Fluid Equations, A Paradigm for Complexity: Regularity vs Blow-up, Deterministic vs Stochastic

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1 Overview of the Field

Fluids are ubiquitous in nature; yet, our understanding remains far from complete. Mathematically rigorous investigation of fluid dynamics goes back to the pioneering work of Leray [13] in 1934. As hydrodynamics model in a spatial domain $D \subset \mathbb{R}^d$, a prominent system of equations is the Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = \nu \Delta u, \tag{1a}$$

$$\nabla \cdot u = 0, \tag{1b}$$

where $u : \mathbb{R}_+ \times D \mapsto \mathbb{R}^d$ represents the velocity vector field, $\pi : \mathbb{R}_+ \times D \mapsto \mathbb{R}$ the pressure scalar field, and $\nu \geq 0$ the viscosity coefficient. The case $\nu = 0$ reduces this model to the Euler equations, and there are many related equations such as the Boussinesq system, magnetohydrodynamics (MHD) system, Hall-magnetohydrodynamics (Hall-MHD) system, surface quasi-geostrophic (SQG) equations, incompressible porous media equations, and transport equation. Different types of variations include the compressible case, as well as the stochastic case. The Navier-Stokes equations has a wide array of real-world applications such as jet flow in aerodynamics, blood flow in medicine, as well as cash flow in finance, and there are decades of study and applications by physicists and engineers.

However, one of the most fundamental properties of any initial value problem of partial differential equations, namely the existence and uniqueness of a solution with bounded kinetic energy starting from an arbitrary smooth initial data with bounded kinetic energy, remains unknown (see [4]). To this day, the complexity of regularity versus blow-up and deterministic versus stochastic give researchers more questions than answers. Very recently, there have been many new developments in the research areas of blow-up, norm inflation, non-uniqueness via convex integration, andsingular stochastic case due to the very rough space-time white noise.

In the past decade, the research direction on PDEs in fluid dynamics is developing at an unprecedented rate, with many breakthrough results on singularity formation, non-uniqueness of weak solutions, etc. It is expected that the combined effort of this workshop participants from various fields with complementary strength will generate new ideas and innovative strategies that can allow us to continue to collaborate and obtain new results, as well as inspire young researchers within the workshop.

2 Recent Developments and Open Problems

There have been many remarkable developments in the areas of ill-posedness in terms of blow-up, norm inflation and non-existence. E.g., Elgindi [8] proved that there exists $\alpha > 0$ and a divergence-free odd initial data in $C^{1,\alpha}(\mathbb{R}^3)$ with certain growth condition such that the unique local odd solution to the Euler equations emanating from such initial data blows up in finite time. This proof is done by reformulating the problem into self-similar variables, and look for an asymptotically self-similar blow-up solution. It is an interesting question whether one can prove blow-up using other mechanisms without relying on the self-similar formulation. Recent progress in this direction has been discussed in the talk by Diego Cordoba.

On the other hand, convex integration is a technique with its roots from the work of Nash [15] in geometry that was adapted to the partial differential equations of fluid mechanics by De Lellis and Székelyhidi Jr. [7] to prove the non-uniqueness of weak solutions to the Euler equations in any dimension at the regularity level of $L^\infty(\mathbb{R} \times \mathbb{R}^d)$. Subsequently, it was extended by many authors and various conjectures that remained open for many decades were solved; the following are only some of such solved problems.

1. Physicist/Chemist Onsager [16] in 1949 conjectured the Hölder regularity of $\frac{1}{3}$ to be the critical threshold concerning the conservation of energy for the Euler equations:

   > "velocity field in such 'ideal' turbulence cannot obey any Lipschitz condition ... for any order n greater than 1/3."

2. Plasma physicist Taylor [19] in 1974 conjectured that the magnetic helicity is conserved in the infinite conductivity limit;

   > "[magnetic helicity] will be almost unchanged so long as departures from perfect conductivity are small."

3. Serrin [18] in 1963 conjectured

   > "whether a non-constant [weak] solution of the [3D Navier-Stokes equations] can ever come to rest in a finite time."

We refer to the surveys [2, 3] and references therein for more details. One of the major open problems is whether the Leray-Hopf weak solutions to the three-dimensional Navier-Stokes equations are unique or not; this is the context of Ladyzhenskaya's conjecture.

Finally, when partial differential equations are forced by random noise, they are called stochastic partial differential equations. Stochastic partial differential equations offer multiple advantages in applications over their deterministic counterpart. First, the stochastic forces are useful in understanding turbulence. Second, effects from external force, such as wind on ocean waves, are unpredictable. Third, for complex models consisting of many variables in real-world such as financial markets, an addition of the noise can turn it to become probabilistic and justify its simplification that is needed to make any mathematical analysis feasible. Finally, fluid in microscopic scales display collisions of molecules which are more accurately characterized as random and chaotic rather than a deterministic function of time and space.

It has been known that certain noise can regularize the solutions allowing one to deduce properties of the corresponding solutions that seem impossible in the deterministic case. A famous example is how a linear multiplicative noise can give a damping effect allowing one to prove probabilistic global well-posedness result starting from small initial data even in the inviscid case. Another famous example is that transport noise has allowed [9] to prove the global well-posedness of a transport equation with a given velocity field with regularity far below the well-known DiPerna-Lions criteria. Yet, using the technique of convex integration, Hofmanová, Zhu, and Zhu [11] were able to prove non-uniqueness even for the stochastic Navier-Stokes equations.

A related major open problem is the the Yang-Mills problem to prove that for any compact simple gauge group, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$ ([4]). In this regard, Parisi and Wu [17] proposed a stochastic quantization approach to prove the existence of the Yang-Mills measure as an invariant measure of the stochastic Yang-Mills equation forced by space-time white noise (STWN) ξ, where STWN ξ is a Gaussian field that is white in both time and space; i.e.,

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$$

and has been utilized in the physics literature very frequently for many decades. Much progress has been made very recently by the group of Chandra, Chevyrev, Hairer, and Shen [5] and there remains great interest to study such stochastic Yang-Mills equation.

3 Presentation Highlights

The conference was organized in a hybrid mode with 29 on-site participants and 26 more virtual participants who joined via Zoom. There were a total of 20 research talks: six on Monday, five on Tuesday, three on Wednesday, and six on Thursday. Each talk was 50 minutes long, leaving ample time for discussions. There were multiple questions and comments from virtual participants via Zoom as well. Roughly speaking, the presentations can be grouped into the following four topics, which we summarize below.

3.1 Singularity v.s. regularity in inviscid fluid equations

1. On Monday morning, Alexander Kiselev from Duke University gave a talk concerning regularity and singularity of vortex and SQG patches. One simple form of the surface quasi-geostrophic equations can be written as

$$\partial_t \theta + (u\cdot\nabla)\theta + \eta(-\Delta)^m\theta = 0, \tag{2a}$$
$$u = \mathcal{R}^\perp\theta, \tag{2b}$$

   where $\mathcal{R}$ represents the Riesz transform and $(-\Delta)^m$ is a fractional Laplacian with $m \in [0,1]$. He first gave an overall review on some recent progress on regularity properties of vortex and SQG patches. Then he presented an example of a vortex patch with continuous initial curvature that immediately becomes infinite but returns to C^2 class at all integer times only. One of the main reasons had to do with the Fourier symbol of a Hilbert transform. Kiselev also discussed another result of how the α-SQG patches interpolating between Euler and SQG cases are ill-posed in L^p for $p \neq 2$ or Hölder based spaces. The proofs involve derivation of a new system describing the patch evolution in terms of arc-length and curvature.

2. On Monday afternoon, Hongjie Dong from Brown University gave a talk on global well-posedness for the one-phase Muskat problem. He considered the free boundary problem for a 2D and 3D fluid filtered in porous media, which is known as the one-phase Muskat problem. He discussed the work by him and his collaborators that proved that if the initial free boundary is the graph of a periodic Lipschitz function, then there exists a unique global Lipschitz strong solution. The proof of the uniqueness relied on a new point-wise $C^{1,\alpha}$ estimate near the boundary for harmonic functions.

3. On Monday afternoon, Javier Gomez Serrano from Brown University talked about smooth imploding solutions for the three-dimensional (3D) compressible fluids. He presented results on singularity formation for the 3D isentropic compressible Euler and Navier-Stokes equations for ideal gases. These equations describe the motion of a compressible ideal gas, which is characterized by a parameter called the adiabatic constant. Finite time singularities for generic adiabatic constants were found in the recent breakthrough [14] of Merle, Raphaël, Rodnianski, and Szeftel. Serrano discussed the result by him and his collaborators that allowed them to drop the genericity assumption and construct smooth self-similar profiles for all values of the adiabatic constant. Part of the proof is very delicate and required a computer-assisted analysis.

4. On Tuesday afternoon, In-Jee Jeong from Seoul National University gave a talk concerning well-posedness for generalized SQG equations with singular velocities. The work by him and his collaborators proved strong ill-posedness for the generalized SQG equations for smooth data, when roughly speaking the stream function is more singular than the advected scalar. The mechanism is due to degenerate dispersion near a quadratic shear flow. The case when the stream function is logarithmically singular is of particular interest ("Ohkitani model"); in this case, there is a well-posedness in a scale of Sobolev spaces with time-decreasing exponents, and this result has immediate applications to long-time dynamics of inviscid and viscous alpha-SQG models.

5. On Thursday morning, Diego Cordoba from Instituto de Ciencias Matematicas-CSIC gave a talk concerning blow-up of classical solutions for the incompressible Euler equations. He discussed recent results on the blow-up problem of classical solutions for the incompressible Euler equations with finite energy. Together with his collaborators, he constructed solutions that have instantaneous gap loss of Sobolev regularity in the plane and finite time singularities in the whole space. Previously, such blow-up results was usually done by reformulating the problem into self-similar setting and look for an asymptotic self-similar solution. The key novelty in their joint works is that the blow-up is not obtained through self-similar ansatz, but a different mechanism, where the interaction between localized solutions played a key role.

6. On Thursday morning, Christophe Lacave from Universite Grenoble Alpes gave a talk concerning point vortex for the lake equations. He presented the lake equations which can be considered as a generalization of the 3D axisymmetric Euler equations without swirl. This 2D model differs from the well-known 2D Euler equations due to an anelastic constraint in the div-curl problem. He explained how this new constraint implies a very different behavior of concentrated vortices: the point vortex moves under its own influence according to a binormal curvature law.

3.2 Ill-posedness, convex-integration and anomalous dissipation in fluid dynamics

1. On Monday afternoon, Jiahong Wu from the University of Notre Dame gave a talk on hyperbolic Navier-Stokes and hyperbolic MHD equations. The MHD system consists of the equations of

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi = \nu\Delta u + (b \cdot \nabla)b, \tag{3a}$$
$$\partial_t b + (u \cdot \nabla)b = \eta\Delta b + (b \cdot \nabla)u, \tag{3b}$$
$$\nabla \cdot u = 0, \tag{3c}$$

where $b : \mathbb{R}_+ \times D \mapsto \mathbb{R}^d$ represents the magnetic vector field and $\eta \geq 0$ the magnetic resistivity. The hyperbolic Navier-Stokes equations contain an extra double time-derivative term $\partial_{tt}u$ while the hyperbolic MHD system differs from the standard MHD system by a double time-derivative term in the magnetic field equation $\partial_{tt}b$. The appearance of these terms is not an artifact but reflects basic physics laws. Mathematically the global regularity problem on these hyperbolic equations is extremely difficult. In fact, even the L^2-norm of solutions to the two-dimensional (2D) equations are not known to be globally bounded in the general case. One of the results Wu discussed was the global existence of a wild solution to the 2D hyperbolic Navier-Stokes equations that doubles its initial kinetic energy size in one unit of time via convex integration technique.

2. On Tuesday morning, Gianluca Crippa from University of Basel gave a talk concerning anomalous dissipation in fluid dynamics. Kolmogorov's K41 theory of fully developed turbulence advances quantitative predictions on anomalous dissipation in incompressible fluids: although smooth solutions of the Euler equations conserve the energy, in a turbulent regime information is transferred to small scales and dissipation can happen even without the effect of viscosity, and it is rather due to the limited regularity of the solutions. In rigorous mathematical terms, however, very little is known. Crippa discussed his receont work in collaboration with Colombo and Sorella that consided passive-scalar advection equation where anomalous dissipation was predicted by the Obukhov-Corrsin theory of scalar turbulence, in similar vein to Onsager's conjecture on the Euler equations. Crippa illustrated the main ideas behind their construction of a velocity field and a passive scalar exhibiting anomalous dissipation in the supercritical Obukhov-Corrsin regularity regime. He also described how the same techniques provide an example of lack of selection for passive-scalar advection under vanishing diffusivity, and an example of anomalous dissipation for the forced Euler equations in the supercritical Onsager regularity regime. The proofs partially used stochastic tools such as random characteristics.

3. On Tuesday morning, Mimi Dai from the University of Illinois at Chicago gave a talk on ill-posedness issues for fluid equations. She discussed some recent progresses in the effort to understand the classical problem by exploring ill-posedness behavior of solutions. The emphasis is on the construction of pathological solutions which either indicate non-uniqueness or develop finite time singularity. She described how the Euler equations, SQG equations, and even the electron-MHD system of the form

$$\partial_t b + \nabla \times ((\nabla \times b) \times b) = \eta\Delta b \tag{4}$$

can be considered as part of a large family of active scalars. In particular, she and her collaborators obtained a result on the forced SQG equations; specifically, one can find a force that is rough such that the non-uniqueness result can be obtained at a higher regularity level.

4. On Tuesday morning, Daniel Faraco from Universidad Autonoma de Madrid discussed the entropy solutions to macroscopic incompressible porous media (IPM) equations. The IPM equations over $\mathbb{T}^d$ may be written as

$$\partial_t \theta + (u \cdot \nabla)\theta + (-\Delta)^m \theta = 0, \tag{5a}$$

$$u = -(\nabla \pi + \gamma e_d), \tag{5b}$$

$$\nabla \cdot u = 0, \tag{5c}$$

   where T is the liquid temperature, and u models Darcy's law. Faraco discussed how the convex integration has proven to be a successful technique for modeling instabilities in fluid dynamics (Kelvin-Helmholtz, Rayleigh-Taylor, or Saffman-Taylor instabilities) with a prime example being the unstable Muskat problem, which is the mathematical treatment of a two-phase, incompressible fluid evolving through porous media. Faraco gave a review of the existence theory and demonstrated how maximizing potential energy dissipation reconciles Otto's minimizing scheme on the Wasserstein space with the convex integration subsolution, leading to a unique equation for macroscopic evolution. Furthermore, he explained that such an equation admits entropy solutions.

5. On Tuesday afternoon, Sauli Lindberg from University of Helsinski gave a talk on the magnetic helicity and weak solutions of ideal MHD. In ideal MHD, smooth solutions conserve total energy, cross helicity and magnetic helicity. Nevertheless, in view of numerical evidence, ideal MHD should possess weak solutions that

   (a) arise at the ideal (inviscid, non-resistive) limit,

   (b) conserve magnetic helicity

   (c) but dissipate total energy and do not conserve cross helicity.

   Lindberg discussed a result directly on ideal MHD, in collaboration with Faraco and Székelyhidi Jr. that there exist bounded solutions of ideal MHD with prescribed total energy and cross helicity profiles and magnetic helicity. The proof used a new convex integration scheme on two-forms consistent with the conservation of magnetic helicity. Lindberg also discussed another result that proved a conjecture of Buckmaster and Vicol: $L^3_{t,x}$ is the L^p-threshold for magnetic helicity conservation in ideal MHD.

3.3 Dynamics and regularity of Navier-Stokes equations

1. On Monday morning, Ian Tice from Carnegie Mellon University gave a talk concerning stationary and slowly traveling solutions to the free boundary Navier-Stokes equations. The stationary problem for the free boundary incompressible Navier-Stokes equations lies at the confluence of two distinct lines of inquiry in fluid mechanics. The first views the dynamic problem as an initial value problem. In this context, the stationary problem arises naturally as a special type of global-in-time solution with stationary sources of force and stress. One then expects solutions to the stationary problem to play an essential role in the study of long-time asymptotics or attractors for the dynamic problem. The second line of inquiry concerns the search for traveling wave solutions. In this context, a huge literature exists for the corresponding inviscid problem, but progress on the viscous problem was initiated much more recently in the work of Ian Tice and co-authors. For technical reasons, these results were only able to produce traveling solutions with nontrivial wave speed. Ian Tice discussed the well-posedness theory for the stationary problem and described how the solutions thus obtained lie along a one-parameter family of slowly traveling wave solutions.

2. On Monday morning, Giusy Mazzone from Queen's University gave a talk concerning fluid-solid interaction problems. Fluid-solid interaction problems are widely studied because of their connections with hemodynamics, geophysical and engineering applications. The differential equations governing this type of interactions feature a dissipative component (typically arising from the fluid, through the Navier-Stokes equations) and a conservative component (due to the solid counterpart, through either Euler equations of rigid body dynamics or Navier equations of elasticity). This dissipative-conservative interplay has a fundamental role in questions related to existence, uniqueness and stability of solutions to the governing equations. Giusy Mazzone discussed results concerning the existence and stability of solutions to equations characterized by the above-mentioned dissipative-conservative interplay, and described the dynamics of different mechanical systems featuring fluid-solid interactions.

3. On Wednesday morning, Alexis Vasseur from the University of Texas, Austin gave a talk concerning boundary vorticity estimate for the Navier-Stokes equation and control of layer separation in the inviscid limit. He discussed the results by him and his collaborators that provided a new boundary estimate on the vorticity for the incompressible Navier-Stokes equation endowed with no-slip boundary condition. The estimate is re-scalable through the inviscid limit and provides a control on the layer separation at the inviscid Kato double limit, which is consistent with the layer separation predictions via convex integration.

4. On Wednesday morning, Tobias Barker from University of Bath gave a talk concerning the dynamics of the 3D Navier-Stokes equations from initial data with zero third component. The study of the 3D Navier-Stokes equations under certain condition only on one of the three velocity field components has attracted much attention recently. In particular, it is known that as long as the third component of the velocity vector field satisfies a certain regularity criterion in a Sobolev scaling-invariant norm, then the solution remains smooth for all time. In 2017 Chemin, Zhang and Zhang posed the question of whether initial data with small third-component (with respect to a scale-invariant norm) implies global regularity of the associated 3D Navier-Stokes solution. Barker discussed results by him and his collaborators concerning the growth properties of solutions for certain initial data with zero third-component.

5. On Thursday morning, Helena Nussenzveig Lopes from Universidade Federal do Rio de Janeiro gave a talk on the conditions for energy balance in 2D incompressible ideal fluid flow. She described anomalous dissipation, flexibility versus rigidity, and focused on the side of rigidity in the 2D case. She pointed out that the Onsager scaling is not the last word on inviscid dissipation. There is a dynamical mechanism to avoid anomalous dissipation in 2D, whereas this is not the case in 3D. Namely, the class of 2D Euler physically realizable weak solution (which are the solutions that can be obtained in vanishing viscosity limit) conserve energy, therefore they are not attainable through convex integration/wild solutions.

6. On Thursday afternoon, Elizabeth Carlson from Caltech gave a talk on learning identifying properties of turbulent flows using analytical techniques in data-driven methods. She described recent developments in the research area of data assimilation that has seen much developments since the pioneering work of [1] by Azouani, Olson, and Titi. To explain how to learn the parameters of a chaotic system using partial observations, she used the Lorenz equations as an example of a chaotic system, and described the algorithm to dynamically learn its parameters from partial observations and its convergence proof. She also discussed a nonlinear-nudging modification of the Continuous Data Assimilation algorithm for the 2D incompressible Navier-Stokes equations, and demonstrated the numerical results.

3.4 Stochastic fluid equations

1. On Wednesday morning, Jonathan Mattingly from Duke University gave a talk concerning random splitting of fluid equations. He described some new models of randomly agitated stochastic dynamics in the context of systems with complex dynamics such as the 2D Euler and Navier-Stokes equations. The models introduced randomness onto the system through a random splitting scheme and can be viewed as a particular class of Random Interested Functions or Piecewise Deterministic Markov Processes. Mattingly went on to explain how the randomly split Galerkin approximations of the 2D Euler equations and other related dynamics can be shown to possess a unique invariant measure that is absolutely continuous with respect to the natural Liouville measure, despite the existence of other invariant measures corresponding to fix points of the PDEs. He also explained how one proves that the dynamics with respect to this measure have positive Lyapunov exponents almost surely. Finally, he discussed recent results that show that the system has a unique invariant measure even when damping is applied to part of the system.

2. On Thursday morning, Gautam Iyer from Carnegie Mellon University gave a talk concerning how mixing can accelerate the convergence of Langevin systems. A common method used to sample from a distribution with density proportional to $p = e^{-V/\kappa}$ is to run Monte Carlo simulations on an overdamped Langevin equation whose stationary distribution is also proportional to p. When the potential V is not convex and the temperature κ is small, this can take an exponentially large (i.e. of order $e^{C/\kappa}$) amount of time to generate good results. Iyer discussed about a method that introduces a "mixing drift" into this system, which allows us to rigorously prove convergence in polynomial time (i.e. a polynomial in $1/\kappa$).

3. On Thursday afternoon, Tommaso Rosati from University of Warwick gave a talk concerning a global-in-time solutions to perturbations of the 2D stochastic Navier-Stokes equations forced by STWN. When the STWN ξ is forced on an equation of the form $\partial_t - \Delta$ such as the Kardar-Parisi-Zhang equation, the spatial regularity of ξ is $C^{\alpha}(\mathbb{T}^d)$ for $\alpha < -\frac{d+2}{2}$ almost surely. Such roughness of the force transmits to the roughness of the solution making the product within the nonlinear term ill-defined according to Bony's paraproducts. The 2D Navier-Stokes equations forced by the STWN has similar difficulties; nevertheless, using Wick products and the explicit knowledge of invariant measure, Da Prato and Debussche [6] in 2002 were able to prove its global well-posedness for almost every initial data. Rosati discussed a new proof, which is a work together with his collaborator, of the global-in-time well-posedness for perturbations of the 2D Navier-Stokes equations driven by STWM.The proof relied on a dynamic high-low frequency decomposition, tools from paracontrolled calculus and an L^2 energy estimate for low frequencies. He described how their arguments require the solution to the linear equation to be a log-correlated field and that their arguments do not rely on (or have) explicit knowledge of the invariant measure: the perturbation is not restricted to the Cameron-Martin space of the noise. Their approach allows for anticipative and critical (L^2) initial data.

4 Scientific Progress Made and Outcome of the Meeting

There were various deep discussions by the participants through the workshop, after talks and during coffee breaks. For example, Rosati and Yamazaki discussed the contents of Rosati's talk in detail and obtained heuristic argument that one can extend [10] to the 3D case up to the exponent of $\frac{5}{4}$, as well as its consequent toward the stochastic Yang-Mills equation from [5]. There has also been some interactions between virtual participants and on-site participants. After some talks, there were some excellent questions raised by virtual participants.

Compared to the other fluid workshops held in the past few years, a key novelty of our workshop is the broad spectrum of people that have brought together with the workshop. We have paid special attention to select researchers that are working on different aspects of fluid equations (singularity v.s. regularity; deterministic v.s. stochastic; fixed-domain v.s. free-boundary; compressible v.s. incompressible; theoretical v.s. numerical...). We have also striven to select both leaders in the field as well as more junior, but up and coming, researchers, and established researchers that are working on areas of renewed interest, but have had less visibility in recent years. Many participants expressed to us that they enjoyed the broad topics covered in this workshop, and they especially appreciated meeting and discussing with other people who they rarely see in their own sub-areas.

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