then will be dictated by the properties of the associated wave in that scalar equation. This potentially will resolve both issues with the non-identity diffusion and the nonlinear stability. The techniques that allow to relate the traveling wave in the full system to the traveling wave in the scalar equation include Geometric Singular Perturbation Theory for the existence, topological methods such as construction of the Stability Index Bundles and the interpretation of the Stability Index Bundle theory to the language of the properties of the projection operators and generated by them semigroups. The group is currently working on a preparation of a manuscript that contains these original technique.

In addition, a number of new research projects were identified and discussed during the workshop. Plans of work for these projects were developed. In particular, one of the projects is related to the possibility of the generalization of the Sattinger's nonlinear stability in exponential weights result. Indeed, the nonlinear stability in an exponentially weighted norm in [5] is inferred for a parabolic system

$$u_t = \mathcal{A}u + f(u, u_x), \;\; u \in R^n,$$

where $\mathcal{A}$ is assumed to be second-order elliptic operator with constant coefficients. There are applications were the waves and the spectrum share the properties with those in [5], but the differential operator $\mathcal{A}$ has variable coefficients. The group discussed issues arising in this case and ways to overcome those.

References

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