Isogeometric Splines: Theory and Applications

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1 Overview of the Field

The allure of simulation-based engineering is that once a model of an engineering system is constructed, its design can be optimized for performance. Unfortunately, this promise of optimal system has still largely not been realized for complex systems. One of the largest challenges preventing this realization is the so-called design-through-analysis bottleneck.

The design-through-analysis bottleneck results from the fact that different geometric representations of engineering systems are utilized in design (geometric modeling) and analysis (numerical simulation). In design, geometric primitives such as Non-Uniform Rational B-Splines (NURBS) are employed to represent a geometry of interest, while in analysis, polygonal meshes are typically utilized to represent the same geometry. Consequently, design optimization requires not only changing the design representation and creating a new analysis mesh at each design iteration, but doing so in a way that is both automatic and tightly integrated (since it must be done repeatedly until convergence) and differentiable (since efficient and scalable optimization methods require gradients of simulation outputs with respect to design parameters).

The framework of Isogeometric Analysis (IGA), introduced in 2005 by Thomas J.R. Hughes and co-authors, has emerged as a very attractive approach to simulation-based engineering [1, 2]. IGA bridges the gap between design and analysis by employing a uniform representation for the geometry of engineering objects and for the physical quantities defined on it. This eliminates the need for slow and error-prone conversion processes between designed geometry and simulation models, and it enables the possibility of widespread use of design optimization tools in simulation-based engineering.

Given its potential to reshape simulation-based engineering, IGA has recently been the subject of a substantial amount of research activity at a global scale. In particular, there has been a near-exponential growth of publications. According to SCOPUS, there were 12 IGA-related papers published in refereed journal proceedings in 2008, 63 in 2011, 110 in 2014, and 360 in 2018.

2 Recent Developments and Open Problems

Despite the growth and popularity of IGA, it still suffers from a severe flaw. Namely, state-of-the-art techniques in geometric modeling (i.e., Bernstein-Bezier representations, Trimmed B-splines/NURBS, and Boundary Representations (BREPS)) are generally unable to be directly employed in analysis, especially for complex three-dimensional geometries of arbitrary topology. Consequently, the vision for this workshop was to develop a unified geometric modeling framework, which we refer to as *isogeometric splines*, that satisfies both the needs of design and analysis *a priori*. In particular, we seek a geometric modeling framework that can:

(i) Represent objects in arbitrary spatial dimension,

(ii) Represent objects of arbitrary topology or genus in a watertight manner,

(iii) Represent canonical objects such as conic sections exactly,

(iv) Represent localized features with minimal disruption and meaningful design parameters,

(v) Represent not just one geometry but rather families of geometries for design space exploration, optimization, and uncertainty quantification,

(vi) Easily incorporate geometric and topological operations and editing, and

(vii) Easily generate volume parameterizations from surface parameterizations.

Moreover, to meet the needs of analysis, the underlying basis should exhibit optimal approximation and conditioning properties and possess fast algorithms for basis evaluation, differentiation, and integration.

It should be noted that we are not looking to reinvent the wheel with isogeometric splines but instead seek to unify existing approaches and extend them. With this in mind, there exist several emerging technologies which we expect isogeometric splines to leverage, including subdivision surfaces [3], T-splines [4, 5], and hierarchical B-splines [6].

3 Presentation Highlights

During the workshop, each of the 21 participants had a 45-minute time slot to present his or her work. All presentations were of high quality and led to inspiring discussions. The following presentations were among the highlights of the week:

- Artem Korobenko (Canada) presented his results about “Isogeometric analysis for fluids, structures and fluid-structure interaction”. He showed an impressive picture of the potential of the isogeometric simulation technology for demanding engineering applications arising in the aerospace, marine, and energy sectors.

- Carla Manni (Italy) presented her work on “Sharp error estimates for spline appoximation”, focusing on a priori error estimates in L^2 with explicit constants for approximation by splines of maximal smoothness. This contribution showed that the advent of IGA inspired new research on classical topics in approximation theory.

- Nelly Villamizar (UK), who talked about the “Dimension of spline spaces and fat points ideals”, clearly demonstrated that results from advanced algebraic geometry are highly significant for the construction of smooth isogeometric spline discretizations.

- Derek Thomas (United States), presenting “U-splines: Splines over unstructured meshes”, showed how a start-up company has begun to transform the latest mathematical results from the IGA community into a commercial product that may change the traditional way of performing numerical simulation in industry.

See also Section 6 for further information about the talks at this workshop.

4 Scientific Progress Made

Several talks during the workshop highlighted recent research advances that may aid in the development of a unified geometric modeling framework, namely isogeometric splines, that satisfies both the needs of geometry and analysis *a priori*, and several talks also identified deficiencies in state-of-the-art geometric modeling approaches toward the realization of a geometric modeling framework. In particular:

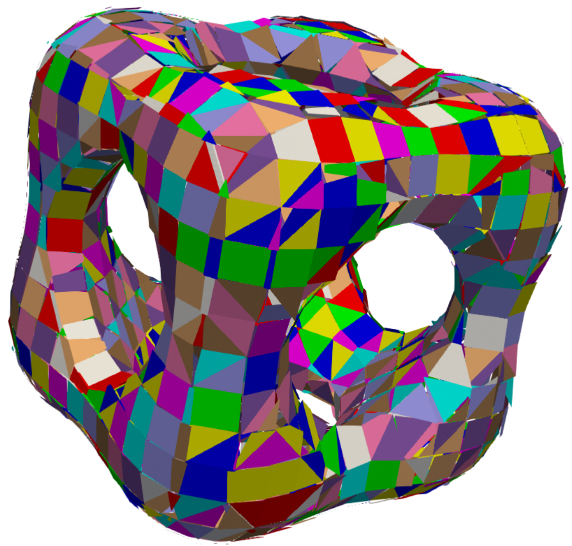

Figure 1: Linearization of a 3D object defined by trimming

(i) Both the state-of-the-art and emerging approaches for geometric modeling in IGA were identified. We specifically mention the talks by Derek Thomas and Jessica Zhang on unstructured spline and T-spline representations.

(ii) The most pressing needs of geometry and analysis were identified. In particular, how to deal with trimmed representations (cf. Fig. 1) is currently one of the main issues facing the IGA community, and this topic was discussed by several participants, including René Hiemstra and Bert Jüttler.

(iii) The workshop also identified needs that are not met by current approaches and challenges the IGA community faces in meeting these needs. Besides trimming, the construction and analysis of globally smooth representations on multi-patch domains is needed for the discretization of high-order problems (e.g., Kirchhoff-Love plates and shells [7], the Cahn-Hilliard equation [8], the Navier-Stokes-Korteweg equations [9], etc.) on complex geometries.

(iv) Several promising approaches, including those originating outside of the computational geometry and numerical analysis communities, were presented. For instance, complementary approaches for addressing the problem of globally smooth representations were presented by Jorg Peters via techniques from classical geometric modeling, by Thomas Takacs based on numerical analysis, and by Nelly Villamizar with the help of results from algebraic geometry.

5 Outcome of the Meeting

Though the workshop did not produce any immediate results, we are convinced that its long-term impact cannot be understated. Design space exploration, optimization, and uncertainty quantification have the potential to dramatically improve the performance, efficiency, and reliability of engineering systems, though these tools have largely been limited to the academic sector. IGA harbors the potential to enable design space exploration, optimization, and uncertainty quantification in engineering practice. Nonetheless, the lack of a unifying mathematical framework which simultaneously addresses the competing needs of geometry and analysis have so far prevented the widespread adoption of IGA. Thus, the primary purpose of the workshop was to focus the vision of the IGA community and unify it toward a set of common goals. The workshop certainly fulfilled this purpose, as the participants identified the most pressing needs of geometry and analysis, the needs which are not met by state-of-the-art approaches, and the challenges the IGA community faces in meeting these needs. In addition, the workshop helped to establish new interdisciplinary collaborations between not only mathematicians and engineers but also between mathematicians across subfields (e.g., between numerical analysts and algebraists).

At the end of the workshop, it was clear that there remain several research advances that must be made before the workshop's vision for isogeometric splines can be realized. First, the problem of trimming must be resolved in a satisfactory manner, for instance, by the removal of trimming curves entirely or by the introduction of robust analysis procedures capable of handling trimmed objects. Second, the problem of constructing globally smooth representations of objects of arbitrary topology must be fully addressed, especially in the three-dimensional setting. Third, the problem of surface-to-volume parameterization remains largely unsolved, even though high quality volume parameterizations are required for the analysis of fluid flow and wave propagation. Fourth, while many new geometric modeling technologies have been introduced to address both the needs of design and analysis, most of these technologies are not able to incorporate topological operations that are common in solid modeling. Finally, virtually none of the geometric modeling technologies that have been introduced directly or naturally allow for the specification of geometric variability, and such a specification is required for the purposes of design space exploration, optimization, and uncertainty quantification.

It should be finally mentioned that the realization of isogeometric splines will not only impact simulation-based engineering but also rapid prototyping. Three-dimensional printing and additive layer manufacturing technology require watertight, three-dimensional input designs in order to produce reliable manufactured geometries. Given the rapid growth of rapid prototyping techniques, there is a pressing need for a change in paradigm to allow for watertight, three-dimensional geometric modeling, and the workshop's vision for isogeometric splines meets this need.

6 Abstracts

For completeness, the abstracts associated with each of the presentations of the workshop are included below.

The impact of parametrization on numerical approximation for isogeometric finite elements
John Evans

High-order finite element methods, including isogeometric finite element methods, harbor the potential to deliver improved accuracy per degree-of-freedom versus low-order methods. Their success, however, hinges upon the use of a curvilinear mesh of not only sufficiently high accuracy but also sufficiently high quality.

In this talk, theoretical results are presented quantifying the impact of mesh parameterization on the accuracy of a high-order finite element approximation, and a formal definition of shape regularity is introduced for curvilinear meshes based on these results. This formal definition of shape regularity in turn inspires a new set of quality metrics for curvilinear finite elements. Computable bounds are established for these quality metrics using the Bernstein-Bézier form, and a new curvilinear mesh optimization procedure is proposed based on these bounds. Numerical results confirming the importance of shape regularity in the context of high-order finite element methods are presented, and numerical results demonstrating the promise of the proposed curvilinear mesh optimization procedure are also provided.

The theoretical results presented in this talk apply to any piecewise-polynomial or piecewise-rational finite element method posed on a mesh of polynomial or rational mapped simplices and hypercubes. As such, they apply not only to classical continuous Galerkin finite element methods but also to discontinuous Galerkin finite element methods and even isogeometric methods based on NURBS, T-splines, or hierarchical B-splines.

This is joint work with Luke Engvall.

Untrimmed splines: Analysis suitable CAD
René Hiemstra

Current CAD technologies describe geometry by means of the boundary representation or simply B-rep. Boolean operations, ubiquitous in computer aided design, use a process called trimming that leads to a non-conforming description of geometry that is un-editable and incompatible with all downstream applications, thereby inhibiting true interoperability across the design-through-analysis process.

We propose a modeling paradigm in which designers are given the tools to create watertight, editable and conforming descriptions of geometry with interactive control over the boundary surface parameterization.

The methodology is based on recent advances in topological vector field design and processing. First, a smooth frame field is computed, by minimizing an appropriate energy functional on a background mesh of the initial B-rep, that is compatible with sparse or dense input constraints on alignment and size. Frame-field singularities, which together satisfy the topological invariant known as the Poincar-Hopf theorem, are automatically placed and can be modified by the user. The frame field is used as a guide for the reparameterization of the initial B-rep into a conforming watertight and editable spline description.

In line with the main rationale of isogeometric analysis, the meshing and reparameterization techniques are applied within CAD as part of the design process, instead of as a post-processing step. The modeling tools actively support the design, analysis and manufacturing process as a whole, enabling true interoperability across these different disciplines and thereby support efficient product development.

This is joint work with Kendrick Shepherd and Thomas Hughes.

Quadrature rules for trimmed domains
Bert Jüttler

A common representation of a Computer-Aided Design (CAD) model is a boundary representation (B-rep), which typically consists of trimmed tensor-product NURBS patches. A trimmed surface patch consists of a tensor-product surface and a set of trimming curves on the surface that specify the boundary of the actual surface. Therefore, it represents only a part of the full tensor-product surface.

Computing integrals over trimmed domains both efficiently and accurately remains a challenging problem, notably for use in the frame of isogeometric analysis (IgA). In this talk, we present a specialized quadrature rule for trimmed domains, where the trimming curve is given implicitly by a real-valued function on the whole domain.

We follow an error correction approach: In a first step, we obtain an adaptive subdivision of the domain in such a way that each cell falls in a predefined base case. We then extend the classical approach of linear approximation of the trimming curve by adding an error correction term based on a Taylor expansion of the blending between the linearized implicit trimming curve and the original one.

This approach leads to an accurate method which improves the convergence of the quadrature error by one order compared to piecewise linear approximation of the trimming curve. It is at the same time efficient, since essentially the computation of one extra one-dimensional integral on each trimmed cell is required. Finally, the method is easy to implement, since it only involves one additional line integral and refrains from any point inversion or optimization operations. The convergence is analyzed theoretically and numerical experiments confirm that the accuracy is improved without compromising the computational complexity. Finally, we show that the method can be extended to trimmed trivariate representations, paving the way to isogeometric simulations on trimmed domains in 3D.

This is joint work with Felix Scholz and Angelos Mantzaflaris.

Multi-patch isogeometric analysis with C^2-smooth functions
Mario Kapl

We present a framework for the construction of a globally C^2-smooth isogeometric spline space over a specific class of planar multi-patch geometries, called bilinear-like G^2 multi-patch geometries. This class of geometries includes the subclass of bilinear multi-patch parameterizations and is characterized by the property to have the same kinds of connectivity functions along the patch interfaces as the bilinear parameterizations. The C^2-smooth isogeometric space is generated as the linear span of three different types of basis functions called patch, edge and vertex functions corresponding to the single patches, edges and vertices of the multi-patch domain. The construction of the single functions is simple, works uniformly for all possible multi-patch configurations and leads to basis functions with small local supports.

The potential of the constructed C^2-smooth space for applications in isogeometric analysis is demonstrated on the basis of several examples. Amongst others, we present a framework for solving the triharmonic equation, a sixth order partial differential equation, over planar multi-patch geometries. This problem requires the use of a C^2-smooth space as discretization space for the corresponding partial differential equation. Moreover, we perform isogeometric collocation to obtain a C^2-smooth solution of Poisson's equation over a planar multi-patch domain. Finally, we numerically show by means of L^2-approximation that the considered space of globally C^2-smooth isogeometric functions possesses optimal approximation properties, and that the generated C^2-smooth basis functions are well-conditioned.

This is joint work with Vito Vitrih.

Isogeometric analysis for fluids, structures and fluid-structure interaction
Artem Korobenko

This talk focuses on application of isogeometric analysis in various problems of computational mechanics. We first start with turbulent flows. The fluid mechanics is governed by incompressible Navier-Stokes equations posed on a moving domain using ALE framework. The equations are discretized in space using quadratic NURBS. The Variational Multiscale (VMS) method is used for turbulence modeling with mesh relaxation on a boundary. It is shown that higher order continuity basis functions provide better turbulence statistics. Also, the complex geometries can be modeled more accurately. The benefits of using IGA is presented on several examples, including atmospheric flow modeling over Bollund hill and Perdigao terrain.

Next, the NURBS-based IGA is applied to the damage modeling in composite structures. The higher order continuity of quadratic NURBS basis function allows the application of thin-shell theory such as Kirchhoff-Love, where your functions should be at least C^1-continuous. This also improves the stress representation which relax the strain localization. The numerical framework is applied to model progressive damage in UAV under extreme flight conditions and fatigue damage in wind turbine blades.

Finally, the computational FSI framework is presented with non-matching discretization on a boundary. The system of non-linear equations are solved iteratively using Newton-Raphson method with the block iterative coupling. The linearized system of equations is solved using GMRES. The framework is applied to simulate wind turbines at full scale under realistic operational conditions.

The show that FSI framework with IGA provides accurate solutions for various challenging problems, providing the data-of-interests that is not readily available or hard to acquire during the experiments.

B-spline-based monotone multigrid methods for the valuation of American options
Angela Kunoth

The valuation of an American option with Heston's stochastic volatility model leads to a free boundary problem in terms of a two-dimensional parabolic partial differential equation with a diffusion, convection and reaction term depending on the price of the underlying asset and its volatility.

We formulate this problem as a parabolic variational inequality on a closed convex set. To determine optimal risk strategies, one is not only interested in the solution of the variational inequality but specifically in the pointwise derivatives of the solution up to order two in space, the so-called Greeks. Initial conditions are commonly given as piecewise linear continuous functions which we approximate with B-splines with coinciding knots at the points where the initial condition is not differentiable. Furthermore, an improvement of the approximations of the spatial derivatives in the initial time steps is achieved by employing Rannacher timestepping. For solving the nonsymmetric discretised variational inequality in each time step and determining the derivative of the solution, we develop a monotone multigrid method for high order B-splines (with possibly coinciding knots) together with a projected iterative scheme. To do so, we construct restriction operators and monotone coarse grid approximations for tensor product B-splines of arbitrary order.

We demonstrate in the numerical experiments that we achieve fast convergence rates of the monotone multigrid method and highly accurate approximations of the Greeks.

This is joint work with Sandra Boschert.

Sharp error estimates for spline approximation
Carla Manni

The emerging field of isogeometric analysis (IGA) triggered a renewed interest in the topic of spline approximation and related error estimates. In particular, isogeometric Galerkin methods aim to approximate solutions of variational formulations of differential problems by using spline spaces of possibly high degree and maximal smoothness.

In this talk we focus on a priori error estimates in L^2 with explicit constants for approximation by splines of maximal smoothness defined on arbitrary knot sequences. We provide accurate estimates, which are sharp or very close to sharp in several interesting cases. These a priori estimates are actually good enough to cover convergence to eigenfunctions of classical differential operators under k-refinement.

The key tools to get these results are the theory of Kolmogorov L^2 n-widths, and related optimal spaces, and the representation of the considered Sobolev spaces in terms of integral operators described by suitable kernels.

This is joint work with Espen Sande and Hendrik Speleers.

Exploring geometrically continuous isogeometric functions in 3D
Angelos Mantzaflaris

One advantage of the framework of isogeometric analysis is that it allows for discretization spaces providing high order smoothness. Using these spaces can be beneficial when solving high order partial differential equations, including the Cahn-Hilliard equation, the Navier-Stokes-Korteweg equation and Kirchhoff-Love shells.

In addition, multi-patch parameterizations are required when considering more complex geometries, and the construction of globally smooth spline functions is a non-trivial problem. This has motivated research on the coupling of isogeometric multi-patch spline spaces across interfaces.

On the one hand, the coupling constraints can be enforced weakly, using variational methods or Lagrangian multiplier-based approaches. On the other hand, the smoothness constraints can be imposed strongly. It turns out that this is particularly well suited for generating C^0–smooth isogeometric splines on multi-patch domains. Furthermore, the construction of multipatch isogeometric discretizations possessing higher order smoothness has attracted lately the attention of several research groups, and has revived the interest of classical works in computer-aided design. Virtually all existing studies refer to the case of two variables, addressing properties such as dimension, suitable discretization bases, approximation and conditioning, etc.

The generalization to the trivariate case of these results is known to be more involved, both from a theoretical and practical viewpoint. In our work we follow a computational approach to discover the first formulas for the dimension of trivariate C^1–smooth isogeometric splines on two-patch domains. We obtain certified results by using exact rational computations on the corresponding linear algebra problem. Moreover, we observe recurring patterns in the nullspace of a collocation matrix expressing the C^1–conditions and we reduce the global problem to local, independent computations of compactly supported basis functions. Preliminary studies of the approximation power of the resulting spaces and discretization bases are encouraging. Nevertheless, we have just scratched the surface of the trivatiate case, since the treatment of domains with more general topology as well as rigorous mathematical theories remain open.

This is joint work with Katharina Berner and Bert Juettler.

Isogeometric analysis for compressible flows in complex industrial geometries
Matthias Möller

In this talk, we describe our IGA framework for the numerical analysis of rotary positive displacement pumps and, in particular, twin-screw compressors. Our approach is based on the overall philosophy that an efficient simulation and, at a later stage, optimization requires the co-design of all components involved in the pipeline, that is, the geometry model and the simulation tools. We present a fully automated algorithm for generating time sequences of analysis-suitable multi-patch parameterizations of counterrotating twin-screw rotor profiles that do not involve topology changes over time and fully exploit the computational potential of modern high-performance computing platforms. The algorithm is based on elliptic grid generation principles and adopts a mixed variational formulation that makes it possible to handle multi-patch parameterizations and parameterizations with C^0 spline basis functions directly. The second part of the talk describes our isogeometric flow solver which, following our co-design philosophy, makes use of auto-generated compute kernels to achieve optimal computational efficiency for each individual patch. The convective term of the Galerkin discretization is stabilized by flux-correction techniques that have been generalized to high-order B-splines in order to suppress the generation of unphysical oscillations in the vicinity of shocks and discontinuities. We finally address the curse of round-off errors which start to become a severe issue for high-order methods. In fact, round-off errors dominate the overall error already for moderate problem sizes if the approximation order is sufficiently large, rendering classical grid convergence studies impractical. We propose a novel a posteriori approach for predicting the optimal mesh width $h_{opt}^{(p)}$ as a function of the approximation order p and perform so-called $h_{opt}^{(p)}p$-refinement to reduce the total error effectively.

Geometrically smooth splines on meshes
Bernard Mourrain

In CAGD, a standard representation of shapes is a boundary representation using parametrized surfaces based on tensor product B-spline functions, which are the basis of the space of piecewise polynomial functions on a grid with a given regularity and degree. However, the complete description of a complex shape by tensor product B-spline patches may require to intersect and trim them, resulting in a geometric model, which is inaccurate or difficult to manipulate or to deform. To circumvent these difficulties, one can consider geometric models composed of quadrangular patches, glued together in a smooth way along their common boundary.

A first objective is to analyze the space of spline functions attached to such constructions. Given a topological complex $\mathcal{M}$ with glueing data along edges shared by adjacent faces, we study the associated space of geometrically smooth spline functions that satisfy differentiability properties across shared edges. We describe algebraic techniques, which involve the analysis of the module of syzygies of the glueing data, to determine the dimension formula of these spaces, for high enough degree. Dimension formula for polynomial patches and B-spline patches are provided. We present a general and algorithmic method for computing the basis. The construction yields basis functions naturally attached respectively to vertices, edges and faces.

The second objective is to construct efficiently geometrically smooth splines on meshes. We present a new subdivision scheme for computing geometrically smooth spline surfaces from a coarse quadrangular mesh. The resulting surface is G^1 everywhere and C^2 except at extraordinary vertices. Each face of the quadrangular mesh is associated to a bi-quintic spline patch. The Catmull-Clark subdivision scheme is used to compute the control points of B-spline patches associated to the faces of the quadrangular mesh. The nearest geometrically smooth biquintic spline surface is then explicitly computed by projection on the space of G^1 splines.

This is joint work with Ahmed Blidia, Nelly Villamizar, and Gang Xu.

Splines on irregular meshes
Jorg Peters

Splines elegantly connect the discrete and continuous computational world via control nets. Their extension to irregular meshes where the tensor-structure breaks down is essential and provides a rich source of mathematical insights. This talk emphasizes the role of splines over irregular meshes in the context of joining geometric design and engineering analysis in the spirit of iso-geometric analysis.

After a brief review of the rich literature since 1984 on using regular splines both for geometry and analysis, the talk presents a classification of techniques for irregular patch layout. Among the smooth surface constructions, the main distinction is between singular constructions (subdivision surfaces, polar layout surfaces, vertex-singular surfaces and rational Gregory surfaces) and regular parameterizations (geometrically smooth, transfinite and generalized barycentric constructions).

With focus on quad meshes and computing on surfaces, the talk discusses in detail: geometrically smooth constructions with T-junctions; subdivision stabilized by guide surfaces and made nearly finite by acceleration; and the class of vertex-singular surface constructions that can be generalized to volumetric hexahedral complexes with irregular edges and points.

Quadrature schemes based on spline quasi-interpolation for Galerkin IgA-BEM
Maria Lucia Sampoli

Boundary Element Method (BEM) is a numerical method to solve PDEs, in which the original problem is reformulated as a system of integral equations defined only on the boundary of the domain. The main advantages of the method are a reduced dimension of the computational domain and the simplicity to solve external problems. One of the important challenges in this topic is to accurately and efficiently solve singular integrals that arise from the boundary integral equations so formulated. Therefore, designing suitable quadrature schemes is one of the main active research topic in BEM. Recently new quasi-interpolation (QI) based quadrature rules have been introduced specifically for IgA-BEM setting. Such quadrature schemes are tailored for B-splines that are the considered basis functions. Quasi-interpolation allows to take advantage of the local support of the basis functions and to provide an approximation using the desired polynomial degree, keeping low the computational costs. The developed quadrature rules, hence provide very good accuracy and optimal convergence rate. Weakly, strongly and hyper-singular kernels related to the 2D integral formulation of the Laplace equation with different types of boundary conditions have been studied giving promising results especially when compared to standard and newest approaches applied in an isogeometric Galerkin BEM context. Moreover local refinability of the approximated solution of the problem is achieved by using hierarchical B-spline spaces.

Bernstein-Bézier techniques for continuous multivariate piecewise harmonic polynomials on simplicial partitions
Tatyana Sorokina

Since the only smooth harmonic splines are polynomials, continuous harmonic splines deserve special attention as possible subspaces for solving PDEs, and modeling harmonic functions. Bernstein-Bézier techniques for analyzing continuous harmonic splines in n variables are developed. Dimension and a minimal determining set for special splits are obtained using the new techniques. We show that both dimension and bases strongly depend on the geometry of the underlying partition. In particular, the angles in the triangulation play an important role. Due to a very small dimension of harmonic polynomials (as opposed to full polynomials), it is impossible to construct harmonic FEMs on unrefined simplicial partitions. We construct quadratic harmonic conforming FEMs on Clough-Tocher refinements and other special partitions.

Construction of smooth B-splines on Powell-Sabin triangulations
Hendrik Speleers

In this talk we will discuss the construction of a suitable B-spline representation for smooth splines on general triangulations. The considered splines have smoothness r and degree $d \geq 3r - 1$, and are defined over a special refinement of the given triangulations. In such a refinement, called Powell-Sabin refinement, every triangle of the triangulation is split into six subtriangles. The B-spline construction can be geometrically interpreted as determining a set of triangles that must contain a specific set of points. The B-spline functions possess several interesting properties:

- local support,
- linear independence,
- nonnegative partition of unity.

This B-spline representation exhibits a natural definition of control points and an intuitive control structure in terms of local triangular nets. These triangular nets locally mimic the shape of the spline surface, and hence they can be used in the geometric design of smooth surfaces. On the other hand, such representation also presents interesting properties for engineering analysis. In particular, the representation allows for:

- stable evaluation and differentiation,
- efficient triangular Bzier extraction,
- optimal approximation and convergence,
- adaptive local mesh refinement.

Overlapping multi-patch domains in IGA
Thomas Takacs

In isogeometric analysis the domain of interest is usually represented by B-spline or NURBS patches, as they are present in standard CAD models. In order to avoid trimming, complicated domains can be represented as a union of simple overlapping subdomains, parameterized by single spline patches. Numerical simulation on such complicated domains is a serious challenge in IGA.

In this talk, we present a non-iterative, robust and efficient method. The computational domain is represented as a collection of B-spline based geometries with overlaps. Consequently, the problem is divided into several sub-problems, which are coupled in an appropriate way. The resulting system can be solved directly in a single step. We compare the proposed method with iterative Schwarz domain decomposition approaches and explore the advantages of our method, especially when handling subdomains with small overlaps.

We will show that the problems can be solved on overlapping patches by a simple non-iterative method, without using trimming. Summing up, our method significantly simplifies the domain parameterization problem. The performance of the proposed method is demonstrated by several numerical experiments in two and three dimensions.

This is joint work with Somayeh Kargaran, Bert Juettler, Stefan Kleiss, and Angelos Mantzaflaris.

Efficient preconditioners for *k*-refined isogeometric analysis
Mattia Tani

In this talk we discuss preconditioning strategies suited for isogeometric analysis, that are robust with respect to the spline degree p and to the mesh size h. Starting with the Poisson problem, we discuss a preconditioner that represents the same problem discretized on the reference domain. The preconditioner can be applied in a very efficient way thanks to the Fast Diagonalization method, that exploits the tensor structure of the basis functions. We then consider the heat equation, and consider a space-time discretization where smooth splines are used both in space and in time. We develop two numerical formulations for this problem, a symmetric high-order least squares formulation, and a nonsymmetric low-order Galerkin formulation. For both approaches, we develop robust preconditioners that can be applied efficiently thanks to a variant to the Fast Diagonalization method. We finally highlight advantages and drawbacks of the two formulations.

This is joint work with Gabriele Loli, Monica Montardini, Mauro Negri, and Giancarlo Sangalli.

U-splines: Splines over unstructured meshes
Derek Thomas

Isogeometric design and analysis is a growing area of research in computational engineering. In an isogeometric approach, the exact CAD representation is adopted as the basis for analysis. To unlock the full potential of isogeometric analysis depends strongly upon the analysis-suitable nature of the underlying geometry. Analysis-suitable geometry possesses a basis that is rich enough for both shape and solution representation. The exact analysis-suitable representation of smooth geometry is essential for correct solution behavior across many application domains. In this talk, we will present motivation and results for algorithms to construct unstructured spline basis functions over unstructured quad meshes (i.e., U-splines) that allow for the presence of T-junctions between mesh faces. We focus particularly on the requirements generality and linear independence of the basis functions. Our construction relaxes the analysis-suitability constraints that have been established for T-splines. We also consider the inclusion of extraordinary points in the mesh.

This is joint work with Luke Engvall, Steven Schmidt, Kevin Tew, and Michael Scott.

Polynomial splines of non-uniform bi-degree on T-meshes: Combinatorial bounds on the dimension
Deepesh Toshniwal

Polynomial splines on triangulations and quadrangulations have myriad applications and are ubiquitous, especially in the fields of computer aided design, computer graphics and computational analysis. Here, focusing on polynomial splines on T-meshes, we study the problem of computation or estimation of the spline space dimension. The general case of splines with polynomial pieces of differing bi-degrees is considered. In particular, using tools from homological algebra, introduced in the context of splines by Billera in 1988, we generalize the approach presented in Mourrain in 2014 and present combinatorial lower and upper bounds on the dimension. We also present sufficient conditions for the lower and upper bounds to coincide. Several examples are provided to illustrate application of the theory developed.

This is joint work with Bernard Mourrain and Thomas Hughes.

Recent developments for isogeometric methods with hierarchical splines
Rafael Vazquez

In this talk I will present several recent results towards the efficient use of hierarchical splines. I will first present a coarsening algorithm for the construction of admissible meshes, and show its advantages in the solution of the transient heat equation with a moving heat source. I will also present the construction of an additive multilevel preconditioner, based on admissible meshes, in such a way that the condition number is bounded and independent of the number of levels. In the last part of the talk I will show results on the construction of hierarchical C^1 basis functions on geometries constructed with two patches.

Dimension of spline spaces and fat points ideals
Nelly Villamizar

In this talk we will address the problem of proving a general formula for the dimension of spline spaces defined on polytopal partitions, particularly triangulations and tetrahedral complexes. We will show the connection between this problem and the study of the Hilbert series of fat points ideals in projective space. Combinatorial upper and lower bounds on the dimension will be presented to illustrate the advances and open problems in spline theory.

A practical unstructured spline modeling platform for isogeometric analysis applications
Jessica Zhang

As a new advancement of traditional finite element method, isogeometric analysis (IGA) adopts the same set of basis functions to represent both the geometry and the solution space, integrating design with analysis seamlessly. In this talk, I will present a practical unstructured spline modeling platform that allows IGA to be incorporated into existing commercial software such as Abaqus and LS-DYNA, heading one step further to bridge the gap between design and analysis. The platform includes all the necessary modules of the design-through-analysis pipeline: pre-processing, surface and volumetric spline construction, analysis and post-processing. Taking IGES files from commercial computer aided design packages, Rhino specific files or mesh data, the platform provides several control mesh generation techniques, such as converting any unstructured quadrilateral/hexahedral meshes to T-meshes, frame field based quadrilateral meshing, and polycube method. Truncated T-splines, hierarchical B-splines and subdivision basis functions are developed, supporting efficient local refinement and sharp feature preservation. To ensure analysis suitability, partition of unity, linear independence and optimal convergence rate of these basis functions are studied in our research.

IGA has very broad engineering applications like the finite element method, and specific application requirements always bring us new research problems and drive the future research directions. At the end of this talk, I will present several practical application problems to demonstrate the capability of our software platform. In addition to mechanics characterization for Navy, NAVAIR and Honda applications, in recent years we also developed novel image registration techniques using truncated hierarchical B-splines, an IGA solver to simulate material transport in complex neuron trees, and a new SimuLearn system to combine finite element method with machine learning for 4D printing.

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